Building an Adjoint code (discrete approach)

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Example: building the adjoint code

Let start with a linear system

$$Ax = b \quad (A \text{ and } b \text{ are known})$$

A simple program is written

$$x = A^{-1}b$$

This is a linear operation with

- **input** $b$
- **output** $x$
Example: building the adjoint code

The program that solves the direct equation (marching) is a sequence of linear operations which are executed in a given order.

\[
\begin{align*}
\text{Sub1}(\text{var1}, \text{var2}) & : \text{var2} = M \cdot \text{var1} \\
\text{Sub2}(\text{var2}, \text{var3}) & \\
\text{Sub3}(\text{var3}, \text{var4}) & \\
\text{Sub4}(\text{var5}, \text{var6}) & \\
\text{Solution} & = \text{var6}
\end{align*}
\]
Example: building the adjoint code

The program that solves the direct equation (marching) is a sequence of linear operations which are executed in a given order. If we want to solve the adjoint equation we have to derive the adjoint of our direct code. This can be done by writing down the adjoint of each direct subroutine (which performs a linear operation between the input and the output) and then calling them in a reversed sequence.

Direct

Initialize var1

Sub1(var1, var2)

Sub2(var2, var3)

Sub3(var3, var4)

Sub4(var5, var6)

Solution = var6

input --------- output ---------
Example: building the adjoint code

**Direct**
- Initialize var1
- Sub1(var1, var2)
- Sub2(var2, var3)
- Sub3(var3, var4)
- Sub4(var5, var6)
- Solution = var6

**Adjoint**
- Solution = avar1
- AdjSub1(avar1, avar2)
- AdjSub2(avar2, avar3)
- AdjSub3(avar3, avar4)
- AdjSub4(avar5, avar6)
- Initialize avar6

input -------- output --------
A simple example: the adjoint of a given subroutine

Sub1(var1, var2)

\[ \text{var2} = M \cdot \text{var1} \]

End Sub1

AdjSub1(avar1, avar2)

\[ \text{avar1} = M^t \cdot \text{avar2} \]

End AdjSub1

Each subroutine performs a linear operation between the input and the output and can be represented through the action of a matrix \( M \).

In a real code, however, this operation can be implemented through a complicated sequence of operations and loops. So care must be taken in deriving the correct calling sequence of operations in the adjoint subroutine which gives the correct results.
A simple example: testing the adjoint subroutine

Considering that each subroutine performs a linear operation we can use the definition of the adjoint

\[ < Mu , v > = < u , M^+ v > \]

to test our adjoint subroutine. In fact giving two random values to \text{var1} and \text{avar2} and calling the direct and adjoint subroutine the following identity must be satisfied to \textit{machine precision}

\[ < \text{var2} , \text{avar2} > = < \text{var1} , \text{avar1} > \]

\begin{center}
\begin{tabular}{c c c}
Initialize \text{var1} & \text{AdjSub1(avar1,avar2)} & \text{Test if}

Initialize \text{avar2} & & \text{var1*avar1=var2*avar2}

\text{Sub1(var1,var2)} & & \\
\end{tabular}
\end{center}