Optimal control of complex flows

J. Pralits[1] and P. Luchini[2]

[1]DICAT, Università di Genova, Italy
[2]DIMEC, Università di Salerno, Fisciano (SA), Italy

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Outline

1. Introduction of classical approach
2. Motivation of new approaches
3. Minimal-energy control feedback + application
4. Riccati-less optimal control + application

   Application: control vortex shedding behind circular cylinder
5. Conclusions

Definitions

Complex flows

here problems with large number of degrees of freedom

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Control

The nonlinear governing equations

$$\frac{\partial \bar{\mathbf{x}}}{\partial t} = N(\bar{\mathbf{x}}, \bar{\mathbf{u}}) \quad \text{on} \quad 0 < t < T, \quad \text{with} \quad \bar{\mathbf{x}} = \bar{\mathbf{x}}_0 \quad \text{at} \quad t = 0.$$

- $\bar{\mathbf{x}}$ is the state vector of dimension n
- $\bar{\mathbf{u}}$ is the control of dimension m

Definitions

Complex flows

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Optimal control

The linearized system with $\mathbf{x} = \mathbf{x}(\bar{\mathbf{x}})$ and $\bar{\mathbf{u}} = 0$

$$\frac{\partial \mathbf{x}}{\partial t} = A\mathbf{x} + B\mathbf{u} \quad \text{on} \quad 0 < t < T, \quad \text{with} \quad \mathbf{x} = \mathbf{x}_0 \quad \text{at} \quad t = 0.$$

- $\mathbf{x}$ has dimension n and $\mathbf{u}$ dimension m
- here $n >> m$
- find $\mathbf{u}$ that minimizes a quadratic cost function J
- consider: full state information, no estimation

The linear optimal control problem

The classical full-state-information control problem is formulated as: find the control $\mathbf{u}$ that minimizes the cost function

$$J = \frac{1}{2}\int_0^T [\mathbf{x}^H Q\mathbf{x} + l^2\mathbf{u}^H R\mathbf{u}]\, dt,$$

where l is the penalty of the control, and the state $\mathbf{x}$ and the control $\mathbf{u}$ are related via the state equation

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The solution depends on: $\mathbf{x}_0$, T, Q, R and l.

Solution approaches

- With a feedback rule $\mathbf{u} = K\mathbf{x}$, and a system which is LTI, then the feedback matrix K is computed once off-line (convenient since K is independent of $\mathbf{x}_0$).
- Optimal control $\mathbf{u}$ corresponding to the state at each time step is computed in real time, normally with a finite horizon (value of T) to make it tractable.
  (Example: Adjoint-based control optimization.)

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Both approaches can be solved using the adjoint of the state equation.

Why introduce Adjoint equations ?

Example gradient computation:

$$J = \mathbf{w}^H \mathbf{x}, \qquad \text{where} \quad A\mathbf{x} = \mathbf{b}, \quad \text{Ex. find} \quad \frac{\partial J}{\partial \mathbf{b}}$$

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$$\left(\frac{\partial J}{\partial \mathbf{b}}\right)_j \approx \frac{J(\mathbf{b} + \epsilon \mathbf{e}_j) - J(\mathbf{b})}{\epsilon}$$

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and

$$\frac{\partial J}{\partial \mathbf{b}} = \mathbf{p}^H \qquad \text{One Solution.}$$

Derivation of adjoint

The adjoint variable $\mathbf{p}$ is introduced as a Lagrange multiplier. The augmented cost function is written

$$J = \int_0^T \frac{1}{2}[\mathbf{x}^H Q\mathbf{x} + l^2\mathbf{u}^H R\mathbf{u}] - \mathbf{p}^H[\frac{\partial \mathbf{x}}{\partial t} - A\mathbf{x} - B\mathbf{u}]\, dt,$$

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integration by parts and $\delta J = 0$ gives

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$$\mathbf{u} = -\frac{1}{l^2}R^{-1}B^H\mathbf{p}. \quad \text{means} \quad \frac{\partial J}{\partial \mathbf{u}} = 0$$

Optimal control using feedback

If we consider a feedback rule $\mathbf{u} = K\mathbf{x}$ then

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How does it work ?

Note that state is often denoted direct

Two-point boundary value problem

Write the direct and adjoint equations on a combined matrix form

$$\frac{d\mathbf{z}}{dt} = Z\mathbf{z} \quad \text{where} \quad Z = Z_{2n\times 2n} = \begin{bmatrix} A & -l^{-2}BR^{-1}B^H \\ -Q & -A^H \end{bmatrix} \tag{1}$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{p} \end{bmatrix}, \quad \text{and} \quad \begin{cases} \mathbf{x} = \mathbf{x}_0 & \text{at} \quad t = 0, \\ \mathbf{p} = 0 & \text{at} \quad t = T. \end{cases}$$

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(Z has a Hamiltonian symmetry, such that eigenvalues appear in pairs of equal imaginary and opposite real part.)

This linear ODE is a two-point boundary value problem and may be solved using a linear relationship between the state vector $\mathbf{x}(t)$ and adjoint vector $\mathbf{p}(t)$ vi a matrix $X(T)$ such that $\mathbf{p} = X\mathbf{x}$, and inserting this solution ansatz into (1) to eliminate $\mathbf{p}$.

The Riccati equation

It follows that matrix X obeys the differential Riccati equation

$$-\frac{dX}{dt} = A^H X + XA - XI^{-2} BR^{-1} B^H X + Q \quad \text{with} \quad X(T) = 0. \quad (2)$$

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Finally, if the system is time invariant (LTI) and we take the limit that $T \to \infty$, the matrix X in (2) may be marched to steady state. This steady state solution for X satisfies the continuous-time algebraic Riccati equation

$$0 = A^H X + XA - X l^{-2} B R^{-1} B^H X + Q,$$

where additionally X is constrained such that $A + BK$ is stable.

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the solutions $\mathbf{z}$ that obey the boundary conditions at $t \to \infty$ are spanned by the first n columns of V. The direct $(\mathbf{x})$ and adjoint $(\mathbf{p})$ parts of the these columns are related as $\mathbf{p} = X\mathbf{x}$, where

$$[\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_n] = X[\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n] \quad \rightarrow \quad X = V_{21} V_{11}^{-1}$$

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- Here, two exact methods which do not rely on such modeling,

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Minimal-energy control feedback

In the limit that $l^2 \to \infty$ we consider

$$J = \int_0^T \frac{1}{2}[l^{-2}\mathbf{x}^H Q\mathbf{x} + \mathbf{u}^H R\mathbf{u}]$$

With this defintion the same derivation as before leads to

$$\frac{d\mathbf{z}}{dt} = Z\mathbf{z} \quad \text{where} \quad Z = Z_{2n\times 2n} = \begin{bmatrix} A & -BR^{-1}B^H \\ -l^{-2}Q & -A^H \end{bmatrix}$$

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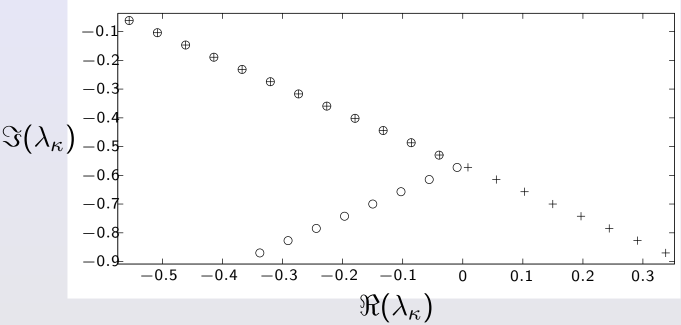
Z becomes block triangular. The direct and adjoint equations are

$$\frac{\partial \mathbf{x}}{\partial t} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{u} = -R^{-1}B^H\mathbf{p}, \qquad \frac{\partial \mathbf{p}}{\partial t} = -A^H\mathbf{p} + 0$$

Minimal-energy control feedback

The eigenvalues of this system is given by the union of the eigenvalues of A and the eigenvalues of $-A^H$.



The eigenvalues of (+) the discretized open-loop system, and (o) the closed-loop system $A + BK$ after minimal-energy control is applied.

Minimal-energy control feedback

Here we know the eigenvalues and only need to compute

$$X = V_{21}V_{11}^{-1}$$

It can be shown that X is only function of V_{21}. K is finally given as a function of the unstable eigenvalues and corresponding left eigenvectors.

$$K = -B^H T_u F^{-1} T_u^H$$

where F has elements

$$f_{ij} = c_{ij}/(\lambda_i + \lambda_j^*)$$

and

$$C = T_u^H B B^H T_u$$

T_u is the matrix containing unstable left eigenvectors

Numerical procedure

- All equations are discretized using second-order finite-differences over a staggered, stretched, Cartesian mesh.
- An immersed-boundary technique is used to enforce the boundary conditions on the cylinder.
- The nonlinear mean-flow equations, along with their boundary conditions, are solved by a Newton-Raphson procedure.
- The linear and nonlinear evolution equations are solved using Adams-Bashforth/Crank-Nicholson.
- The eigenvalue problems are solved using an Inverse Iteration algorithm
- Discrete adjoint equations (accurate to machine precision).

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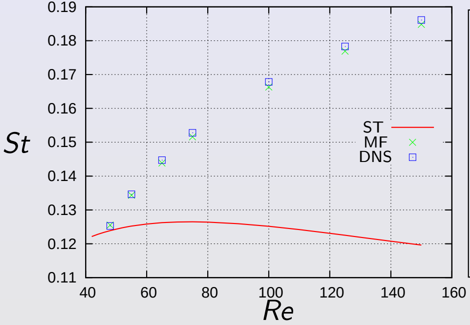
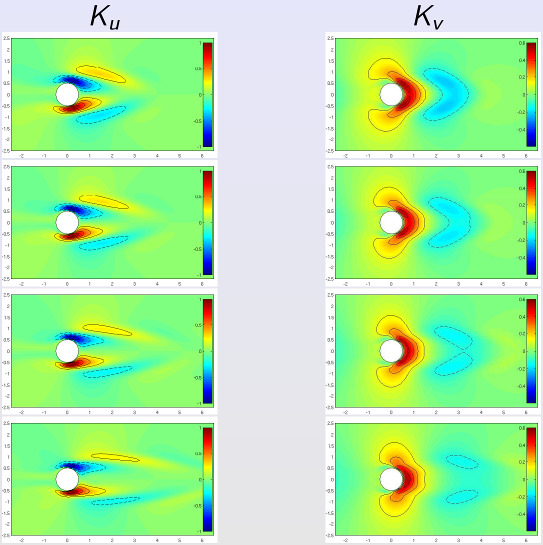
Full state information, Actuator: angular oscillation, $Re = UD/\nu$
Dimension of control $\mathbf{u}$ is $m = 1$



NoControl

The feedback matrix K ($\mathbf{u} = K\mathbf{x}$)

- $Re = 55$
- $Re = 75$
- $Re = 100$
- $Re = 150$

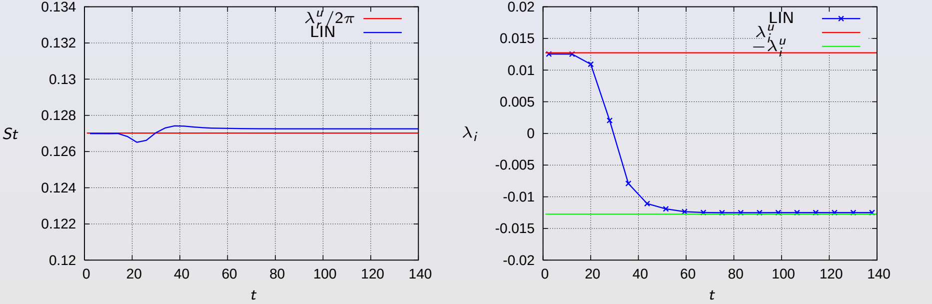


Results: linearized N-S equations

The temporal evolution of the frequency and growth rate is compared with the eigenvalue λ

- The Strouhal number: $St = fD/U$ compared to $St = \lambda_r/2\pi$
- The growth rate: $\sigma = \frac{d}{dt} log(u(t))$ compared to λ_i

Test case: $Re = 55$, control is turned on at $t = 18$



Control of vortex shedding: $Re = 55$

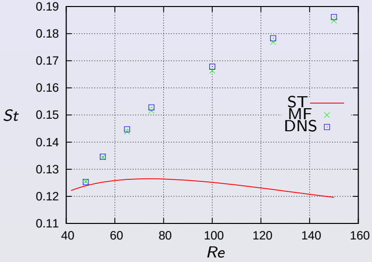
WithControl

Control of vortex shedding: $Re = 55$

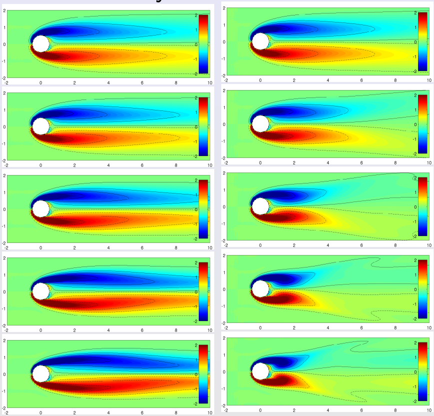
WithControl

Stationary vs. mean flow

St for limit cycle coincide with mean-flow eigenfrequency



Re	stationary	mean
48		
55		
75		
100		
150		



K_u stationary vs. mean

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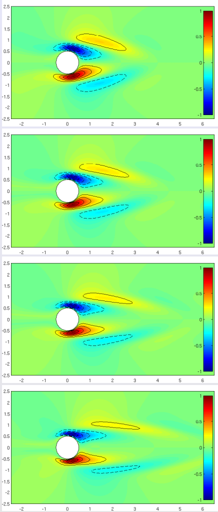
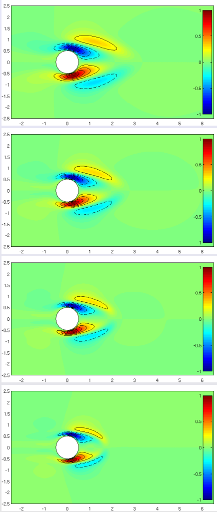
K_u stationary



K_u mean



K_v stationary vs. mean

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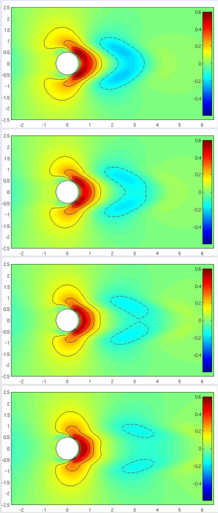
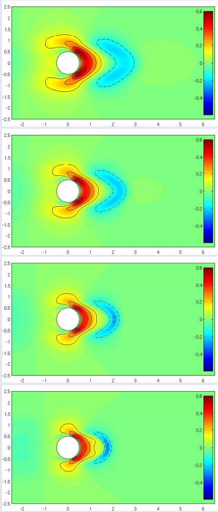
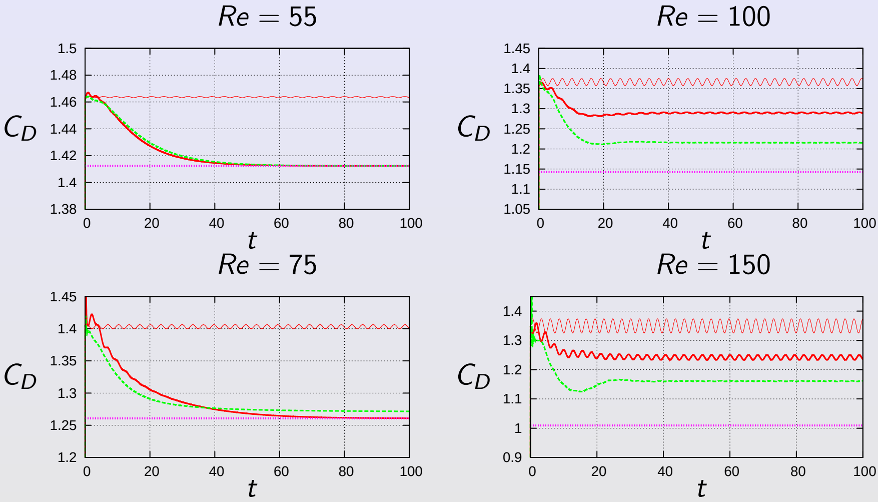
K_v stationary



K_v mean



Control of vortex shedding: stationary vs. mean



Red: stationary flow, Green: mean flow

Riccati-less optimal control

The aim is to compute the solution for K, which is independent of x_0 and time invariant. This can be solved using an iterative procedure to "try" different $\mathbf{x}_0$ (computationally expensive).

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For a converged solution at $t = 0$ we can write

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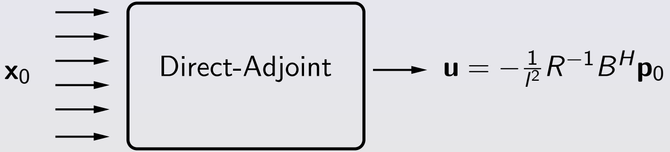
This is a linear relation between the input $\mathbf{x}_0$ and output $\mathbf{u}$.

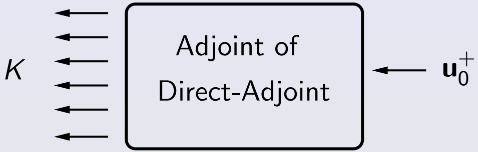


The input has a large dimension and the output a small dimension.

Riccati-less optimal control

Such a problem is efficiently solved using the adjoint equations.

The adjoint input has a small dimension and the output a large dimension.



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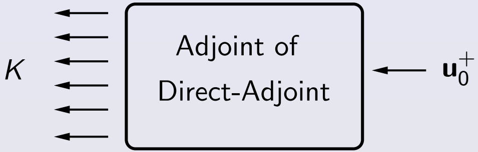

K is obtained from the solution of the adjoint of the direct-adjoint system.

Adjoint of the Direct-Adjoint system

Introduce the adjoint variables $\mathbf{x}^+$ and $\mathbf{p}^+$ and multiply with the direct-adjoint equations, then integrate in time from $t = 0$ to $t = T$. Obs! here we consider that $\mathbf{u}$ has dimension $m = 1$.

$$\int_0^T \mathbf{x}^{+H} \left(\frac{\partial \mathbf{x}}{\partial t} - A\mathbf{x} + \frac{1}{l^2} BR^{-1}B^H \mathbf{p} \right) dt + \int_0^T \mathbf{p}^{+H} \left(\frac{\partial \mathbf{p}}{\partial t} + A^H \mathbf{p} + Q\mathbf{x} \right) dt = 0.$$

Adjoint of the Direct-Adjoint system

Using integration by parts, and considering that both R and Q are symmetric, we obtain

$$-\int_0^T \mathbf{p}^H \left(\frac{\partial \mathbf{p}^+}{\partial t} - A\mathbf{p}^+ - \frac{1}{l^2} BR^{-1}B^H\mathbf{x}^+ \right) dt - \int_0^T \mathbf{x}^H \left(\frac{\partial \mathbf{x}^+}{\partial t} + A^H\mathbf{x}^+ - Q\mathbf{p}^+ \right) dt$$

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with $\mathbf{x}^+(t=T)=0$ and $\mathbf{p}(t=T)=0$, the remaining terms are

$$\mathbf{x}^{+H}(0)\mathbf{x}(0)+\mathbf{p}^{+H}(0)\mathbf{p}(0)=0.$$

Recall that the original linear relation was

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- we can identify one row of K as $\mathbf{x}^{+H}(0)$. $(m=1)$

Riccati-less optimal control: solution procedure

If we let $\mathbf{x}^+ \to -\mathbf{p}$ and $\mathbf{p}^+ \to \mathbf{x}$ we easily obtain the original (Direct-Adjoint) system. (self-adjoint)

Finally: solve the original linear system with new b.c.

$$\frac{\partial \mathbf{x}}{\partial t} = A\mathbf{x} - \frac{1}{l^2} B R^{-1} B^H \mathbf{p} \quad \text{on} \quad 0 < t < T, \quad \mathbf{x}^H(0) \quad \text{is one row of} \quad \frac{1}{l^2} R^{-1} B^H,$$
$$\frac{\partial \mathbf{p}}{\partial t} = -A^H \mathbf{p} - Q\mathbf{x} \quad \text{on} \quad 0 < t < T, \quad \text{with} \quad \mathbf{p}(T) = 0.$$

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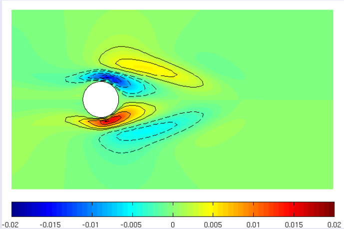
IMPORTANT

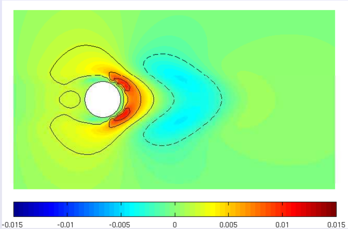
Avoid solving $X_{n\times n}$ solve original system $\mathbf{x}_{n\times 1}$ m times

Results: K for $Re = 55$

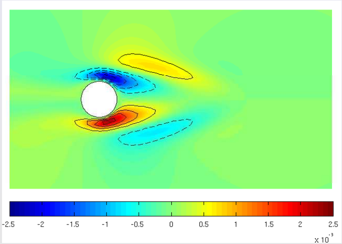
$K_u,\ l^2 = 1$



$K_v,\ l^2 = 1$



$K_u,\ l^2 \to \infty$



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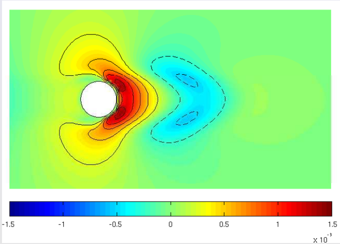
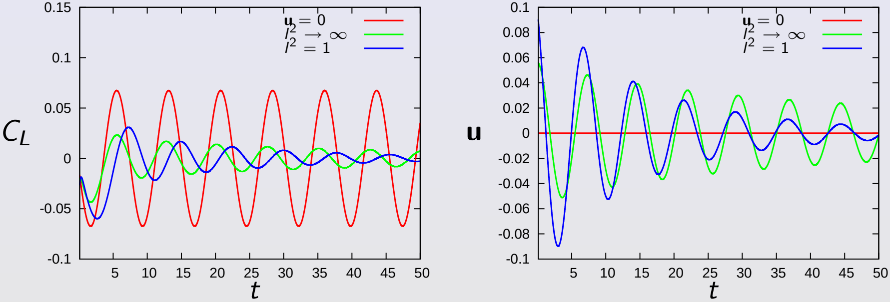


Control of vortex shedding

In the temporal evolution of the lift (C_L) and control **u**:

- C_L and **u** tend to zero as the control is applied
- Control **u** strengthens as l^2 decrease

Test case: $Re = 55$, control is turned on at $t = 0$

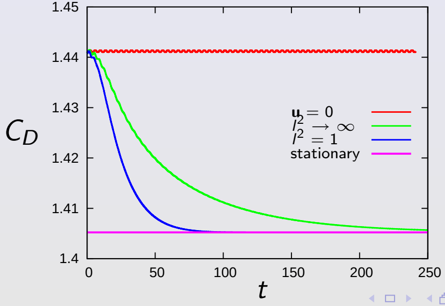


Control of vortex shedding

In the temporal evolution of drag (C_D) coefficient:

- As the control is applied C_D tends to the constant value corresponding to the steady state solution
- The control acts more quickly as l^2 is decreased

Test case: $Re = 55$, control is turned on at $t = 0$



Summary and Conclusions

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- Riccati-less optimal control: The feedback matrix K for the general problem (any value of l^2), can be obtained from the solution of Adjoint of the Direct-Adjoint system. This is equivalent of solving the original system with particular initial condition.
- The methods have been applied to control vortex shedding behind a cylinder.

EXTRA SLIDES

Background: control using rotational oscillation

Aim: reduce C_D

Exp. Tokumaru & Dimotakis (1991), -20%, $Re = 15000$

Feedback control:

Exp. Fujisawa & Nakabayashi (2002) -16% (-70% C_L), $Re = 20000$

Exp. Fujisawa et al.(2001) "reduction", $Re = 6700$

Optimal control (using adjoints):

Num. He et al.(2000) -30 to -60% for $Re = 200 - 1000$

Num. Protas & Styczek (2002) -7% at $Re = 75$, -15% at $Re = 150$

Bergmann et al.(2005) -25% at $Re = 200$ (POD)

Aim: reduce vortex shedding

Feedback control:

Num. Protas (2004) reduction, "point vortex model", $Re = 75$

Optimal control (using adjoints):

Num. Homescu et al.(2002) reduction, $Re = 60 - 1000$

Minimal-energy control feedback

Denoting:

- $\mathbf{x}^i$ and λ^i the i-th right eigenvector and eigenvalue of A,
- $\mathbf{y}^i$ and $-\lambda^{i*}$ the i-th right eigenvector and eigenvalue of $-A^H$,
- $\mathbf{y}^{i*}$ is left eigenvector of A,

we see that the stable eigenvectors of

$$\frac{\partial \mathbf{x}}{\partial t} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{u} = -R^{-1}B^H\mathbf{p}, \qquad \frac{\partial \mathbf{p}}{\partial t} = -A^H\mathbf{p}$$

are of two possible types:

$$\begin{array}{lll} \mathbf{p} = 0,\ \mathbf{x} = \mathbf{x}^i & \text{if} \quad \Re(\lambda^i) < 0 & \text{(stable)} \\ \mathbf{p} = \mathbf{y}^i,\ \mathbf{x} = (\lambda^{i*} + A)^{-1}BR^{-1}B^H\mathbf{y}^i & \text{if} \quad \Re(\lambda^i) > 0 & \text{(unstable)} \end{array}$$

We now project an arbitrary initial condition $\mathbf{x}_0$ onto these modes,

$$\mathbf{x}_0 = \sum_{\text{stable}} d_j \mathbf{x}^j + \sum_{\text{unstable}} f_j (\lambda^{j*} + A)^{-1} B R^{-1} B^H \mathbf{y}^j \qquad (4)$$

and note that in order to reconstruct $\mathbf{p}$ we only need the f_j's, because the stable modes have $\mathbf{p} = 0$. The coefficients d_j can be eliminated from (4) by projecting the left eigenvectors:

$$\mathbf{y}^{i*}\mathbf{x}_0 = \mathbf{y}^{i*} \sum_{\text{unstable}} f_j (\lambda^{j*} + A)^{-1} B R^{-1} B^H \mathbf{y}^j = \sum_{\text{unstable}} c_{ij} f_j$$

where, since $\mathbf{y}^{i*}$ is also a left eigenvector of $(\lambda^{j*} + A)^{-1}$,

$$c_{ij} = \frac{\mathbf{y}^{i*} B R^{-1} B^H \mathbf{y}^j}{\lambda^i + \lambda^{j*}}$$

Only the unstable eigenvalues and left eigenvectors are needed.

The main theorem

Summarizing, the solution of the minimal-energy stabilizing control feedback problem can be written in terms of the unstable left eigenvectors only.

Theorem 1. *Consider a stabilizable system* $\dot{x} = A\mathbf{x} + B\mathbf{u}$ *with no pure imaginary open-loop eigenvalues. Determine the unstable eigenvalues and corresponding left eigenvectors of* A *such that* $T_u^H A = \Lambda_u T_u^H$ *(equivalently, determine the unstable eigenvalues and corresponding right eigenvectors of* A^H *such that* $A^H T_u = T_u \Lambda_u^H$*). Define* $\bar{B}_u = T_u^H B$ *and* $C = \bar{B}_u \bar{B}_u^H$*, and compute a matrix* F *with elements* $f_{ij} = c_{ij}/(\lambda_i + \lambda_j^*)$*. The minimal-energy stabilizing feedback controller is then given by* $\mathbf{u} = K\mathbf{x}$*, where* $K = -\bar{B}_u^H F^{-1} T_u^H$*.*