Exact Coherent Structures in Pipe Flow: Travelling Wave Solutions

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Nonlinear travelling wave solutions are found for a pressure-driven flow through a circular pipe. The numerical solutions consist of three flow structures that sustain each other against viscous decay and are streamwise rolls, streaks and wavelike perturbations. All obtained solutions are characterised by their \( m_0 \)-fold rotational symmetry about the pipe axis and are represented by \( m_0 = 1,2,3,4,5 \) and 6. The travelling waves are born out of saddle node bifurcations and start to exist at a mean Reynolds number of 1251 for \( m_0 = 3 \). The appearance of these solutions is believed to be a relevant precursor to the turbulent transition and have recently been observed in laboratory experiments in very good agreement with the above numerical solutions. The solutions are found by a homotopy approach preceded by an idea called the self-sustaining process, which has proved to be successful in other flow configurations.

1. Introduction
The stability of laminar pressure-driven flow or Hagen-Poiseuille Flow (HPF) through a long circular pipe is one of the most classical and intriguing problems in fluid mechanics. This flow configuration is now believed to be linearly stable for all Reynolds numbers \( (Re) \), see e.g. Sexl (27), Sexl (28), Corcos & Sellars (4), Davey & Drazin (7), Salwen & Grosch (25), Lessen, Sadler & Liu (20) and Meseguer & Trefethen (22), who discovered recently that Hagen-Poiseuille flow was linearly stable up to \( Re = 10^7 \). Osborne Reynolds did the first flow visualisation of the transition from laminar to turbulent flow of water in a pipe in 1883 (24), where the fluid was driven by a constant pressure gradient. The critical Reynolds number \( Re_c \) found by Reynolds turned out to possess the lower limit of about 2000. If the Reynolds number \( Re < Re_c \) transition to turbulence cannot be maintained, and if \( Re > Re_c \), a critical amplitude of a perturbation is required to cause transition. The lower limit of the Reynolds number has been confirmed experimentally by e.g. Wygnanski & Champagne (36), Wygnanski et al. (37) and Darbyshire & Mullin (5). The fact that the lower limit is \( Re_c \approx 2000 \) means that a very large perturbation will decay, as long as \( Re < Re_c \). A recent discovery by Hof et al. (12) indicates that the perturbation amplitude required to cause transition is of the order \( O(Re^{-1}) \) over a range in \( Re \) of 2000-18000. The required finite amplitude to cause transition highlights the nonlinear origin of the observed transition to turbulence.

Since all infinitesimal perturbations are believed to be damped in Hagen-Poiseuille flow for all Reynolds numbers, the Navier-Stokes equations, linearised around this particular solution, can not be employed to find alternative solutions straight forwardly. Recent thinking now views transition in pipe flow to be related to the existence of alternative solutions or travelling wave solutions to Navier-Stokes equations, without any connection to the Hagen-Poiseuille flow. Pipe flow can be considered as a nonlinear dynamical system \( \partial u/\partial t = f(u, Re) \), defined by the governing Navier-Stokes equations and appropriate boundary conditions and the system being parametrized by \( Re \). In pipe flow HPF acts as a linearly stable fixed point for all Reynolds numbers which is a global attractor for \( Re < Re_c \) (nonlinearly stable) and a local attractor for \( Re > Re_c \) (nonlinearly unstable). Any disturbance imposed on HPF at \( Re < Re_c = 81.49 \) will experience monotonic decay, which is the lower limit where short-lived growth of disturbance energy is possible (Joseph & Carmi (14)). A disturbance can therefore initially grow but will eventually decay as long as \( Re_e \leq Re < Re_c \). Another fixed point is possible when finite perturbations are introduced at \( Re_e \leq Re < Re_c \). Here a new limit in phase space is born, represented by the travelling wave solutions. It is believed that the finite amplitude travelling wave solutions provide a skeleton around which time-dependent trajectories (in phase space) can tangle and thus prevent the flow to go back to the laminar flow (Schmiegel & Eckhardt (26)). These new solutions are usually unstable from the onset (Faisst & Eckhardt (9)). It is believed that these solutions are saddle points in phase space, to which the flow dynamics approaches via the attracting stable trajectory. The system spends a substantial fraction of time orbiting around these points before being ejected in the unstable direction towards another (unstable) travelling wave solution. The experimental observation of travelling waves (transiently observable) by Hof et al. (13) strengthens the belief of these states being the building blocks of turbulence. In theory it has been discovered that as the Reynolds number is approaching the \( Re_c \), an increasing number of unstable travelling wave solutions appear (Wedin & Kerswell (35), Faisst & Eckhardt (9)), that might form a turbulent transient with long lifetimes, before decaying to the Hagen-Poiseuille flow (Faisst & Eckhardt (10)). Thus, the basin of attraction of the Hagen-Poiseuille flow is progressively decreasing with increasing \( Re \) (Hof et al. (12)) so that small but finite perturbations can lead to transition for any \( Re \geq Re_c \).

The existence of alternative solutions to the Navier-Stokes equations has now been shown to exist in several wall-bounded shear flows. Nagata (23) found steady nonlinear solutions for plane Couette flow (PCF) down to \( Re=125 \) and later refined by Waleffe (33) to \( Re=127.7 \). These values can be compared to a transitional value

1 The exact dimensional Hagen-Poiseuille flow is \( u^*(s^*) = \frac{2Re}{\pi s^*_0^2} (s^*2 - s^*_0^2) \) in the streamwise direction, \( s^* \) being the radius of the pipe.
of $Re=325-360$ obtained from experiments (Tillmark & Alfredsson (29), Bottin et al (2), Dauchot & Daviaud (6)) and $Re=375$ by direct numerical simulations by Lundblad & Johansson (21). For plane Poiseuille nonlinear solutions have been traced down to $Re≈1000$ by Ehrenstein & Koch (8) and Waleffe (33) down to $Re=977$, where transition has been observed to occur at $Re=1000$ (Carlson et al. (39) and $Re=1100$ (Alavoyoon, Henningson and Alfredsson (4)). The travelling wave solutions have been shown to capture main turbulent statistics by Waleffe (33) and Kawahara & Kidu (16).

The customary method to find alternative solutions to linearly stable flows is homotopy. This was first adopted by Nagata (23) who found steady nonlinear solutions for plane Couette flow (PCF) down to $Re=125$. To circumvent the problem of linear stability, Nagata discovered the three-dimensional solutions (disconnected from the laminar state) by smooth transformation by starting from the neighbouring problem of rotating Taylor-Couette flow (TCF), for which nonlinear solutions are known. By involving a continuation parameter, the rotation rate, and progressively deforming TCF through approaching zero rotation, provided Nagata with the means of finding nonlinear solutions to PCF.

Recently a new theoretical approach, named the self-sustaining process (SSP) has emerged which can lead to nonlinear travelling wave solutions in the form of streamwise rolls, streamwise streaks and wavelike perturbations. It was pioneered by Waleffe (31, 30) and Waleffe et al. (11, 18), and arose out of efforts to understand how turbulence is maintained rather than initiated at low Reynolds numbers for plane Couette flow. This idea does not focus the ideas behind the SSP, Waleffe developed a homotopy approach on plane Couette flow and plane Poiseuille flow (32, 33), aiming at finding nonlinear solutions to the Navier-Stokes equations. By involving a continuation parameter in the form of an artificial forcing function, he replaced the physical system with the forced one and located a bifurcation point from where a new branch of solutions could be traced towards the original system with vanishing forcing.

We present six nonlinear solutions with respect to a one to sixfold rotational symmetry according to a transformation $\tilde{R}^m$ of $\rho^*, \nu^*$ and $s^*_{0}$, which is uniquely realised at low Reynolds numbers $Re = s^*_{0}W^*/\nu^*$, where $W^*$ is the centreline speed. The constant pressure gradient is

$$w^*(s^*) = W^*(1 - s^2/s^*_{0}^2)$$

(1)

which is uniquely realised at low Reynolds numbers $Re = s^*_{0}W^*/\nu^*$. The mean Reynolds number can be defined in terms of the mean streamwise speed $\bar{W}^*$

$$\bar{W}^* = \frac{1}{s^*_{0}^2} \int_0^{s^*_{0}} \int_0^{2\pi} w^* s^* ds^* d\phi$$

(3)

where the mean Reynolds number is $Re_m = 2s^*_{0}\bar{W}^*/\nu^*$. The mean Reynolds number is not known a priori, $Re_m$ also serves as a measure of how far the realised flow has deviated from the laminar flow, where $Re$ and $Re_m$ is identical. The governing equations (non-dimensionalized using the centreline velocity $W^*$ and pipe radius $s^*_{0}$) are

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + \frac{1}{Re} \nabla^2 v,$$

(4)

$$\nabla \cdot v = 0.$$  

(5)

The boundary condition at the pipe wall is $v(s = 1, \phi, z) = 0$ (no-slip condition) and regularity conditions at the pipe axis. We seek neutrally stable three-dimensional solutions in the form of travelling waves represented by a perturbation velocity $\check{v}$ and pressure $\check{p}$ away from Hagen-Poiseuille flow or

$$v = (1 - s^2)\check{z} + \check{v},$$

(6)

$$p = (P_0 - 4z/Re) + \check{p},$$

(7)

where $\check{v} = (\check{u}, \check{v}, \check{w})$. Computationally, it is preferable to work with the perturbation velocity, that is $\bar{v} = v - (1 - s^2)\check{z}$ which then satisfies homogeneous boundary conditions at the pipe wall. The velocity $\bar{v}$ and pressure $\check{p}$ are presumed periodic along the pipe. The governing equations, (4) and (5), rewritten for these new variables and used henceforth are

$$\frac{\partial \bar{v}}{\partial t} + (1 - s^2)\frac{\partial \bar{v}}{\partial z} + 2s v \cdot \check{z} + \check{v} \cdot \nabla \bar{v} = -\nabla \check{p} + \frac{1}{Re} \nabla^2 \bar{v},$$

(8)

$$\nabla \cdot \bar{v} = 0.$$  

(9)

The travelling waves possess a number of symmetries, the most important, which is a discrete $m_0$-fold symmetry in the azimuthal direction $\phi$ so that the transformation

$$\mathcal{R}_{m_0}: (\check{u}, \check{v}, \check{w}, \check{p})(s, \phi, z) \rightarrow (\check{u}, \check{v}, \check{w}, \check{p})(s, \phi + 2\pi/m_0, z)$$

(10)

leave them unchanged for some integer $m_0$. Henceforth we shall refer to travelling waves with $\mathcal{R}_{m_0}$ symmetry as simply $\mathcal{R}_{m_0}$-waves. The axis of the pipe ($s=0$) can cause numerical problems
unless specific efforts are made to desensitise the code to this singularity. Therefore care must be taken to guarantee that the solutions are smooth or have a distinct value on the pipe axis. Here we consider a transformation $S_1$

$$S_1 : (s, \phi, z) \rightarrow (-s, \phi + \pi, z),$$

$$S_1 : (u, \bar{v}, \bar{w}, \bar{p}) \rightarrow (-\bar{u}, -\bar{v}, \bar{w}, \bar{p}).$$

(11)

The coordinates $(s, \phi, z)$ and $(-s, \phi + \pi, z)$ represent the same point in the pipe. The relation for the unit vectors under $S_1$ is $s(-s, \phi + \pi) = -s(s, \phi), \phi(-s, \phi + \pi) = -\phi(s, \phi)$ and $\bar{z}(-s, \phi + \pi) = \bar{z}(s, \phi)$. Transformation $S_1$ gives the parity conditions for the velocity and pressure respectively in $s$ (see for example the appendix of Kerswell & Davey (17)) and also takes care of the singularity that arises at the centre of the pipe. As the pipe is described by the intervals $-1 \leq s \leq 1, 0 \leq \phi < \pi$ and the fact that $(s, \phi, z)$ and $(-s, \phi + \pi, z)$ is the same point in physical space we only need to consider the interval $0 < s \leq 1$ as the solution in $-1 \leq s < 0$ can be constructed from the solution obtained in $0 < s \leq 1$. All governing equations were imposed by using a Galerkin projection over $\phi$ and $z$ and a collocation over the positive zeros of $T_{2N}(s)$ on the following collocation points in $s$

$$s_i = \cos\left(\frac{2i - 1 - \pi}{4N}\right), \quad i = 1, \ldots, N$$

(12)

where $T_n$ is the Chebyshev polynomial and $N$ the truncation level in $s$. The nonlinear system (8)-(9) is solved by the Newton-Raphson’s method.

3. The Self-Sustaining Process

The SSP defines three flow structures that maintain each other against viscous decay and are two-dimensional streamwise rolls $[U(s, \phi), V(s, \phi), 0]$ of finite amplitude $\sigma$, two-dimensional streaky flow $W(s, \phi)$ of order one amplitude and three-dimensional waves $\bar{\phi}$ of amplitude $\epsilon$. The SSP consists of the velocity field and pressure (away from HPF) according to equation (13)

$$\begin{bmatrix} \bar{\phi} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} U(s, \phi) \bar{s} \\ V(s, \phi) \bar{\phi} \\ 0 \\ P(s, \phi) \end{bmatrix}_{\text{rolls}} + \begin{bmatrix} 0 \\ 0 \\ W(s, \phi) \bar{z} \end{bmatrix}_{\text{streaks}} + \begin{bmatrix} \bar{u}(t, s, \phi, z)\bar{s} \\ \bar{v}(t, s, \phi, z)\bar{\phi} \\ \bar{w}(t, s, \phi, z)\bar{z} \end{bmatrix}_{\text{waves}}. \tag{13}$$

The SSP has three phases, see figure 1, where in the first phase the rolls $[U(s, \phi), V(s, \phi), 0]$ cause a $\phi$-dependency, or $w(s) \rightarrow W(s, \phi)$ ($\bar{W}$ is defined as $1 - s^2 + W(s, \phi)$, where $W$ (streaks) is yet to be determined). In other words, the rolls redistribute the Hagen-Poiseuille flow in such a way that they drag slow moving fluid into faster moving fluid and lift faster moving fluid into slower moving flow, causing azimuthal fluctuations, or streaks, away from a mean. A consequence of $z$-independent rolls is that the rolls do not have an energy source and the flow would undergo a viscous decay back to laminar (Joseph & Tao (15)). The azimuthal fluctuations cause in turn a development of a wavy time-periodic three-dimensional perturbation, with wavenumber $\alpha$, of the form $\bar{\psi} = v_1(s, \phi) e^{i\alpha z}$. The final step of the SSP is called the feedback which is the most crucial phase, since it brings about guidance to pick the appropriate wavelike perturbation to sustain the initial rolls and the streaks. A positive feedback closes the self-sustaining process and provides us with a linear SSP-solution that will aid us in finding nonlinear solutions to pipe flow. From the equations involved in the self-sustaining process one finds that a roll amplitude of $O(Re^{-1})$ generates streaks of $O(1)$. At this amplitude the streaks become linearly unstable, where wavelike perturbations of $O(Re^{-1})$ counterbalance the viscous decay of the rolls and keep the flow away from the Hagen-Poiseuille flow. We will now consider the SSP in detail to motivate the search for nonlinear solutions.

Choosing Streamwise Rolls

The equations that govern the rolls are

$$\frac{\partial U}{\partial t} + \frac{\partial P}{\partial s} - \frac{1}{Re} \bar{s} \cdot \nabla^2 \bar{V} = - \bar{s} \cdot (\nabla \bar{V} + \bar{\psi} \nabla \bar{\phi}). \quad (15)$$

$$\frac{\partial V}{\partial t} + \frac{1}{Re} \frac{\partial P}{\partial \phi} - \frac{1}{Re} \bar{\phi} \cdot \nabla^2 \bar{V} = - \bar{\phi} \cdot (\nabla \bar{V} + \bar{\psi} \nabla \bar{\phi}). \quad (16)$$

These equations are decoupled from the streaks $W(s, \phi)$ hence, in this circumstance, we have $V = [U, V, 0]$. Linearising with respect to $V$ and $\bar{\psi}$ and choosing the least decaying eigenfunction as the initial streakwise structure, hence, setting $\lambda^2/Re$ as the decay rate and choosing a single Fourier mode, $(U, V) = [u(s) \cos \Theta \phi, \bar{\psi}(s) \sin \Theta \phi]$, the equations for the rolls reduce to

$$\nabla^2 (\lambda^2 + \bar{\psi}) (\bar{\psi} + i\bar{\psi}) e^{i(m_0 + 1)\phi} = 0. \tag{17}$$

The full solution is then

$$U(s, \phi) = (J_{m_0 + 1}(\lambda s) + J_{m_0 - 1}(\lambda s) - J_{m_0 - 1}(\lambda s) s^{m_0 - 1}) \cos m_0 \phi, \quad (18)$$

$$V(s, \phi) = (J_{m_0 + 1}(\lambda s) - J_{m_0 - 1}(\lambda s) + J_{m_0 - 1}(\lambda s) s^{m_0 - 1}) \sin m_0 \phi. \quad (19)$$

The rolls are assumed to have an amplitude $\sigma$, defined as the maximum amplitude of $U$. The decay rate $\lambda$ is defined from the following eigenvalue relation

$$J_{m_0 + 1}(\lambda m_0) = 0. \tag{20}$$

Here $J$ is the Bessel function of the first kind and $m = 1, 2, 3, \ldots$ is the order of the zero of $J_{m_0 + 1}$. As we are aiming to construct a steady self-sustaining solution we will keep the rolls at their initial amplitude to see if the velocity disturbance $\bar{u}(t, s, \phi, z)$ is able balance the decay with time of the rolls.
Formation of Streaks

Taking the \( z \)-component of the \( z \)-averaged Navier-Stokes equation yields equation (21) for the streaks \( W \)

\[
U \frac{\partial W}{\partial s} + \frac{V}{s} \frac{\partial W}{\partial \phi} - \frac{1}{Re} \nabla^2 W = 2sU - \hat{v} \cdot \nabla \hat{w} \tag{21}
\]

with no-slip boundary condition at the wall. By assuming that the wave-like disturbances are small and currently unknown, we linearise with respect to \( \hat{v} \). Next we impose a transformation \( Z \) on equation (21) in the line \( \phi = 0 \) to get the appropriate symmetry for \( W(s, \phi) \). Transformation \( Z \) is defined as

\[
Z : (s, \phi, z) \rightarrow (s, -\phi, z), \quad Z : (u, v, w, p) \rightarrow (u, -v, w, p). \tag{22}
\]

This transformation leads to an even Fourier series for \( W \)

\[
W(s, \phi) = \sum_{m=0}^{\infty} w^{(m)}(s) \cos mm\phi \tag{23}
\]

with the parity \( w^{(m)}(-s) = (-1)^{mm} w^{(m)}(s) \). The rolls and the streaks are displayed in figure 2 for \( m_0=2 \) and \( 3 \) and \( \lambda = \lambda_{m_0} \). Here one notices that the rolls (arrows) have generated an azimuthal inflectional velocity profile, giving \( m_0 \) high speed streaks (white/light contour levels), located close to the wall, and slow speed streaks (dark/red), at the centre of the pipe.

Instability of the Streaky Flow

To arrive at the equations for the wave-like perturbations linearised around the new basic state \((1 - s^\alpha)\hat{z} + V \hat{V} \) and \( \hat{V} = [U(s, \phi), V(s, \phi), W(s, \phi)] \), the following approximations are made. The full velocity and pressure field in equation (13) are substituted into the Navier-Stokes equations (8) together with the contiunity equation (9), giving equations referred to as \( \hat{s} \cdot (\hat{s}, \phi, \hat{S}) \) and \( \hat{V} \cdot \hat{v} \). Then subtracting these equations by the equations that already have been solved for the rolls and the streaks, gives us \( \hat{s} \cdot (\hat{s}, -\phi, -\hat{S}) \) and \( \hat{V} \cdot \hat{v} \). The resulting equations for the wave-like perturbations are as follows

\[
\frac{\partial \hat{v}}{\partial t} + (1 - s^\alpha) \frac{\partial \hat{v}}{\partial z} - 2s \hat{u} \hat{z} + V \cdot \nabla \hat{v} + \hat{v} \cdot \nabla V + \nabla \hat{p} - \frac{1}{Re} \nabla^2 \hat{v} = -\hat{v} \cdot \nabla \hat{v} - \left[ \begin{array}{c} V \cdot \nabla U - V^2/s \\ V \cdot \nabla V + UV/s \end{array} \right], \tag{24}
\]

\[
\nabla \cdot \hat{v} = 0. \tag{25}
\]

To recover the linear stability problem for the perturbation we drop the right hand side of equation (24), yielding linear equations for the waves. By considering the transformation \( Z \), defined in equation (22), on the Navier-Stokes equations and continuity equation, we can divide the problem into an \( Z \)-odd and \( Z \)-even symmetry of the linear waves. This will cut down the required computer space by a factor of four. The symmetries are defined from the equality

\[
(\hat{u}(x, t), \hat{v}(x, t), \hat{w}(x, t), \hat{p}(x, t)) = \pm Z (\hat{u}(Z^{-1}x, t), \hat{v}(Z^{-1}x, t), \hat{w}(Z^{-1}x, t), \hat{p}(Z^{-1}x, t)) \tag{26}
\]

where \( x = (s, -\phi, z) \) and \( Z^{-1}x=(s, \phi, z) \). This gives the following symmetries as \( (s, \phi, z) \rightarrow (s, -\phi, z) \)

\[
(\hat{u}, \hat{v}, \hat{w}, \hat{p}) \rightarrow (-\hat{u}, \hat{v}, -\hat{w}, -\hat{p}) \quad Z \text{ - odd.} \tag{27}
\]

\[
(\hat{u}, \hat{v}, \hat{w}, \hat{p}) \rightarrow (\hat{u}, -\hat{v}, \hat{w}, -\hat{p}) \quad Z \text{ - even.} \tag{28}
\]

Another symmetry, defined as the \( S_2 \)-transformation, will also be used but has more importance in the nonlinear approach. This transformation reveals the same symmetries as the \( Z \)-transformation. Transformation \( S_2 \) is defined as

\[
S_2 : (s, \phi, z) \rightarrow (s, -\phi, z + \pi/\alpha), \quad S_2 : (u, v, w, p) \rightarrow (u, -v, w, p). \tag{29}
\]

\( A \)-\( Z \)-even instability possesses the same symmetry in \( \phi \) as an \( S_2 \)-odd and \( Z \)-odd instability equals an \( S_2 \)-even mode. The basic flow is both \( Z \)-even and \( S_2 \)-even. As one of the symmetries for the instability is \( Z \)-odd (or equally \( S_2 \)-even) and the basic state is \( Z \)-even, we need to bring into play the \( S_2 \)-even symmetry in order to satisfy both symmetries. Using the \( S_2 \)-transformation will also reduce the required level of truncation for the nonlinear solution. We build in the vanishing boundary conditions at the wall for the velocities in terms of Chebyshev polynomials in \( s \) according to (31) - (33). The boundary condition for the pressure perturbation is not known in advance, thus we set the \( s \)-dependency of the pressure to be a function of \( T_0(s) \) only. Here we focus on the solutions for the fundamental \( Z \)-odd, or symmetric in \( S_2 \)

\[
\begin{align*}
\hat{u} & = e^{i\alpha(z-ct)} \sum_{m=0}^{M} \sum_{n=1}^{N} \left[ u^{m,n}_{un}(s; mm) \cos mm\phi + v^{m,n}_{un}(s; mm) \sin mm\phi \right] + c.c. \\
\hat{v} & = \sum_{m=0}^{M} \sum_{n=1}^{N} \left[ v^{m,n}_{un}(s; mm) \sin mm\phi + p^{m,n}_{un}(s; mm) \cos mm\phi \right] + c.c. \\
\hat{w} & = \sum_{m=0}^{M} \sum_{n=1}^{N} \left[ w^{m,n}_{un}(s; mm) \cos mm\phi + \hat{p}^{m,n}_{un}(s; mm) \sin mm\phi \right] + c.c.
\end{align*} \tag{30}
\]

The boundary condition \( \hat{v} = 0 \) at the wall \((s = 1) \) and the parity conditions are satisfied by setting the following expressions in \( s \) for each velocity component and pressure,

\[
\Theta_n(s; mm) = \begin{cases} T_{2n}(s) - T_{2n-2}(s) & mm_{n0} \text{ odd,} \\
T_{2n+1}(s) - T_{2n-1}(s) & mm_{n0} \text{ even.} \end{cases} \tag{31}
\]

\[
\Phi_n(s; mm) = \begin{cases} T_{2n+1}(s) - T_{2n-1}(s) & mm_{n0} \text{ odd,} \\
T_{2n}(s) - T_{2n-2}(s) & mm_{n0} \text{ even.} \end{cases} \tag{32}
\]

\[
\Psi_n(s; mm) = \begin{cases} T_{2n-1}(s) & mm_{n0} \text{ odd,} \\
T_{2n-2}(s) & mm_{n0} \text{ even.} \end{cases} \tag{33}
\]
The growth rates for \( m_0 = 2 \) and \( 3 \) that subsequently gave a good feedback, are presented in figure 3 as a function of the streamwise wavenumber \( \alpha \) and the roll amplitude \( \sigma \). Owing to the azimuthal inflectional velocity profile of the streaks, an instability sets in at a certain roll amplitude \( \sigma \). According to figure 3 no instability occurs if the amplitude of the rolls \( \sigma \) is too low (see blue curve). The remaining two curves (red and black curves) represent higher values of \( \sigma \), which destabilises the streaks. The wave indicated by the dashed line at neutral stability \( (\alpha \varepsilon_1 = 0) \) gave the best feedback and was used for the nonlinear approach.

Fig. 3 The growth rate \( \alpha \varepsilon_1 \) of the streaks as a function of the streamwise wavenumber \( \alpha \) and roll amplitude \( \sigma \). \( \alpha \) is constant along each individual curve. The left plot displays \( m_0 = 2 \), \( Re = 1700 \), \( \lambda = \lambda_{21} \) at \( \sigma = 0.009188 \) (upper red curve), \( \sigma = 0.007111 \) (middle black) and \( \sigma = 0.005011 \) (lower blue). At \( \alpha = 1.55 \) a neutral wave comes into play (vertical dashed line) which proved to give the best feedback of the four zeros shown. The right figure represents \( m_0 = 3 \), \( Re = 1800 \), \( \lambda = \lambda_{31} \) at \( \sigma = 0.00938 \) (upper red curve), \( \sigma = 0.007800 \) (middle black) and \( \sigma = 0.00625 \) (lower blue). The best feedback is obtained by the neutral wave at \( \alpha = 2.44 \), indicated by the vertical dashed line.

**Nonlinear Feedback on the Rolls**

The interesting feature of this phase, which is the last and most crucial phase in the self-sustaining process, is to elucidate if the wavelike perturbations are able to reenergise the rolls to their initial shape. If the perturbations prove to feedback on the initial rolls, we start with the roll equations introduced in equations (15)-(16), linearised with respect to \( U(s, \phi) \) and \( V(s, \phi) \). To arrive at an equation for the radial component \( U \) of the reenergised rolls we take \( \hat{s} \cdot \nabla \times \nabla \times \) to equations (15)-(16). The final equation for the feedback of the rolls is then

\[
\frac{1}{Re} \left( \frac{\partial}{\partial s} \frac{2}{s} \frac{\partial}{\partial s} + \frac{1}{s^2} \right) U(s, \phi) = \hat{s} \cdot \nabla \times \nabla \times (\hat{v} \cdot \nabla \hat{v}).
\]

(34)

The interesting mode of the wavelike perturbations is the one that is able to offset the decay with time of the initial rolls, hence, we balance the time derivative of the rolls \( \partial U / \partial t \) (decay rate \( \lambda^2 / Re \) of the rolls is included here) with that of the nonlinear term \( \hat{v} \cdot \nabla \hat{v} \) of the linear waves, computed in the stability analysis. To establish whether the mode of the wavelike perturbation is acceptable, a match of the \( s \)-dependent functions of the reenergised rolls in (34) and the initial rolls (18) was sought. This also determined the amplitude \( \epsilon \) of the wavelike perturbation. The wave amplitude is of \( O(\sqrt{\sigma / Re}) \) or \( O(Re^{-1}) \) since \( \sigma = O(Re^{-1}) \). If a positive feedback is obtained, see figure 4, the cycle of the self-sustaining process closes, and we have found a SSP-solution, comprising of the initial rolls and the associated pressure field, streaks and wavelike perturbations. The purpose of the SSP-analysis is that it gives approximate solutions to the Navier-Stokes equations and approximates the position of the important bifurcation point in parameter space \( (Re, \alpha, \sigma) \), for constant \( m_0 \) and \( \lambda_{m0n} \), from where finite-amplitude solutions can be traced. Next we search for nonlinear solutions guided by the SSP-analysis.

4. Exact Solution via Smooth Continuation

Here we convert the SSP-solution by introducing an artificial forcing function \( f = (f_s, f_\phi) \) to the Navier-Stokes equations (8) (Waleffe (32)). The new nonlinear roll equations are

\[
\hat{s} \cdot (\hat{V} \cdot \nabla \hat{V}) = \frac{1}{Re} \hat{s} \cdot \nabla^2 \hat{V} + \frac{\partial P}{\partial s} = f_s(s, \phi) - \hat{s} \cdot (\hat{v} \cdot \nabla \hat{v}),
\]

(35)

\[
\hat{\phi} \cdot (\hat{V} \cdot \nabla \hat{V}) = \frac{1}{Re} \hat{\phi} \cdot \nabla^2 \hat{V} + \frac{\partial P}{\partial \phi} = f_\phi(s, \phi) - \hat{\phi} \cdot (\hat{v} \cdot \nabla \hat{v}),
\]

(36)

where \( \hat{V} = U(s, \phi), V(s, \phi) \) are the nonlinear streamwise rolls and \( P(s, \phi) \) is the associated pressure. The converted SSP analysis represents the flow at the bifurcation point, approximated by the SSP. The purpose of the forcing function is that it will, at the outset, sustain the initial rolls against viscous decay, without the presence of the wavelike perturbations \( \hat{v} \) in equations (35)-(36).

The aim is to bring \( f \) down to zero, where nonlinear solutions to pipe flow exist. We let the forcing function take the place of the time derivatives \( \partial U / \partial t, \partial V / \partial t \) of the linear rolls in (18)-(19) (note that the decay rate \( \lambda^2 / Re \) is accounted for here) and possesses thus the same shape in \( s \) and \( \phi \) as the rolls considered in the self-sustaining process. The linear rolls are converted to nonlinear rolls by the forcing function in equations (35)-(36), using the same values of the involved parameters obtained in the SSP. The components of the forcing function are

\[
f_s(s, \phi) = 2 f(J_{m_0+1}(\lambda s) + J_{m_0-1}(\lambda s) - J_{m_0-1}(\lambda s^{m_0-1})) \cos m_0 \phi,
\]

(37)
Nonlinear Travelling Wave Solutions

We look for the following type of travelling wave solution

\[
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{w} \\
\tilde{\rho}
\end{bmatrix} =
\begin{bmatrix}
\text{const} \\
\text{const} \\
\text{const} \\
\text{const}
\end{bmatrix} e^{i\lambda z + i\sigma t} + c.c.
\]

(40)

\[
= \sum_{m=0}^{M} \sum_{n=1}^{N} \left\{ \sum_{l=1}^{L} \begin{bmatrix}
\tilde{u}^{mn} \Theta_{n}^{m}(z; \pm m\Omega) \sin \pm m\Omega \phi \\
\tilde{v}^{mn} \Theta_{n}^{m}(z; \pm m\Omega) \cos \pm m\Omega \phi \\
\tilde{w}^{mn} \Phi_{n}^{m}(z; \pm m\Omega) \sin \pm m\Omega \phi \\
\tilde{p}^{mn} \Psi_{n}^{m}(z; \pm m\Omega) \cos \pm m\Omega \phi
\end{bmatrix}
\right\} e^{i\lambda z + i\sigma t} + c.c.
\]

The final system of equations is then represented as

\[ F(u^{mn}, v^{mn}, w^{mn}, p^{mn}; Re, f, \alpha) = 0. \]

(41)

The wave speed \( c \) is also unknown, hence we need an extra equation in form of a phase condition which fixes the phase of the solution. The phase condition is

\[ Im\left\{ \sum_{n=1}^{N} u^{1n} \Theta_{n}(0.1; -m_{0}) \right\} = 0, \]

and

\[ 3Damp = \sqrt{\sum_{n=1}^{N} |u^{1n}|^2 + |u^{11n}|^2 + |u^{12n}|^2} \]

(43)

is a measure of the amplitude of the wave-like perturbation. The Navier-Stokes equations are solved for in each step toward \( f=0 \)

\[ f_{0}(s, \phi) = 2f(J_{m_{0}+1}(\lambda s) - J_{m_{0}-1}(\lambda s) + J_{m_{0}-1}(\lambda) s^{m_{0}-1}) \sin m_{0}\phi. \]

(38)

The amplitude \( \sigma \) of the forced streamwise rolls is the maximum value of the radial component \( U(s, \phi) \) and is determined by the forcing amplitude \( f \) in the nonlinear roll equations (35)-(36). Assuming that \( \sigma \ll 1 \) we can neglect the nonlinear terms of the rolls and hence, the forced roll amplitude is defined as

\[ \sigma = \frac{2fRe}{\lambda^2} \max[J_{m_{0}+1}(\lambda s) + J_{m_{0}-1}(\lambda s) - J_{m_{0}-1}(\lambda) s^{m_{0}-1}], \]

(39)

where \( s \in [0, 1] \) and \( \lambda = \lambda_{m_{0}} \) is the decay rate eigenvalue of the rolls determined by \( J_{m_{0}+1}(\lambda) = 0 \) in equation (20). At a certain forcing amplitude \( f_{c} \), the basic flow \( \sigma \) becomes neutrally stable to infinitesimal 3-dimensional wavelike perturbations for constant \( Re \) and \( \alpha \), as predicted by the SSP analysis. The fact that this instability is known to have positive feedback on the rolls makes it very likely that this bifurcation will be subcritical towards \( f = 0 \). The next move is to increase the amplitude \( \epsilon \) of the waves to a finite value, with the aim to find a nonlinear solution to pipe flow.

Fig. 5 The continuation curve for \( m_{0}=3, \alpha=2.44 \) at \( Re=1800 \) and truncation \( M, N, L=8, 24, 5 \) \( f_{crit} = 1.7962 \times 10^{-4} \) as opposed to \( f_{crit} = 1.7474 \times 10^{-4} \) predicted from the SSP analysis. The blue solid line shows how the forcing amplitude can be decreased all the way down to \( f=0 \). The measure of the wave amplitude 3Damp is on the left vertical axis. There are two crossings at \( f=0 \), labelled \( g \) and \( h \), representing solutions at a upper and a lower branch of travelling wave solutions in \( c-Re \) parameter space. The green dashed line represents the phase speed \( c \) (right vertical axis) and illustrates that the lower crossing of the solid blue curve has faster travelling waves than the upper crossing.

Fig. 6 The streamwise-averaged (away from HPF) plots for \( m_{0}=3 \) over increasing \( Re \) (branch 3a in fig.7). Top plot shows the velocity field at \( Re=1631 \) (at the lowest \( Re_{m}=1251 \)) and the bottom two at \( Re=1914 \) at the upper and the lower branch of solution curve 3a in fig.7. The plots lying side by side have the same \( Re \) in order to compare the two solution branches away from the same laminar flow. The contour levels of the plots lying side by side range between -0.5 to 0.08 in steps of 0.031. The contours at \( Re_{m}=1251 \) range between -0.31 to 0.057. The colour of the background represent zero velocity.

\[^{22} \text{The basic flow consists of the same flow structures as in the SSP, i.e } 1 - s^2 \text{ and rolls+streaks.} \]
at \( f=0 \), the aim was to find the lowest possible value of \( Re_m \). A smooth search over \( \alpha \) and \( Re \), for a constant value of \( m_0 \), was hence performed until

\[
\min_{Re,\alpha} Re_m(m_0)
\]

was found. Examples of the velocity field of the \( R_3 \) travelling wave solution are shown in figure 6 for increasing \( Re \). The three solution branches for this particular symmetry are shown in figure 7. At \( Re_m = 1251 \) (fig.6) there are \( 2m_0 \) high-speed streaks (light/white contour levels), whereas on the upper branch, at \( Re_m=1512 \), the high-speed streaks seem to merge into a \( m_0 \) number of high-speed streaks, giving a much more widespread patch of high-speed fluid. An actual transition to \( m_0 \) high-speed streaks is not confirmed since the upper branch is cut-off at this particular Reynolds number, due to lack of accuracy for this particular truncation level. The lower branch at \( Re_m=1362 \) we still have \( 2m_0 \) high-speed streaks. For all covered Reynolds numbers there are \( m_0 \) slow-speed streaks (dark/red) in the centre of the pipe. There is a difference in \( Re_m \), between the solution on the upper- and the lower branch and is mainly related to the difference in the magnitude of the slow-speed streaks in the centre of the pipe.

Figure 7 shows the discovered travelling wave solutions for the \( R_1-R_6 \) symmetries and are shown as far as they are guaranteed to be accurately resolved. The accuracy was checked by tracing the individual curves in figure 7 to higher \( Re_m \) for different truncations. Where the various truncation runs ceased to overlap, gave an indication of the upper limit in \( Re_m \). The \( R_3 \)-wave has the largest value of the three-dimensional amplitude (3Damp=0.0068)

and explains why this solution has the largest degrees of freedom (\( =19390 \)) at \( Re_m = 1251 \) in relation to the remaining symmetries (see table 1). For example, the \( R_6 \)-wave was the easiest to resolve with a value of 3Damp of 0.0025 (at \( Re_m=2869 \)) and the number of degrees of freedom being 15120. Table 1 shows selected data for the travelling wave solutions where they first onset. The characteristic pattern is that the phase speed \( C \), see figure 7, is strictly decreasing with increasing \( m_0 \). The high-speed streaks move closer to the wall with increasing \( m_0 \) (fig.8), and they thus undergo an enhanced dissipation of energy, which explains the increasing critical \( Re_m \) with increasing \( m_0 \). Figure 8 shows the solutions where they first onset and are in general linearly unstable (Faisst and Eckhardt 2003).

<table>
<thead>
<tr>
<th>( m_0 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \min_{R,Re} Re_m )</td>
<td>3046</td>
<td>1358.5</td>
<td>1250.9</td>
<td>1647</td>
<td>2485.5</td>
<td>2869</td>
</tr>
<tr>
<td>( \alpha^* )</td>
<td>2.17</td>
<td>1.95</td>
<td>2.44</td>
<td>3.23</td>
<td>4.11</td>
<td>4.73</td>
</tr>
<tr>
<td>( C(=cW*/W^*) )</td>
<td>1.56</td>
<td>1.44</td>
<td>1.28</td>
<td>1.16</td>
<td>1.08</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Corresponding \( Re \) | 3800 | 1663 | 1631 | 2280 | 3427 | 4069 |

Estimated error in \( Re_m \) | \( \pm 0.5 \) | \( \pm 0.5 \) | \( \pm 3 \) | \( \pm 0.5 \) | \( \pm 2 \) |

Truncation | (9,35.6) | (8,30.8) | (9,35.6) | (7,35.6) | (7,35.7) | (7,40.5) |

Degrees of freedom | 16200 | 19260 | 19390 | 15470 | 17710 | 15120 |

<table>
<thead>
<tr>
<th>( W^<em>/W^</em> )</th>
<th>0.401</th>
<th>0.408</th>
<th>0.384</th>
<th>0.362</th>
<th>0.363</th>
<th>0.353</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \max(u^*) )</td>
<td>0.21</td>
<td>0.076</td>
<td>0.057</td>
<td>0.048</td>
<td>0.064</td>
<td>0.060</td>
</tr>
<tr>
<td>( \min(\overline{w^*}) )</td>
<td>-0.25</td>
<td>-0.32</td>
<td>-0.31</td>
<td>-0.33</td>
<td>-0.33</td>
<td>-0.32</td>
</tr>
<tr>
<td>( \max(\overline{z^*}) )</td>
<td>0.0091</td>
<td>0.014</td>
<td>0.018</td>
<td>0.020</td>
<td>0.0092</td>
<td>0.0087</td>
</tr>
</tbody>
</table>

Table 1 Optimal properties of the travelling wave solutions where they first onset. The value of \( Re \) corresponding to \( \min_{R,Re} Re_m(m_0) \) is given although this is not the minimum value of this parameter only close to it. The cross-stream motion \( v_\perp = [u, v, 0] \).

Fig. 7 The phase speed \( C \) normalised by the averaged streamwise speed (\( C = cW*/W^* \)) as a function of \( Re_m \) at optimal streamwise wavenumber \( \alpha^* \). Each branch is labelled to denote the value of \( m_0 \) and is shown only as far as it is assured to be resolved. Different truncations were compared in order to check the accuracy. For \( m_0=1 \) the truncations \( (M, N, L) = (9,35.5) \) and \( (9,36,5) \) were compared. Truncations \( (8,25.5) \) and \( (9,26.6) \) for \( m_0=2 \), \( (8,24.5), (8,24.7), (9,28.5),(8,28.6) \) and \( (9,26.6) \) for all involved solutions curves for \( m_0=3 \), \( (6,32.6) \) and \( (7,35.5) \) for the two curves of \( m_0=4 \), \( (7,32.4) \) and \( (7,35.5) \) for \( m_0=5 \) and \( (7,32.4) \) and \( (7,40.5) \) for \( m_0=6 \). A characteristic pattern is apparent for the shown solutions, i.e as \( m_0 \) increases the phase speed gradually decreases.

Fig. 8 The streamwise averaged travelling wave solutions at their onset. The solutions are characterised by their symmetries \( (R_1-R_6) \) about the pipe axis.
5. Travelling waves in Experiments

Travelling waves have recently been observed experimentally by Hof et al.\(^\text{(13)}\) in pipe flow in the range of \(Re_m\) between 2000 to 5300. According to figure 9, the numerically obtained \(R_2, R_3\) and \(R_6\)-solutions are in good to very good agreement with the experimental ones with respect to the spatial distribution of the low and high speed streaks, and also to their axial dependence, i.e. stationary high-speed streaks close to the wall and fluctuating low-speed streaks in the centre of the pipe. The travelling waves discovered by Hof et al. were identified as transient objects, but still in excellent agreement with the numerical solutions, and prove the relevance of these nonlinear states. In general on the numerical states, the solutions did not change much with increasing \(Re_m\), which justifies the shown difference in \(Re_m\) (for fixed \(m_0\) in figure 9.

![Fig. 9 Plots comparing the z-averaged numerical solutions (bottom) and their experimental counterpart (top) in pipe flow (away from HPF). The red areas correspond to the high speed streaks, blue the low-speed streaks and arrows the streamwise rolls.](image)

6. Conclusion

In this study we have found three-dimensional finite-amplitude travelling wave solutions for pressure-driven pipe flow. The discovered steady state solutions are characterised by a certain rotational symmetry \(S_{m_0}\) about the pipe axis and an even shift and reflect symmetry \(S_2\). We used the self-sustaining process to find approximate solutions to the Navier-Stokes equations. In the following step we converted the self-sustaining analysis to a continuous environment by adding a forcing function to the Navier-Stokes equations. Upon reaching vanishing forcing amplitude, we discovered steady finite-amplitude solutions to pure pipe flow.

The fundamental travelling waves were found for the first time for \(R_2, R_3\) and \(R_6\) and a subharmonic \(R_1\). We confirm and improve the \(R_2\) and the \(R_3\)-branch obtained by Faisst and Eckhardt, due to the higher degrees of freedoms used, while also discovering two additional \(R_3\)-solution branches. All travelling waves solutions appeared out of saddle node bifurcations, where according to Faisst and Eckhardt, at least the \(R_2\) and the lowest solution branch of our three \(R_3\)-solution branches are unstable from the onset. The \(R_3\)-symmetry survived down to a mean Reynolds number of 1251, using 19400 degrees of freedom (or 3GB) to polish this particular critical \(Re_m\).

The general feature with the increasing \(m_0\) is that the rolls get closer to the wall, hence getting more damped. In other words, higher critical mean Reynolds numbers were required to make the flow unstable and to keep the flow from decaying back to the laminar flow. The same occurrence was noted for the high-speed streaks that moved closer to the wall with increasing \(m_0\), implying that they would undergo an enhanced dissipation of energy.

Moreover, all travelling wave solutions moved at a speed faster than the bulk speed, apart from the lower branch solutions of the \(R_6\)-symmetry that travelled at a speed slightly lower than the bulk speed. The relevance of the \(R_2, R_3\) and \(R_6\) travelling wave solutions to turbulent flow has been verified by the experimental work by Hof et al., where excellent agreement of the spatial distribution of the velocity field has been shown.

参考文献

(14) Joseph, D.D., Carmi, S. Stability of Poiseuille Flow in Pipe,


(24) Reynolds O. An Experimental Investigation of the Circumstances which Determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels. Phil. Trans. 174, 935, (1883).


