Hydrodynamic stability

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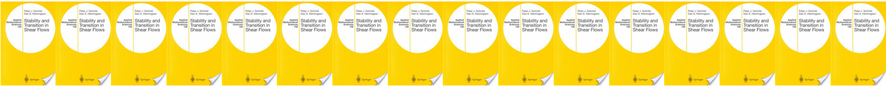
Corso di dottorato in Scienze e Tecnologie per l'Ingegneria (STI):

Fluidodinamica e Processi dell'Ingegneria Ambientale (FPIA)

Outline stability analysis

- **Topic** : Hydrodynamic stability
- **Hours** : 10h
- **Content** :
  1. Introduction
  2. Definitions
  3. Modal analysis (2h)
  4. Nonmodal analysis (2.5h)
  5. Optimal perturbations (Constrained optimization) (2.5h)
  6. Exercises : (3h)
- **Aim** : **Overview** of main concepts; Provide you with tools and **let you test them**
- **Book** : Schmid P. J. & Henningson D. S., *Stability and Transition in Shear Flows*, Springer



Poiseuille flow

The evolution of the linearized equations give us the dynamics of infinitesimal perturbations, potentially leading to transition.

Q1: What is the behaviour for $t \to \infty$?

A1: Modal analysis will give the answer.

Q2: How large can the amplification be for finite t ?

A2: Nonmodal analysis will give the answer.

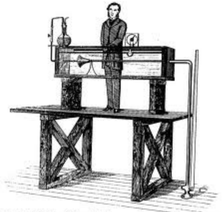
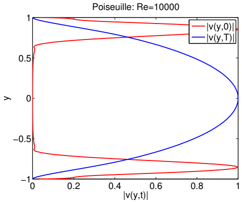
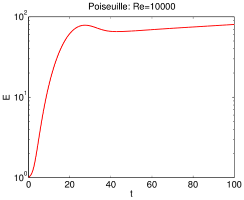


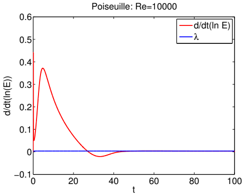
Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper



Perturbations



Energy



Growth rate

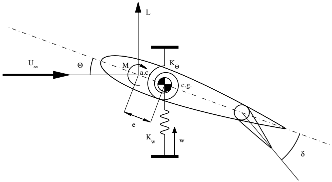
Aeroelasticity

Q1: What is the behaviour for finite and infinite t ?

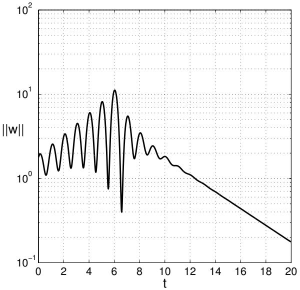
A1: Answer from nonmodal and modal stability analysis.

Q2: Can we determine an optimal way to control instabilities ?

A2: Constrained optimization is a useful tool.
Optimal perturbations ↔ Nonmodal growth



Movie 2



Hydrodynamic stability

Hydrodynamic stability theory is concerned with the **respons of laminar flow to a disturbance of small or moderate amplitude.**

The flow is generally defined as

Stable : If the flow returns to its original laminar state.

Unstable: If the disturbance grows and causes the laminar flow to change into a different state.

Stability theory deals with the **mathematical analysis** of the evolution of disturbances superposed to a laminar base flow.

In many cases one assumes the disturbances to be small so that further simplifications can be justified. In particular, a **linear equation governing the evolution of disturbances is desirable.**

As the disturbance velocities grow above a few % of the base flow, **nonlinear effects** become important and linear equations no longer accurately predict the disturbance evolution.

Although the linear equations have a limited region of validity they are important in detecting physical growth mechanisms and identifying dominant disturbance types.

Reynolds pipe flow experiment (1883)

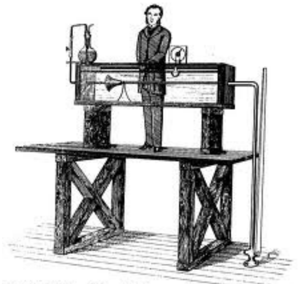
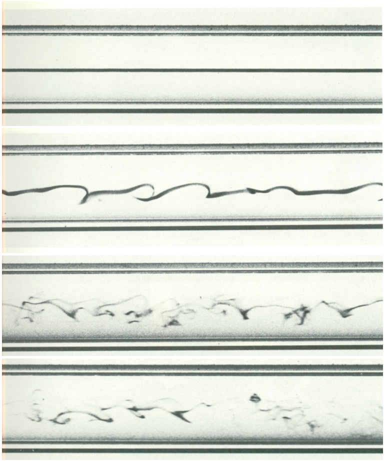


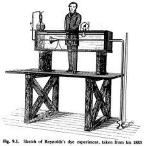
Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper

- Original 1883 appartus
- Dye into center of pipe
- Critical $Re = 13.000$
- Lower today due to traffic



History of shear flow stability and transition

- Reynolds pipe flow experiment (1883)
- Rayleigh's inflection point criterion (1887)
- Orr (1907) Sommerfeld (1908) viscous eq.
- Heisenberg (1924) viscous channel solution
- Tollmien (1931) Schlichting (1933) viscous Boundary Layer solution
- Schubauer & Skramstad (1947) experimental TS-wave verification
- Klebanoff, Tidström & Sargent (1962) 3D breakdown



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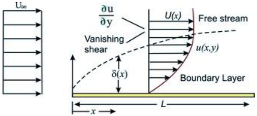
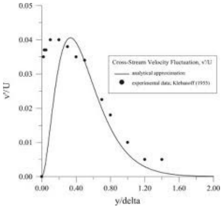
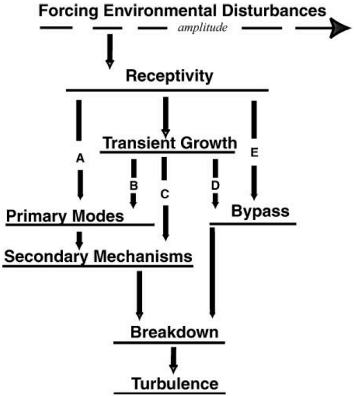
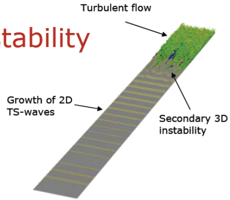




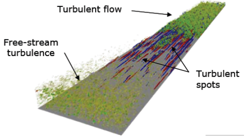
Routes to transition : highly dependent on Tu



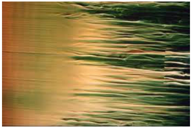
stability



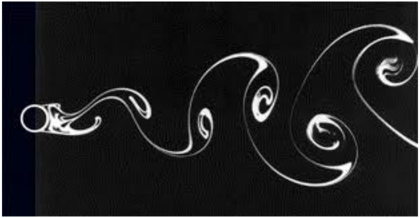
$Tu \sim 0.1\%$



$Tu \sim 10\%$



More examples of instabilities I

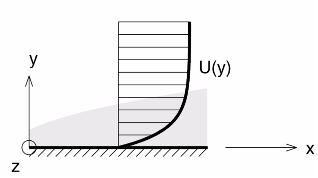


More examples of instabilities II

Movie 2

Disturbance equations I

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i \\
\frac{\partial u_i}{\partial x_i} &= 0 \\
u_i(x_i, 0) &= u_i^0(x_i) \\
u_i(x_i, t) &= 0 \quad \textbf{on solid boundaries}
\end{aligned}
$$



$$
\begin{aligned}
Re &= U_\infty^* \delta^* / \nu^* \\
u_i &= U_i + u_i' \quad \textbf{decomposition} \\
p &= P + p'
\end{aligned}
$$

Introduce decomposition, drop primes, subtract eq's for $\{U_i, P\}$

$$
\begin{aligned}
\frac{\partial u_i}{\partial t} &= -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j} \\
\frac{\partial u_i}{\partial x_i} &= 0
\end{aligned}
$$

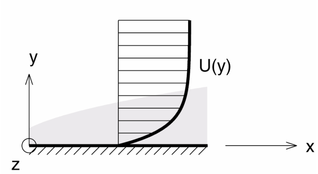
Disturbance equations II

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0$$ **on solid boundaries**



$$Re = U_\infty^* \delta^* / \nu^*$$

$$u_i = U_i + u_i'$$ **decomposition**

$$p = P + p'$$

Introduce decomposition, drop primes, linearize

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

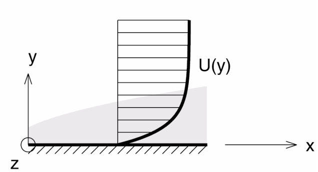
Disturbance equations III

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0$$ **on solid boundaries**



$$Re = U_\infty^* \delta^* / \nu^*$$

$$u_i = U_i + u_i'$$ **decomposition**

$$p = P + p'$$

Linearised Navier-Stokes equations,

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Stability definitions I

$$E(t) = \frac{1}{2}\int_{\Omega} u_i(t)u_i(t)\, d\Omega$$

Stable : $\lim_{t\to\infty} \frac{E(t)}{E(0)} \to 0$

Conditionally stable : $\exists\ \delta > 0 :\ E(0) < \delta \Rightarrow \text{stable}$

Globally stable : Conditionally stable with $\delta \to \infty$

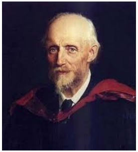
Monotonically stable : Globally stable and $\frac{dE}{dt} \leq 0 \quad \forall t > 0$

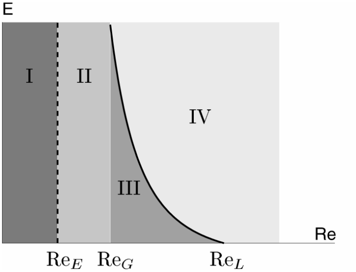
Critical Reynolds numbers

Re_E : $Re < Re_E$ flow monotonically stable

Re_G : $Re < Re_G$ flow globally stable

Re_L : $Re < Re_L$ flow linearly stable ($\delta \to 0$)



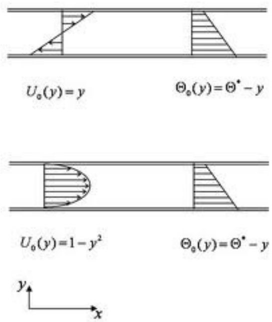


Initial energy **E** vs the Reynolds number **Re**

Critical Reynolds numbers

Flow	Re_E	Re_G	Re_{tr}	Re_L
Hagen-Poiseuille	81.5	–	2000	∞
Plane Poiseulle	49.6	–	1000	5772
Plane Couette	20.7	125	360	∞

Critcial Reynolds numbers for a number of wall-bounded shear flows compiled from the literature.



Reynolds-Orr equation

Scalar multiplication of linearised Navier-Stokes equations with u_i

$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{Re} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j} \left[-\frac{1}{2} u_i u_i U_j - \frac{1}{2} u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{Re} u_i \frac{\partial u_i}{\partial x_j} \right]
\end{aligned}
$$

integrate in space (Ω), vanishing perturbation at the boundaries $\Rightarrow$

$$
\frac{dE}{dt} = \int_\Omega -u_i u_j \frac{\partial U_i}{\partial x_j} \, d\Omega - \frac{1}{Re} \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} \, d\Omega
$$

Nonlinear terms have dropped out
RHS : **exchange of energy with the base flow** and **energy dissipation due to viscosity**

Theorem : Linear mechanisms required for energy growth

Proof : $\frac{1}{E} \frac{dE}{dt}$ independent of disturbance amplitude

Inviscid Analysis

Parallel shear flows : $U_i = U(y)\delta_{1i}$ I

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + &= -\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + &= -\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

Initial conditions :

$$\{u, v, w\}(x, y, z, t = 0) = \{u_0, v_0, w_0\}(x, y, z)$$

Boundary conditions :

$$\mathbf{v}(x, y = y_1, z, t) \cdot \mathbf{n} = 0 \quad \textbf{solid boundary 1}$$

$$\mathbf{v}(x, y = y_2, z, t) \cdot \mathbf{n} = 0 \quad \textbf{solid boundary 2}$$

Parallel shear flows : $U_i = U(y)\delta_{1i}$ II

We can reduce the original **4 eq's & 4 unknowns** to a system of **2 eq's and 2 unknowns**
This is in two steps

1. Take the divergence of the momentum equations. This yields

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}.$$

2. The new pressure equation is introduced in the momentum equation for v. This yields

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} \right] v = 0.$$

The three-dimensional flow is then analyzed introducing the normal vorticity

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right] \eta = -U' \frac{\partial v}{\partial z}.$$

with the boundary conditions

$v = \eta = 0$ at a solid wall and in the far field (or second solid wall)

The Rayleigh equation I

Assume **wave-like solutions**:

$$v(x, y, z, t) = \tilde{v}(y) \exp\, i(\alpha x + \beta z - \omega t)$$

Introduce the ansatz in the v equation.
We limit ourselves to study the v-equation. This yields

$$(-i\omega + i\alpha U)(D^2 - k^2)\tilde{v} - i\alpha U'' \tilde{v} \quad = \quad 0$$

$$\text{substitute} \quad \omega = \alpha c \qquad \Rightarrow$$

$$\left(D^2 - k^2 - \frac{U''}{U - c}\right)\tilde{v} \quad = \quad 0$$

Here, $k^2 = \alpha^2 + beta^2$ and $D^i = \partial^i / dy^i$, and the boundary conditions are

$$\tilde{v}(y = y_1) = \tilde{v}(y = y_2) = 0 \qquad \text{solid boundaries}$$

The Rayleigh equation II

- The Rayleigh equation poses an **eigenvalue problem** of second order with c as the complex eigenvalue. The coefficients of the operator are real. Any complex eigenvalue will therefore appear as complex conjugate pairs. So, if c is an eigenvalue, so is c^*.
- It has a **regular singular point** at $U(y_c) = c$, a condition where the order of the equation is reduced (**critical layer**).
- **Analytical solution** for the eigenfunctions exists (*Tollmien, 1928*)

Instability must depend on $U(y)$ (only parameter). U can be **any** base flow

- We look for base flows where the perturbations become unstable
- By definition perturbations in time behave as $\sim \exp(-i\alpha c_r t)\exp(\alpha \mathbf{c_i t})$
- Take $\alpha > 0$. If $\alpha c_i > 0$ the corresponding mode **grows exponentially in time**

Rayleigh's inflection point criterion (1887) I

Here we consider a parallel shear flow in a domain $y \in (-1, 1)$ and prove a **necessary condition** for instability.

THEOREM : If there exist perturbations with $c_i > 0$, then $U''(y)$ must vanish for some $y_s \in [-1, 1]$

PROOF :
The proof is given by multiplying the Rayleigh equation by $\tilde{v}^*$ and integrating y from -1 to 1. This yields

$$
\begin{aligned}
-\int_{-1}^{1} \tilde{v}^* \left(D^2\tilde{v} - k^2\tilde{v} - \frac{U''}{U-c}\tilde{v} \right) dy &= \\
\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2 \right) dy + \int_{-1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 dy &= 0
\end{aligned}
$$

The first integral is positive definite. The equation equals zero if the second integrand of the second equation changes sign.

Rayleigh's inflection point criterion (1887) II

This is analyzed by multiplying and dividing the second integral with $U - c^*$. This yields

$$\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy + \int_{-1}^{1} \frac{U''(U - c^*)}{(U - c)(U - c^*)} |\tilde{v}|^2 dy \quad = \quad 0$$

The real part is

$$\int_{-1}^{1} \frac{U''(U - c_r)}{|U - c|^2} |\tilde{v}|^2 dy = -\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy,$$

the imaginary part states : U'' **must change sign to render the integral equal to zero if** $c \neq 0$.

$$\int_{-1}^{1} \frac{U'' c_i}{|U - c|^2} |\tilde{v}|^2 dy = 0.$$

Fjortofts criterion (1950) I

Here we consider the same flow as in the Rayleigh's criterion.

THEOREM : Given a monotonic mean velocity profile $U(y)$, a necessary condition for instability is that $U''(U - U_s) < 0$ for some $y \in [-1, 1]$, with $U_s = U(y_s)$ as the mean velocity at the inflection point, i.e. $U''(y_s) = 0$

PROOF : Consider again the real part

$$\int_{-1}^{1} \frac{U''(U - c_r)}{|U - c|^2} |\tilde{v}|^2 dy = - \int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2 |\tilde{v}|^2 \right) dy,$$

We add to the left side the following integral which is identically 0 (Rayleigh's i.p. criteria)

$$(c_r - U_s) \int_{-1}^{1} \frac{U''}{|U - c|^2} |\tilde{v}|^2 dy = 0.$$

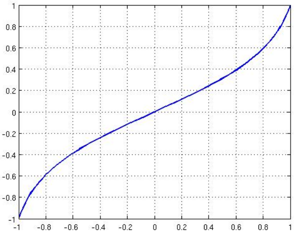
We then get

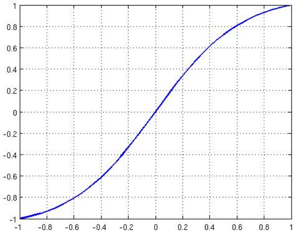
$$\int_{-1}^{1} \frac{U''(U - U_s)}{|U - c|^2} |\tilde{v}|^2 dy = - \int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2 |\tilde{v}|^2 \right) dy,$$

For the integral on the LHS to be negative the value of $U''(U - U_s)$ must be negative somewhere in the flow.

Fjortofts criterion (1950) II

Here are two examples of parallel monotonic shear flow.





Both profiles lead to unstable solutions according to Rayleigh's criterion; however the inflection point has to be a maximum of the spanwise vorticity (not a minimum).

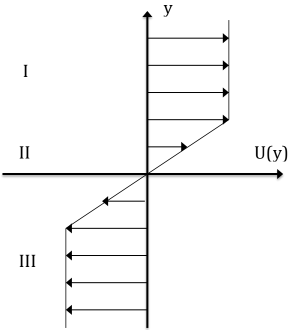
LEFT : unstable according to Fjortoft **RIGHT :** stable according to Fjortoft

Solutions to piecewise linear velocity profiles I

Before computers were available to researchers in the field of hydrodynamic stability theory, a common technique to solve inviscid stability problems was to **approximate continuous mean velocity profiles by piecewise linear profiles**. It allows to find **analytical expression for the dispersion relation** $c(\alpha, \beta)$ and the eigenfunctions.

General considerations:

- $U'' = 0$ which simplifies the Rayleigh equation (except at the connecting points)
- **Matching conditions** must be imposed where U is continuous but U'' is discontinuous



Solutions to piecewise linear velocity profiles II

Matching condition

We can rewrite the Rayleigh equation as

$$D[(U-c)D\tilde{v} - U'\tilde{v}] = (U-c)k^2\tilde{v}$$

and integrating over the discontinuity in U and/or U' located at y_D we get

$$[(U-c)D\tilde{v} - U'\tilde{v}]_{y_D-\epsilon}^{y_D+\epsilon} = k^2\int_{y_D-\epsilon}^{y_D+\epsilon}(U-c)\tilde{v}\,dy$$

As $\epsilon \to 0$ the RHS $\to 0$ which gives the **first** matching condition

$$[\![(U-c)D\tilde{v} - U'\tilde{v}]\!] = 0, \qquad \textbf{Condition 1}$$

which is equivalent to **matching the pressure** across the discontinuity which in Fourier-transformed form reads

$$\tilde{p} = \frac{i\alpha}{k^2}(U'\tilde{v} - (U-c)D\tilde{v}).$$

Solutions to piecewise linear velocity profiles III

A second condition is derived by dividing the pressure $\tilde{p}$ by $i\alpha(U-c)/k^2$. This yields

$$-\frac{k^2\tilde{p}}{i\alpha(U-c)^2} = \frac{D\tilde{v}}{U-c} - \frac{U'\tilde{v}}{(U-c)^2} = D\left[\frac{\tilde{v}}{U-c}\right]$$

Integrating across the discontinuity in the velocity profile gives

$$\left[\frac{\tilde{v}}{U-c}\right]_{y_D-\epsilon}^{y_D+\epsilon} = -\frac{k^2}{i\alpha}\int_{y_D-\epsilon}^{y_D+\epsilon}\frac{\tilde{p}}{(U-c)^2}dy$$

Again, as $\epsilon \to 0$ we obtain the second matching condition

$$\left[\!\left[\frac{\tilde{v}}{U-c}\right]\!\right] = 0, \qquad \textbf{Condition 2}$$

which, for continuous U, corresponds to matching $\tilde{v}$.

Solutions to piecewise linear velocity profiles IV

Summary :

To solve the Rayleigh equation for a piecewise linear velocity profile we need to solve

$$(D^2 - k^2)\tilde{v} = 0$$

in **each subdomain** and impose **boundary** and **matching** conditions

$$\begin{aligned} [\![(U - c)D\tilde{v} - U'\tilde{v}]\!] &= 0, \\ \left[\!\!\left[\frac{\tilde{v}}{U - c}\right]\!\!\right] &= 0, \end{aligned}$$

to determine the coefficients of the fundamental solution and finally the **dispersion relation** $c(k)$.

Solutions to piecewise linear velocity profiles V

Exercise : piecewise linear mixing layer

Velocity profile

$$U(y) = \begin{cases} 1 & \text{for} \quad y > 1 \\ y & \text{for} \quad -1 \leq y \leq 1 \\ -1 & \text{for} \quad y < -1 \end{cases}$$

Boundary conditions

$$\tilde{v} \to 0 \quad \text{as} \quad y \to \pm\infty$$

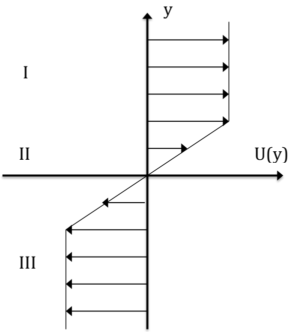
A general solution can be written

$$\begin{aligned} \tilde{v}_I &= A\exp(-ky) & \text{for} \quad & y > 1 \\ \tilde{v}_{II} &= B\exp(-ky) + C\exp(ky) & \text{for} \quad & -1 \leq y \leq 1 \\ \tilde{v}_{III} &= D\exp(ky) & \text{for} \quad & y < -1 \end{aligned}$$

Derive

$$c = c(k)$$

Make a plot of $c(k)$ for $k \in [0, 2]$ and discuss the results.

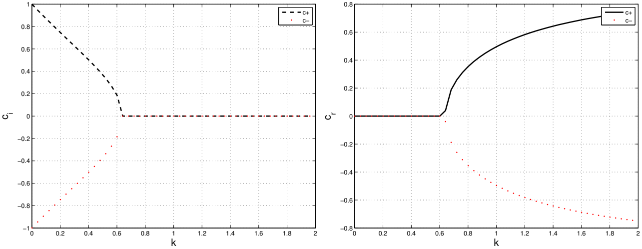


Solutions to piecewise linear velocity profiles VI

Results : Piecewise mixing layer

$$c = \pm\sqrt{\left(1 - \frac{1}{2k}\right)^2 - \left(\frac{1}{4k^2}\right)\exp(-4k)}$$

- For $0 \leq k \leq 0.6392$ the expression under the square root is negative resulting in purely imaginary eigenvalues
- For $k > 0.6392$ the eigenvalues are real, and all disturbances are neutral
- As the wave number goes to zero, the wavelength associated with the disturbances is much larger than the length scale associated with $U(y)$. The limit of small k is equivalent to the limit of zero thickness of region II.



Viscous Analysis

- Only **linear** or **parabolic** velocity profiles satisfy the steady viscous equations (Couette, Poiseuille)
- Inviscid criteria state that Poiseuille flow is stable
- Common sense would suggest that viscosity acts as a **damping**

However, viscous Poiseuille flow undergoes transition: **viscosity destabilizes the flow**

Parallel shear flows : $U_i = U(y)\delta_{1i}$ I

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + &= -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

Initial conditions :

$$\{u, v, w\}(x, y, z, t = 0) = \{u_0, v_0, w_0\}(x, y, z)$$

Boundary conditions : **depend on flow case**

$$\{u, v, w\}(x, y = y_1, z, t) = 0$$ **solid boundaries**

Semi-infinite domain :

$$\{u, v, w\}(x, y \to \infty, z, t) \to 0$$ **free stream**

Closed domain :

$$\{u, v, w\}(x, y = y_2, z, t) = 0$$ **solid boundary 2**

Parallel shear flows : $U_i = U(y)\delta_{1i}$ II

We can reduce the original **4 eq's & 4 unknowns** to a system of **2 eq's and 2 unknowns**
This is in two steps

1. Take the divergence of the momentum equations. This yields

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}.$$

2. The new pressure equation is introduced in the momentum equation for v. This yields

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] v = 0.$$

The three-dimensional flow is then analyzed introducing the normal vorticity

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2 \right] \eta = -U' \frac{\partial v}{\partial z}.$$

with the boundary conditions

$$v = v' = \eta = 0 \qquad \text{at a solid wall and in the far field (or second solid wall)}$$

Orr-Sommerfeld and Squire equations

Assume **wave-like solutions**:

$$v(x,y,z,t) = \tilde{v}(y)\exp\, i(\alpha x + \beta z - \omega t)$$

Introduce the ansatz in the equations for $\{v, \eta\}$. This yields

$$\begin{aligned} \left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} &= 0 \\ \left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right]\eta &= -i\beta U'\tilde{v} \end{aligned}$$

Here, $k^2 = \alpha^2 + \beta^2$ and $D^i = \partial^i/dy^i$.

Orr-Sommerfeld modes : $\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\}_{n=1}^N$

Squire modes : $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^M$

Squire modes I

THEOREM : **Squire modes are always damped**, *i.e.* $c_i < 0 \ \forall \alpha, \beta, Re$

Rewriting the homogeneous Squire equation we get

$$(U - c)\tilde{\eta} = \frac{1}{i\alpha Re}(D^2 - k^2)\tilde{\eta}$$

Multiplying by $\tilde{\eta}^*$ and integrating

$$c \int_{-1}^{1} |\tilde{\eta}|^2 dy = \int_{-1}^{1} U|\tilde{\eta}|^2 dy - \frac{1}{i\alpha Re} \int_{-1}^{1} \tilde{\eta}^*(D^2 - k^2)\tilde{\eta} dy$$

Taking the imaginary part and integrating by parts yields

$$c_i \int_{-1}^{1} |\tilde{\eta}|^2 dy = -\frac{1}{\alpha Re}\left(k^2|\tilde{\eta}|^2 + |D\tilde{\eta}|^2\right) < 0$$

Squire's transformation and theorem I

Let's consider 3D and 2D Orr-Sommerfeld equation with $\omega = \alpha c$

$$
\begin{aligned}
(U-c)(D^2-k^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha Re}(D^2-k^2)^2\tilde{v} &= 0 \\
(U-c)(D^2-\alpha_{2D}^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha_{2D} Re_{2D}}(D^2-\alpha_{2D}^2)^2\tilde{v} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{2D} &= k = \sqrt{\alpha^2+\beta^2} \\
\alpha_{2D} Re_{2D} &= \alpha Re \\
&\Rightarrow \\
Re_{2D} &= Re\frac{\alpha}{k} < Re
\end{aligned}
$$

Squire's transformation and theorem II

Each 3D Orr-Sommerfeld mode corresponds to a 2D Orr-Sommerfeld mode at a **lower** *Re, i.e.*

$$Re_{2D} = Re\frac{\alpha}{k} < Re$$

We can therefore define a **critical Reynolds number** for parallel shear flows as

$$Re_c \equiv \min_{\alpha,\beta} Re_L(\alpha,\beta) = \min_{\alpha} Re_L(\alpha,0)$$

since the growth rate increases with the Reynolds number.

Discretization of the equations in y

The Orr-Sommerfeld equations

$$\begin{aligned} \left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} &= 0 \\ \left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right]\eta &= -i\beta U'\tilde{v} \end{aligned}$$

including boundary conditions $\tilde{v} = D\tilde{v} = \eta = 0 \ y = \pm 1$, can, after suitable discretization (Chebyshev polynomials, finite-differences), be written on the following compact form

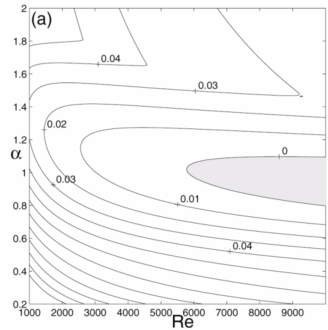
$$\omega\tilde{q} = A\tilde{q} \quad \text{with} \quad \tilde{q} = (\tilde{v}, \tilde{\eta})$$

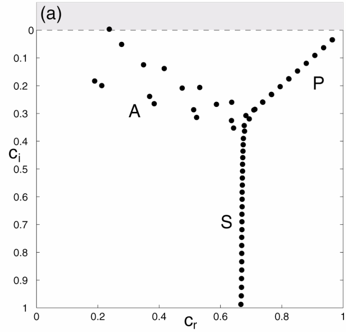
where A is a matrix $\in \mathbb{C}^{2N \times 2N}$. This is an eigenvalue problem from which a solution is obtained for the **eigenvalue** ω_n and **eigenvector** $\tilde{q}_n$. Note that N is the number of discrete points in the wall-normal direction.

Solutions of Eigenvalue analysis I

Plane Poiseuille flow

Neutral curve & spectrum ($Re = 10.000$, $\alpha = 1$, $\beta = 0$)



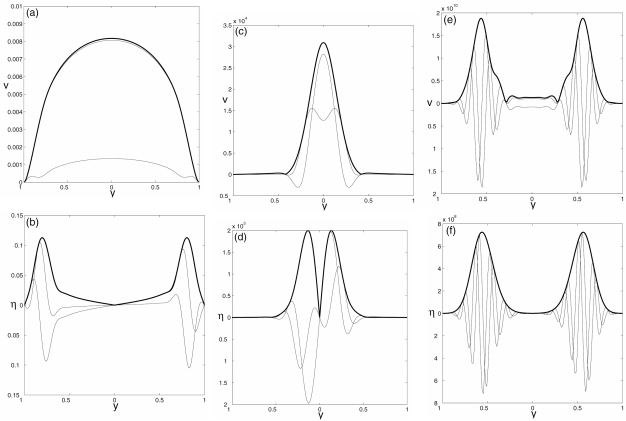


A ($c_r \to 0$), P ($c_r \to 1$), S ($c_r = 2/3$), *Mack (1976)*

Solutions of Eigenvalue analysis II

A, P, S- Eigenfunctions for PPF

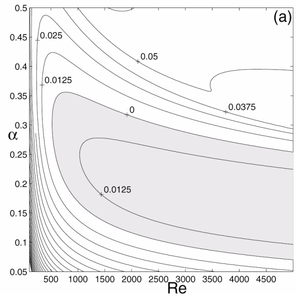
$Re = 5000$, $\alpha = 1$, $\beta = 1$

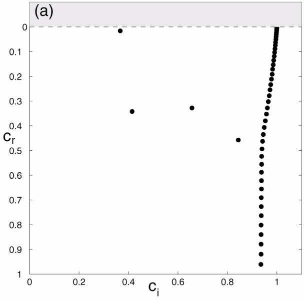


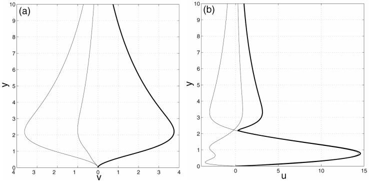
Solutions of Eigenvalue analysis III

Blasius boundary layer

Neutral curve & spectrum ($Re = 500$, $\alpha = 0.2$, $\beta = 0$)



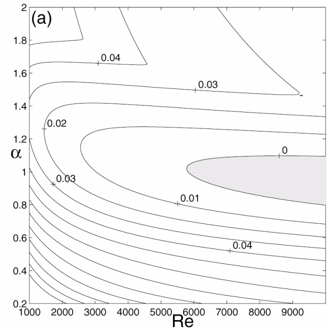


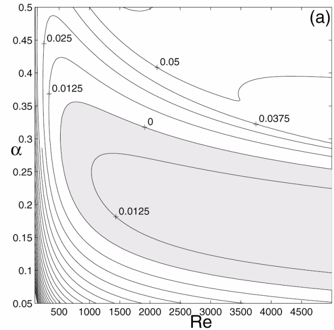


Critical Reynolds numbers

Flow	α_{crit}	Re_{crit}	$c_{r_{crit}}$
Plane Poiseulle	1.02	5772	0.264
Blasius boundary layer flow	0.303	519.4	0.397

Plane Poiseuille Flow & Blasius boundary layer





Continuous spectrum

As $y \to \infty$ the OSE reduces to

$$(D^2 - k^2)^2 \tilde{v} = i\alpha Re[(U_\infty - c)(D^2 - k^2)]\tilde{v}$$

If we assume that

$$\tilde{v}(y) = \hat{v}\exp(\lambda_n y)$$

then the solution is analytical with eigenvalues

$$\lambda_{1,2} = \pm\sqrt{i\alpha Re(U_\infty - c) + k^2}, \quad \lambda_{3,4} = \pm k$$

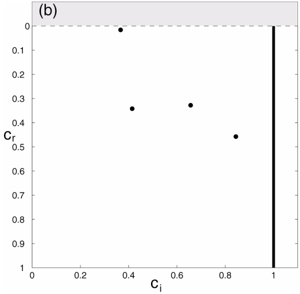
Assuming that $i\alpha Re(U_\infty - c) + k^2$ is **real and negative** which means that $\tilde{v}$ and $D\tilde{v}$ are bounded, $\lambda_{1,2} = \pm iC$

$$\Rightarrow \quad \alpha Re c_i + k^2 < 0, \quad \alpha Re(U_\infty - c_r) = 0$$

From which we can derive analytically $c(k, Re)$

$$c = U_\infty - i(1 + \xi^2)\frac{k^2}{\alpha Re}$$

Example : Blasius boundary layer



Summary

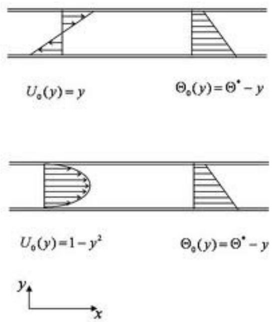
- We are considering the stability of the linearized system
- Stability in the limit in which $E(0) \to 0$
- Reynolds-Orr equation: linear mechanism required for energy growth
- Modal analysis: we consider $v \sim \exp(i\alpha x + i\beta z - i\omega t)$
- Eigenvalue problem
- Rayleigh & Fjortoft: Inflection point criteria for instability
- Piecewise linear profiles: approximate analytical solutions exist
- Squire's theorem: 2D perturbations are more unstable
- Finite domain (ex. channel): all discrete modes
- Semi-infinite domain (ex. boundary layer): discrete and continuous modes
- Re_L sometimes far from Re_{tr}. Modal analysis cannot tell the whole story

Nonmodal stability analysis

Critical Reynolds numbers

Flow	Re_E	Re_G	Re_{tr}	Re_L
Hagen-Poiseuille	81.5	–	2000	∞
Plane Poiseulle	49.6	–	1000	5772
Plane Couette	20.7	125	360	∞

Critcial Reynolds numbers for a number of wall-bounded shear flows compiled from the literature.



Time scale of viscous linear instability

Maximum growth rate in plane Poiseuille flow occurs at $Re \approx 46950$.

It takes $\approx$ 90 time units, corresponding to a propagation about $\approx$ 54 times the channel half width, for the wave to double its amplitude.

Viscous instability acts on a slow time scale

Are we missing some faster dynamics ?

Comments on classical stability theory

- Looking at eigenvalues of the linear stability operator gives us information about the asymptotic behavior of the solution, as $t \to \infty$
- No information is provided about the short-time dynamics if t remains finite
- What if the linear solution experiences transient amplifications before eventually going to zero ?
- Is linearization still valid in this case ?

Linearization & diagonalization I

To analyze the failure of linear stability theory for the case of plane Poiseuille flow, we need to scrutinize the steps involved in the analysis. Linear stability theory is a two-step procedure, consisting of a linearization and a diagonalization step.

Linearization

The linearization step decomposes the flow field into a (steady) base flow and a small amplitude perturbation of order $\mathcal{O}(\epsilon)$

$$\mathcal{Q}(\mathbf{x}, t) = \mathbf{Q}(\mathbf{x}) + \epsilon \mathbf{q}(\mathbf{x}, t) + \mathcal{O}(\epsilon^2).$$

Substituting into the Navier-Stokes equations and extracting the terms of order $\mathcal{O}(\epsilon)$ yields the linearized Navier-Stokes equations governing the evolution of small disturbances

$$\frac{\partial \mathbf{q}}{\partial t} = \mathcal{L}\mathbf{q}.$$

Note that $\mathcal{L} = \mathcal{L}(\mathbf{Q})$.

Linearization & diagonalization II

Diagonalization The formal solution of the linearized Navier-Stokes equations can be written

$$\mathbf{q} = \exp(t\mathcal{L})\mathbf{q}_0$$

where $\mathbf{q}_0$ is the initial condition. The operator exponential propagates the initial condition forward in time.

Note:

$$\exp(t\mathcal{L}) = I + \frac{1}{1!}t\mathcal{L} + \frac{1}{2!}t^2\mathcal{L}^2 + ... = \sum_{n=0}^{\infty}\frac{1}{n!}(t\mathcal{L})^n.$$

We simplify the linear operator $\mathcal{L}$ by transforming it into diagonal form, thus decoupling the degrees of freedom. This allows the analysis of individual modes. If $\mathcal{L}\mathcal{S} = \mathcal{S}\Lambda$ then we have

$$\mathcal{L} = \mathcal{S}\Lambda\mathcal{S}^{-1}$$

where Λ represents a diagonal operator of eigenvalues, and $\mathcal{S}$ consists of the eigenfunctions.

Linearization & diagonalization III

So far $\exp(t\mathcal{L})$ has been diagonalized as $\mathcal{L} = \mathcal{S}\Lambda\mathcal{S}^{-1}$.

Most conclusions about the behavior of $\exp(t\mathcal{L})$ are drawn from Λ with little regard given to the similarity transformation based on $\mathcal{S}$ that diagonalized the linear operator $\mathcal{L}$.

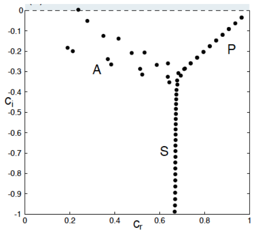


Figure 2: Spectrum for plane Poiseuille flow for $\alpha = 1, \beta = 0, Re = 10000$.

Questions

1. When is the above two-step procedure appropriate and accurate?
2. When can we deduce the behavior of $\exp(t\mathcal{L})$ entirely from Λ?

Evaluating the bounds on $\exp(t\mathcal{L})$ can help us.

Bounds on the operator exponential

Let's determine a lower and upper bound of the operator exponential norm.

$$e^{t\lambda_{max}} \leq \| \exp(t\mathcal{L})\| = \|\mathcal{S}\exp(t\Lambda)\mathcal{S}^{-1}\| \leq \|\mathcal{S}\|\|\mathcal{S}^{-1}\|e^{t\lambda_{max}}$$

Note

- Lower bound: It cannot decay faster than the least stable mode λ_{max}
- The term $\|\mathcal{S}\|\|\mathcal{S}^{-1}\| = \kappa(\mathcal{S})$ is called the condition number and $\kappa(\mathcal{S}) \geq 1$.

Classification

- If $\kappa(\mathcal{S}) = 1$ then the upper and lower bound coincide. The temporal behavior is governed by the exponential behavior for all times.
- If $\kappa(\mathcal{S}) > 1$ then only the asymptotic behavior is given by the least stable mode.

Short explanations:

If $\mathcal{L} = \mathcal{S}\Lambda\mathcal{S}^{-1}$ and $\mathcal{L}^2 = (\mathcal{S}\Lambda\mathcal{S}^{-1})(\mathcal{S}\Lambda\mathcal{S}^{-1}) = \mathcal{S}\Lambda^2\mathcal{S}^{-1}$ then $\mathcal{L}^n = \mathcal{S}\Lambda^n\mathcal{S}^{-1}$

So $I + t\mathcal{L} + 1/(2!)t^2\mathcal{L}^2 + \ldots = \mathcal{S}(I + t\Lambda + 1/(2!)t^2\Lambda^2 + \ldots)\mathcal{S}^{-1} = \mathcal{S}\exp(t\Lambda)\mathcal{S}^{-1}$

Definition of non-normality

- Linear operators with $\kappa(\mathcal{S}) = 1$ are called Normal and have orthogonal eigenvectors
- Linear operators with $\kappa(\mathcal{S}) > 1$ are called Non-normal and have non-orthogonal eigenvectors

Alternatively

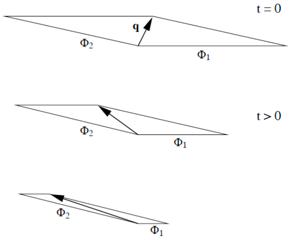
- An operator is non-normal if $\mathcal{L}\mathcal{L}^\star \neq \mathcal{L}^\star\mathcal{L}$
- Linear operators which satisfy $\mathcal{L} = \mathcal{L}^\star$ are called self-adjoint.

Summary common measures

- $\kappa(\mathcal{S}) = \|\mathcal{S}\|\|\mathcal{S}^{-1}\|$
- $\|\mathcal{L}\mathcal{L}^\star - \mathcal{L}^\star\mathcal{L}\|$

Short explanations:
Definition of adjoint: $\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^* v, u\rangle$ for two arbitrary fields u and v, and $\langle , \rangle$ denotes a chosen inner product.
If $\mathcal{L}$ is a real valued matrix then $\mathcal{L}^* = \mathcal{L}^T$

Non-orthogonal superposition



Let us assume that we expand an initial condition q of unit length in a non-orthogonal (two-dimensional) basis as shown in the Figure.

Φ_1 and Φ_2 are two solutions which decay in time. In terms of eigenvalues they are **both stable**.

- The non-orthogonal superposition of exponentially decaying solutions can give rise to short-term transient growth.
- Eigenvalues alone only describe the asymptotic fate of the disturbance, but fail to capture transient effects.
- The source of the transient amplification of the initial condition lies in the nonorthogonality of the eigenfunction basis.

Norm of the operator exponential: Definition of Gain

The correct way to analyze the behavior of $\exp(t\mathcal{L})$ is to compute its potential to amplify a given disturbance over time.

We will measure the size of the disturbance by an appropriate norm (see below) and define as the maximum amplification the ratio of disturbance size to its initial size optimized over all possible initial conditions.

We have

$$\max_{\forall q_0} \frac{\|q\|}{\|q_0\|} = \max_{\forall q_0} \frac{\|\exp(t\mathcal{L})q_0\|}{\|q_0\|} = \|\exp(t\mathcal{L})\| \equiv G(t)$$

The quantity $G(t)$ represents the maximum possible amplification of unit-norm initial conditions over a time period t and is denote the gain.

Short explanations:
Using the inequality $\|\exp(t\mathcal{L})q_0\| \leq \|\exp(t\mathcal{L})\|\|q_0\|$

then

$$\frac{\|\exp(t\mathcal{L})q_0\|}{\|q_0\|} \leq \frac{\|\exp(t\mathcal{L})\|\|q_0\|}{\|q_0\|} = \|\exp(t\mathcal{L})\|$$

General properties of the norm

The choice of inner product will quantitatively influence the maximum amplification potential of the underlying operator. Therefore, the norm and inner product have to be chosen carefully.

Ex. in shear flows the disturbance kinetic energy is normally chosen.

Basic requirements

$$\|q\| \geq 0$$

and

$$\|q\| = 0 \qquad \text{if and only if} \qquad q = 0.$$

Note that the norm has to include all components of q. Otherwise, infinite transient growth is possible, by choosing a disturbance with infinite amplitudes in components that are not accounted for in the norm.

Algorithm : gain (simple)

1. Compute the first N eigenvalues (λ) and eigenvectors (q) of the flow, where $\mathcal{L} = \mathcal{S}\Lambda\mathcal{S}^{-1}$

$$\mathcal{S} = [q_1,\ q_2,\ \dots q_N] \quad \text{and} \quad \Lambda = diag(\lambda_1,\ \lambda_2, \dots, \lambda_N)$$

2. Invert $\mathcal{S}$
3. Form the matrix

$$\mathcal{S} \begin{pmatrix} \exp(t\lambda_1) & & \\ & \ddots & \\ & & \exp(t\lambda_N) \end{pmatrix} \mathcal{S}^{-1}$$

4. Compute the norm of the above matrix

$$G(t) = \|\mathcal{S}\exp(t\Lambda)\mathcal{S}^{-1}\|$$

5. Advance in time and go back to step (3)

The energy norm

From now on we consider disturbances which behave as

$$q(y,t)\exp i(\alpha x+\beta z) \qquad \text{and} \qquad k=\sqrt{\alpha^2+\beta^2}$$

We choose a formulation of the linearized Navier-Stokes equations in terms of the normal velocity v and the normal vorticity $\eta = \partial u/\partial z - \partial w/\partial x$. The linear operator is similar to the classical Orr-Sommerfeld operator.

Our state vector is $q = (v,\eta)^T$ and the kinetic energy in these variables is

$$\begin{aligned} E(t) &= \frac{1}{2k^2}\int_\Omega \left(|\mathcal{D}v|^2 + k^2|v|^2 + |\eta|^2\right) d\Omega \\ &= \|q\|_E^2 = \frac{1}{2k^2}\int_\Omega \begin{pmatrix} v \\ \eta \end{pmatrix}^H \begin{pmatrix} -\mathcal{D}^2 + k^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \eta \end{pmatrix} \\ &= \frac{1}{2k^2}\int_\Omega q^H M\, q\, d\Omega \end{aligned}$$

Here, $\mathcal{D}$ denotes differentiation, k wave-number modulus and M a positive definite weight matrix.

Reduction to a 2-norm

The problem is simplified by transforming the energy norm to a standard L_2-norm. Using Cholesky decomposition we can write $M = F^H F$. We then get

$$\|q\|_E^2 = \frac{1}{2k^2} \int_\Omega q^H F^H F q \, d\Omega = \frac{1}{2k^2} \int_\Omega (Fq)^H F q \, d\Omega$$

Recalling the definition of $G(t)$ we have

$$\begin{aligned} G(t) &= \max_{\forall q_0} \frac{\|q\|_E^2}{\|q_0\|_E^2} = \max_{\forall q_0} \frac{\|Fq\|_2^2}{\|Fq_0\|_2^2} = \max_{\forall q_0} \frac{\|F \exp(t\mathcal{L}) q_0\|_2^2}{\|Fq_0\|_2^2} \\ &= \max_{\forall q_0} \frac{\|F \exp(t\mathcal{L}) F^{-1} F q_0\|_2^2}{\|Fq_0\|_2^2} = \|F \exp(t\mathcal{L}) F^{-1}\|_2^2 \end{aligned}$$

Note that the energy weight is accounted for by the matrices F^{-1} and F.

Projections onto eigenvectors

Computationally, it is not practical to compute $\exp(t\mathcal{L})$. A better solution is to decompose q into a large, but finite, number of eigenvectors of $\mathcal{L}$. This can be written

$$q(y,t) = \sum_{n=1}^{N} \kappa_n(t)\bar{q}_n(y)$$

for the first N eigenfunctions of $\mathcal{L}$. In the following we need to consider the expansion coefficients κ and the matrix exponential $\exp(t\Lambda)$. The latter is much easier to compute.
The energy norm is now written

$$\|q\|_E^2 = \frac{1}{2k^2}\int_\Omega q^H M\, q\, d\Omega = \frac{1}{2k^2}\int_\Omega \left(\sum_{n=1}^{N} \kappa_n^*(t)\bar{q}_n^H\right) M \left(\sum_{m=1}^{N} \kappa_m(t)\bar{q}_m\right) d\Omega$$

$$= \frac{1}{2k^2}\sum_{n,m=1}^{N} \kappa_n^*(t) M_{mn}\, \kappa_m(t),$$

where

$$M_{mn} = \int_\Omega \bar{q}_n^H\, M\, \bar{q}_m\, d\Omega.$$

Finally, with $M_{mn} = F^H F$, we get

$$G(t) = \|F\exp(t\Lambda)F^{-1}\|_2^2$$

Algorithm : gain (energy norm)

1. Compute the first N eigenvalues and eigenvectors of the flow ($\mathcal{L}$)

$$\bar{q}_j,\ \lambda_j \qquad \text{for} \qquad j = 1, ..., N$$

2. Compute the entries of the matrix M_{mn}

$$M_{mn} = \int_\Omega \bar{q}_n^H\, M\, \bar{q}_m\, d\Omega$$

3. Decompose M_{mn} into $F^H F$

$$\begin{aligned} M_{mn} &= U \Sigma U^H \qquad \text{(SVD)} \\ F &= U \Sigma^{1/2} \end{aligned}$$

4. Invert F
5. Form the matrix

$$F \begin{pmatrix} \exp(t\lambda_1) & & \\ & \ddots & \\ & & \exp(t\lambda_N) \end{pmatrix} F^{-1}$$

6. Compute the L_2-norm of the above matrix

$$G(t) = \| F \exp(t\Lambda) F^{-1} \|_2^2$$

7. Advance in time and go back to step (5)

A note on the result

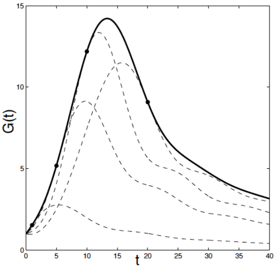


Figure : Amplification $G(t)$ for Poiseuille flow with $Re = 1000$, $\alpha = 1$ (solid line) and growth curves of selected initial conditions (dashed lines).

The quantity $G(t)$ gives the maximum amplification optimized over all possible initial conditions. In general, each point on the curve $G(t)$ is arrived at by a different initial condition, and $G(t)$ represents the envelope of individual growth curves, see figure.

Optimal disturbances

The initial condition yielding the gain at a specific time (t_{spec}) is called optimal disturbance. It can be evaluated by performing a Singular Value Decomposition as

$$F \exp(t\Lambda) F^{-1} = U \Sigma V^H,$$

or equivalently

$$F \exp(t\Lambda) F^{-1} V = U \Sigma.$$

We identify the left singular vector associated with the largest singular value (which is identical to the norm of the matrix exponential) as the desired initial condition that will result in maximum amplification at time t_{spec}. Note : U and V are unitary matrices.

Algorithm : optimal disturbance at $t = t_{spec}$

1. Compute the first N eigenvalues and eigenvectors of the flow ($\mathcal{L}$)

$$\bar{q}_j,\ \lambda_j \qquad \text{for} \qquad j = 1, \dots, N$$

2. Compute the entries of the matrix M_{mn}

$$M_{mn} = \int_\Omega \bar{q}_n^H \, M \, \bar{q}_m \, d\Omega$$

3. Decompose M_{mn} into $F^H F$

$$\begin{aligned} M_{mn} &= U \Sigma U^H \qquad \text{(SVD)} \\ F &= U \Sigma^{1/2} \end{aligned}$$

4. Invert F
5. Form the matrix

$$F \begin{pmatrix} \exp(t_{spec}\lambda_1) & & \\ & \ddots & \\ & & \exp(t_{spec}\lambda_N) \end{pmatrix} F^{-1}$$

6. Compute the singular value decomposition of the above matrix

$$F \exp(t_{spec}\Lambda) F^{-1} = U \Sigma V^H$$

7. Extract the first column of V as the optimal initial condition at $t = t_{spec}$

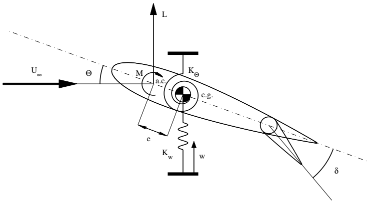
Constrained Optimization

Motivation

Q: Why is constrained optimization useful in problems concerning stability analysis ?

A1: Gives a general framework to compute optimal perturbations. Alternative to the previously shown nonmodal stability analysis and can be applied to nonlinear state equations.

A2: Gives a framework to compute optimal control of instabilities



Movie 2

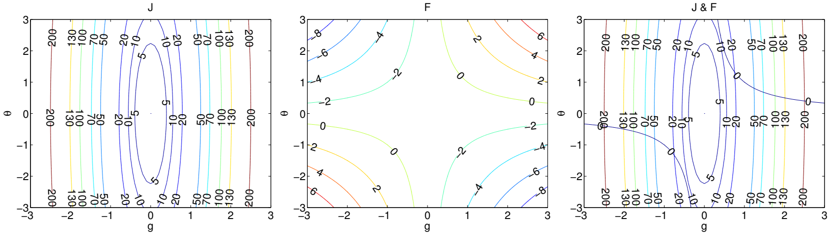
Definition of the optimization problem

Given the **state vector**	$\boldsymbol{\phi} \in \mathcal{R}^N$
and the **control vector**	$\mathbf{g} \in \mathcal{R}^K$
minimize the **cost function**	$\mathcal{J}(\boldsymbol{\phi}, \mathbf{g})$
constrained by the **state equation**	$\mathbf{F}(\boldsymbol{\phi}, \mathbf{g}) = 0$

The goal is to reach a **local minimum** of $\mathcal{J}(\boldsymbol{\phi}, \mathbf{g})$ acting on the control variables $\mathbf{g}$.

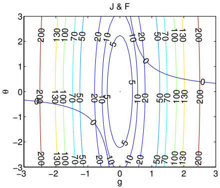
The solution of the constrained problem is usually very different from the solution of the unconstrained problem as seen from the example below.

Exercise *Minimize the cost function* $\mathcal{J}(\phi, g) = \phi^2 + 32g^2$ *constrained by* $F(\phi, g) = \phi g - 1 = 0$.



What is the value of $\boldsymbol{\phi}$ and $\mathbf{g}$ in the constrained case ?

Lagrangian and optimality condition



Scope: descend as low as possible on the $\mathcal{J}$ level curves, remaining on the path given by $F = 0$. If the level lines of $\mathcal{J}$ and the path are continuous, then at the point where the minimum is reached, **the path is tangent to the level curve of the optimal $\mathcal{J}$**.

This implies that at optimality the **gradient of the cost function and the gradient of F are parallel in the $\phi - g$ plane**, i.e.

$$\left\{\frac{\partial \mathcal{J}}{\partial g}, \frac{\partial \mathcal{J}}{\partial \phi}\right\} = a \left\{\frac{\partial F}{\partial g}, \frac{\partial F}{\partial \phi}\right\}$$

The above relation gives an **Optimality System**:

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial g} - a\frac{\partial F}{\partial g} &= 0 \\ \frac{\partial \mathcal{J}}{\partial \phi} - a\frac{\partial F}{\partial \phi} &= 0 \\ F &= 0 \end{aligned}$$

Lagrange remarked: *if the new cost function $\mathcal{L} = \mathcal{J} - a\,F$ is considered, then the above conditions coincide with the optimality conditions for the* **unconstrained optimization** *of $\mathcal{L}(\phi, g, a)$ if the all the variables are considered as* **independent**.

$\mathcal{L}$ is usually referred to as the **Lagrangian** and a is usually called **Lagrange multiplier**.

Lagrangian and optimality condition: a variational approach

We again consider minimizing $\mathcal{J}(\phi, g)$ constrained by $F(\phi, g)$. The Lagrangian $\mathcal{L}$ is written

$$\mathcal{L}(\phi, g, a) = \mathcal{J}(\phi, g) - a\, F(\phi, g)$$

where ϕ, g and a are considered independent variables. We set the variation of $\mathcal{L}$ equal to zero

$$\delta\mathcal{L}(\phi, g, a) = \frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial g}\,\delta g + \frac{\partial \mathcal{L}}{\partial a}\,\delta a = 0$$

By definition:

$$\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi = \lim_{\epsilon \to 0} \frac{\mathcal{L}(\phi + \epsilon\delta\phi, g, a) - \mathcal{L}(\phi, g, a)}{\epsilon} = 0, \quad \forall \delta\phi$$

In practice:

$$\frac{\partial \mathcal{L}}{\partial \phi}\,\delta\phi = \left[\frac{\partial \mathcal{J}}{\partial \phi} - a\frac{\partial F}{\partial \phi}\right]\delta\phi = 0 \quad \rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{\partial \mathcal{J}}{\partial \phi} - a\frac{\partial F}{\partial \phi} = 0, \quad \forall \delta\phi$$

Applied to all terms yields

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial g} &= \frac{\partial \mathcal{J}}{\partial g} - a\frac{\partial F}{\partial g} = 0 \\
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\partial \mathcal{J}}{\partial \phi} - a\frac{\partial F}{\partial \phi} = 0 \\
\frac{\partial \mathcal{L}}{\partial a} &= F = 0
\end{aligned}$$

This is exactly the system we obtained in the previous example !!!

Lagrangian and optimality condition

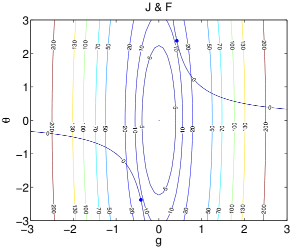
Application of the Lagrangian to the model problem:
We again consider the problem of minimizing the cost function $\mathcal{J}(\phi, g) = \phi^2 + 32g^2$ constrained by $F(\phi, g) = \phi g - 1 = 0$.

The optimality system, using the Lagrangian as defined previously, can be written

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial g} &= \frac{\partial \mathcal{J}}{\partial g} - a\frac{\partial F}{\partial g} = 64g - a\,\phi = 0 \\
\frac{\partial \mathcal{L}}{\partial \phi} &= \frac{\partial \mathcal{J}}{\partial \phi} - a\frac{\partial F}{\partial \phi} = 2\phi - a\,g = 0 \\
\frac{\partial \mathcal{L}}{\partial a} &= F = \phi g - 1 = 0
\end{aligned}
$$

This system of 3 unknowns and 3 equations can be solved analytically.

The solution is
$(\phi, g)_1 = (2.38, 0.42)$
$(\phi, g)_2 = (-2.38, -0.42)$



Lagrangian and optimality condition in N dimensions I

So fare we have looked at the static case in 1 dimension. The above approach can easily be generalized to a N-dimensional state vector $\boldsymbol{\phi}$ and a K-dimensional control vector $\mathbf{g}$. We therefore have to consider a Lagrange multiplier vector $\mathbf{a}$ with the same dimension as the vector of state equations $\mathbf{F}$, i.e. N.

The corresponding Lagrangian can now be written

$$\mathcal{L}(\boldsymbol{\phi}, \mathbf{g}, \mathbf{a}) = \mathcal{J}(\boldsymbol{\phi}, \mathbf{g}) - \mathbf{a} \cdot \mathbf{F}(\boldsymbol{\phi}, \mathbf{g})$$

where $\cdot$ denotes a scalar product. Optimality conditions are given on $\mathcal{L}$ *considering* $\boldsymbol{\phi}, \mathbf{g}$ *and* $\mathbf{a}$ *as independent variables* and therefore enforcing that

$$\frac{\partial \mathcal{L}}{\partial \phi_j} = 0, \ (j = 1, \ldots, N), \ \ \frac{\partial \mathcal{L}}{\partial g_k} = 0, \ (k = 1, \ldots, K), \ \ \frac{\partial \mathcal{L}}{\partial a_i} = 0, \ (i = 1, \ldots, N)$$

When enforced these conditions using the variational approach. The system reads:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}} = 0 \quad &\rightarrow \quad \left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\phi}}\right]^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \boldsymbol{\phi}} \qquad &\textbf{adjoint equations} \\
\frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad &\rightarrow \quad \left[\frac{\partial \mathbf{F}}{\partial \mathbf{g}}\right]^T \mathbf{a} = \frac{\partial \mathcal{J}}{\partial \mathbf{g}} \qquad &\textbf{optimality condition} \\
\frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0 \quad &\rightarrow \quad \mathbf{F} = 0 \qquad &\textbf{state equation}
\end{aligned}$$

Lagrangian and optimality condition in N dimensions II

What is the adjoint equation ?

By definition the **adjoint** of a linear operator A is a **linear** operator A^* which satisfies the following identity:

$$\langle v, A\,u\rangle = \langle A^* v, u\rangle.$$

The $\langle , \rangle$ denotes an inner product.

In our case the state equation, in general, is written as $\mathbf{F}(\boldsymbol{\phi}, \mathbf{g})$ and the linear operator can be written $\partial \mathbf{F}/\partial \boldsymbol{\phi}$. If we define the inner product as $\langle \mathbf{a}, \mathbf{b}\rangle = \mathbf{a}^T \mathbf{b}$, then the adjoint identity can be written

$$\langle \mathbf{a}, \frac{\partial \mathbf{F}}{\partial \boldsymbol{\phi}} \delta\boldsymbol{\phi}\rangle = \langle \left[\frac{\partial \mathbf{F}}{\partial \boldsymbol{\phi}}\right]^T \mathbf{a}, \delta\boldsymbol{\phi}\rangle$$

The **adjoint** operator does not really have any physical meaning but is very useful in different fields of analysis.

Lagrangian and optimality condition in N dimensions III

An example why the adjoint is useful

Consider the following optimization problem where $\boldsymbol{\phi}$, $\mathbf{c}$ and $\mathbf{g}$ have dimension N.

$$\mathcal{J}(\boldsymbol{\phi}, \mathbf{g}) = \mathbf{c}^T \boldsymbol{\phi}, \quad (1)$$

$$A\boldsymbol{\phi} = \mathbf{g}, \quad (2)$$

A simple optimization update (steepest descent) is given by:

$$\mathbf{g}^{i+1} = \mathbf{g}^i - \rho \left(\frac{\partial \mathcal{J}}{\partial \mathbf{g}} \right)^i$$

A straightforward way to compute $\partial \mathcal{J} / \partial \mathbf{g}$ is given by finite differences

$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} \cdot \mathbf{e}^n = \frac{\mathcal{J}(\boldsymbol{\phi}, \mathbf{g} + \epsilon \mathbf{e}^n) - \mathcal{J}(\boldsymbol{\phi}, \mathbf{g})}{\epsilon}$$

where $n = 1, .., N, \epsilon \ll 1$ and $\mathbf{e}^n$ is a Cartesian unit vector.

This has **computational cost N**. This means that you must solve (2) N times.

Instead, solve an additional linear system

$$A^T \mathbf{a} = \mathbf{c}.$$

By simple linear algebra we find

$$\mathcal{J} = \mathbf{c}^T \boldsymbol{\phi} = (A^T \mathbf{a})^T \boldsymbol{\phi} = \mathbf{a}^T A \boldsymbol{\phi} = \mathbf{a}^T \mathbf{g}$$

Now $\mathcal{J}$ depends explicitly on the vector $\mathbf{g}$, and

$$\frac{\partial \mathcal{J}}{\partial \mathbf{g}} = \mathbf{a}.$$

Computational cost 1, independently of N.

Lagrangian and optimality condition: IVP (ODE systems) I

The initial value problem of an ODE system describes the dynamics of a system which evolves in time. For simplicity let us consider the linear system

$$\begin{aligned} \mathbf{F}(\boldsymbol{\phi}, \mathbf{g}) = \frac{d\boldsymbol{\phi}}{dt} - \mathbf{L}\boldsymbol{\phi} &= 0, \qquad 0 \leq t \leq T \\ \boldsymbol{\phi}(0) &= \mathbf{g} \end{aligned}$$

Let us optimize the initial condition **g** in order to maximize the "energy" of $\boldsymbol{\phi}$ at the final time T to the input "energy". In a similar manner we can define a minimization problem where the cost function is

$$\mathcal{J} = \frac{\mathbf{g} \cdot \mathbf{g}}{\boldsymbol{\phi}(T) \cdot \boldsymbol{\phi}(T)}$$

The Lagrangian of the unconstrained problem can now be written, by first introducing the Lagrange multipliers $\mathbf{a}(t)$ and $\mathbf{b}$, as

$$\mathcal{L}(\boldsymbol{\phi}, \mathbf{g}, \mathbf{a}, \mathbf{b}) = \mathcal{J}(\boldsymbol{\phi}, \mathbf{g}) - \int_0^T \mathbf{a} \cdot \left[\frac{d\boldsymbol{\phi}}{dt} - \mathbf{L}\boldsymbol{\phi} \right] dt - \mathbf{b} \cdot [\boldsymbol{\phi}(0) - g]$$

In general this problem definition considers **optimal (transient) energy growth** and the corresponding **optimal perturbation**. This analysis coincides with the analysis of **maximum nonmodal growth for a prescribed final time** T. With a converged solution we have the so called gain as $G(T) = \mathcal{J}^{-1}$.

Lagrangian and optimality condition: IVP (ODE systems) II

The optimality system is derived using a **variational** approach. Further, integration by parts must be used to "move" the derivatives from $\boldsymbol{\phi}$ to $\mathbf{a}$.

This derivation will be shown on the white board...

The optimality system finally reads

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{a}} = 0, \frac{\partial \mathcal{L}}{\partial \mathbf{b}} = 0 \quad &\rightarrow \quad \frac{d\boldsymbol{\phi}}{dt} - \mathbf{L}\boldsymbol{\phi} = 0, \quad \boldsymbol{\phi}(0) = g \qquad &\textbf{state equation} \\
\frac{\partial \mathcal{L}}{\partial \boldsymbol{\phi}} = 0 \quad &\rightarrow \quad -\frac{d\mathbf{a}}{dt} - \mathbf{L}^T\mathbf{a} = 0, \quad \mathbf{a}(T) = \frac{-2\boldsymbol{\phi}(T)\mathbf{g}\cdot\mathbf{g}}{(\boldsymbol{\phi}(T)\cdot\boldsymbol{\phi}(T))^2} \qquad &\textbf{adjoint equations} \\
\frac{\partial \mathcal{L}}{\partial \mathbf{g}} = 0 \quad &\rightarrow \quad \mathbf{g} = -\mathbf{a}(0)\frac{\boldsymbol{\phi}(T)\cdot\boldsymbol{\phi}(T)}{2} \qquad &\textbf{optimality condition}
\end{aligned}
$$

Note that the adjoint equation is integrated "backwards" in time. How about the solution procedure ?

Lagrangian and optimality condition: IVP (ODE systems) III

Solution procedure

Initial: $i = 0$, $g = g^1$, $\mathcal{J}^0 = 10^{15}$, err$=10^{-10}$

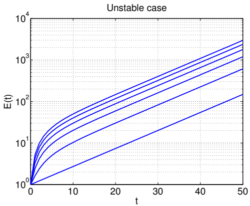
do

1. $i = i + 1$
2. $\dfrac{d\boldsymbol{\phi}}{dt} - \mathbf{L}\boldsymbol{\phi} = 0,\ 0 \le t \le T,$
   with $\boldsymbol{\phi}(0) = g$
3. $\mathcal{J}^i = \dfrac{\mathbf{g}\cdot\mathbf{g}}{\boldsymbol{\phi}(T)\cdot\boldsymbol{\phi}(T)}$
4. $-\dfrac{d\mathbf{a}}{dt} - \mathbf{L}^T\mathbf{a} = 0,\ 0 \le t \le T,$
   with $\mathbf{a}(T) = -2\boldsymbol{\phi}(T)\dfrac{\mathbf{g}\cdot\mathbf{g}}{(\boldsymbol{\phi}(T)\cdot\boldsymbol{\phi}(T))^2}$
5. $\mathbf{g} = -\mathbf{a}(0)\dfrac{\boldsymbol{\phi}(T)\cdot\boldsymbol{\phi}(T)}{2}$

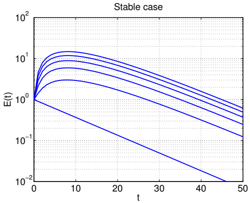
while $(\mathcal{J}^i - \mathcal{J}^{i-1})/\mathcal{J}^i >$ err

Example

$$\mathbf{L} = \begin{bmatrix} 0.1 & p \\ 0 & -0.15 \end{bmatrix}$$



$$\mathbf{L} = \begin{bmatrix} -0.1 & p \\ 0 & -0.15 \end{bmatrix}$$



$T = 50$ and $p = 0, 1, 2, 3, 4, 5$