Hydrodynamic stability

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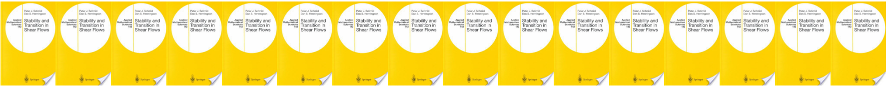
Corso di dottorato in Scienze e Tecnologie per l'Ingegneria (STI):

Fluidodinamica e Processi dell'Ingegneria Ambientale (FPIA)

Course outline

- **Topic** : Hydrodynamic stability **(*Linear, temporal, parallel shear flows*)**
- **Hours** : 10
- **Lectures** : Aula A12 (Ex-DISEG)
  - Wednesday 13/06 9-11 & 14-16
  - Thursday 14/06 9-11 & 14-16
  - Friday 15/06 9-11 & **11-13** (*Exercise optional*)

  Please bring your laptop for the numerical analysis
- **Credits** : 2
- **Content** :
  1. Introduction
  2. Definitions
  3. Inviscid analysis
  4. Viscous analysis
  5. Exercises : **analytical & numerical**
- **Book** : Schmid P. J. & Henningson D. S., *Stability and Transition in Shear Flows*, Springer



Hydrodynamic stability

Hydrodynamic stability theory is concerned with the **respons of laminar flow to a disturbance of small or moderate amplitude.**

The flow is generally defined as

Stable : If the flow returns to its original laminar state.

Unstable: If the disturbance grows and causes the laminar flow to change into a different state.

Stability theory deals with the **mathematical analysis** of the evolution of disturbances superposed to a laminar base flow.

In many cases one assumes the disturbances to be small so that further simplifications can be justified. In particular, a **linear equation governing the evolution of disturbances is desirable.**

As the disturbance velocities grow above a few % of the base flow, **nonlinear effects** become important and linear equations no longer accurately predict the disturbance evolution.

Although the linear equations have a limited region of validity they are important in detecting physical growth mechanisms and identifying dominant disturbance types.

Reynolds pipe flow experiment (1883)

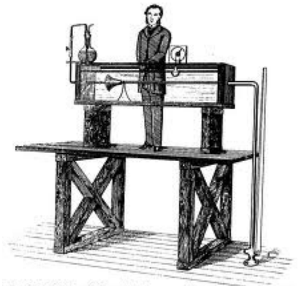
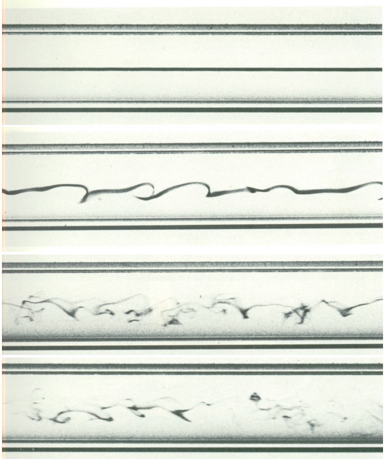


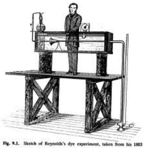
Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper



- Original 1883 appartus
- Dye into center of pipe
- Critical $Re = 13.000$
- Lower today due to traffic

History of shear flow stability and transition

- Reynolds pipe flow experiment (1883)
- Rayleigh's inflection point criterion (1887)
- Orr (1907) Sommerfeld (1908) viscous eq.
- Heisenberg (1924) viscous channel solution
- Tollmien (1931) Schlichting (1933) viscous Boundary Layer solution
- Schubauer & Skramstad (1947) experimental TS-wave verification
- Klebanoff, Tidström & Sargent (1962) 3D breakdown



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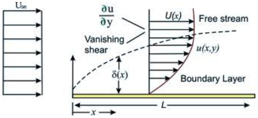
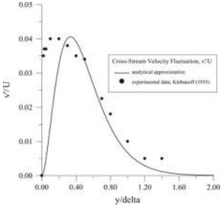
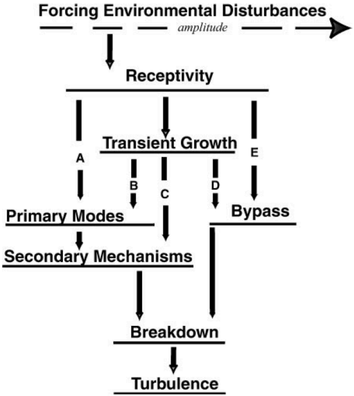
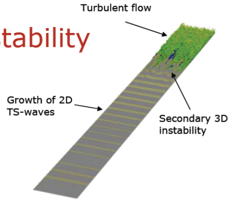




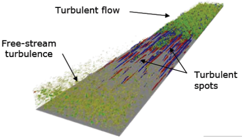
Routes to transition : highly dependent on Tu



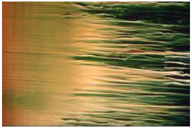
stability



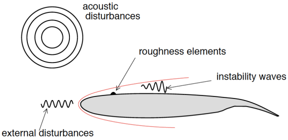
$Tu \sim 0.1\%$



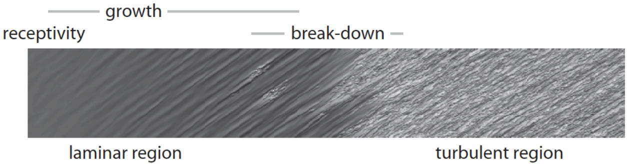
$Tu \sim 10\%$



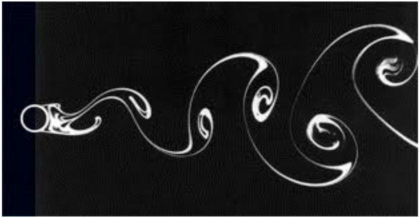
Classical route to transition : low Tu, Modal analysis



1. Receptivity: Initial amplitudes of unstable waves need to be estimated to capture transition "location"
2. Disturbance growth is initially linear and accurately predicted by Linear Stability Theory (LST)
3. Breakdown of disturbances, nonlinear process, finally leading to turbulence



More examples of instabilities I



More examples of instabilities II

Movie 2

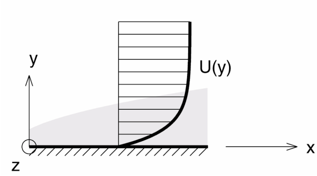
Disturbance equations

$$\frac{\partial u_i}{\partial t} = -u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i(x_i, 0) = u_i^0(x_i)$$

$$u_i(x_i, t) = 0$$ **on solid boundaries**



$$Re = U_\infty^* \delta^* / \nu^*$$

$$u_i = U_i + u_i'$$ **decomposition**

$$p = P + p'$$

Introduce decomposition, drop primes, subtract eq's for $\{U_i, P\}$

$$\frac{\partial u_i}{\partial t} = -U_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j \frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Disturbance equations

$$\frac{\partial u_i}{\partial t} = -u_j\frac{\partial u_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i$$

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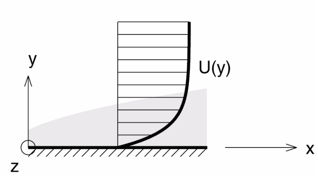


$$Re = U_\infty^*\delta^*/\nu^*$$

$$u_i = U_i + u_i' \quad \textbf{decomposition}$$

$$p = P + p'$$

Introduce decomposition, drop primes, linearize

$$\frac{\partial u_i}{\partial t} = -U_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial U_i}{\partial x_j} - \frac{\partial p}{\partial x_i} + \frac{1}{Re}\nabla^2 u_i - u_j\frac{\partial u_i}{\partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Stability definitions I

$$E(t) = \frac{1}{2} \int_\Omega u_i(t) u_i(t) \, d\Omega$$

Stable : $\lim_{t \to \infty} \frac{E(t)}{E(0)} \to 0$

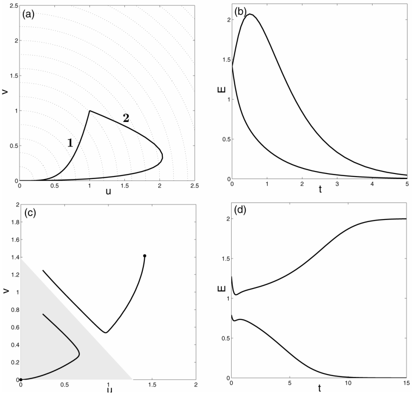
Conditionally stable : $\exists \; \delta > 0 : \; E(0) < \delta \Rightarrow$ stable

Globally stable : Conditionally stable with $\delta \to \infty$

Monotonically stable : Globally stable and $\frac{dE}{dt} \leq 0 \quad \forall t > 0$

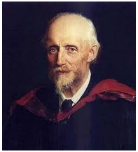
Stability definitions II

Monotonical



Conditional

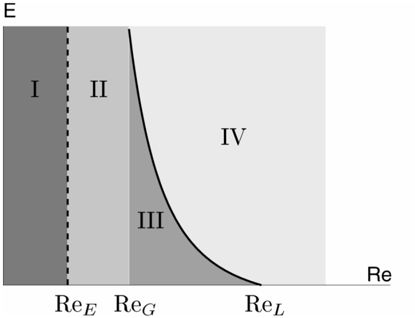
Critical Reynolds numbers



Re_E : $Re < Re_E$ flow monotonically stable

Re_G : $Re < Re_G$ flow globally stable

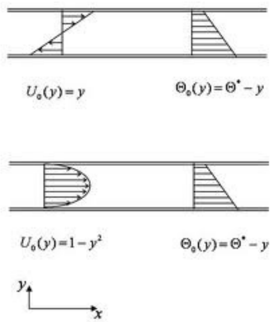
Re_L : $Re < Re_L$ flow linearly stable ($\delta \to 0$)



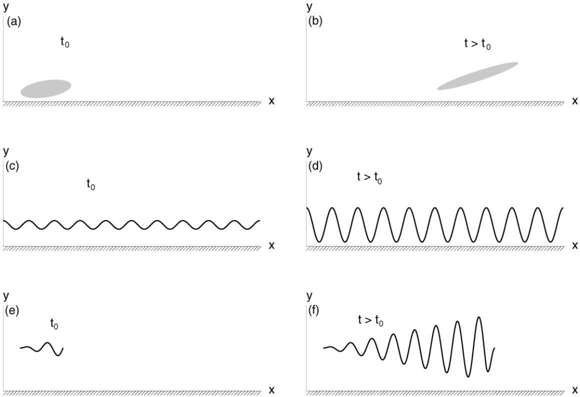
Critical Reynolds numbers

Flow	Re_E	Re_G	Re_{tr}	Re_L
Hagen-Poiseuille	81.5	–	2000	∞
Plane Poiseulle	49.6	–	1000	5772
Plane Couette	20.7	125	360	∞

Critcial Reynolds numbers for a number of wall-bounded shear flows compiled from the literature.



Evolution of disturbances in shear flows



Reynolds-Orr equation

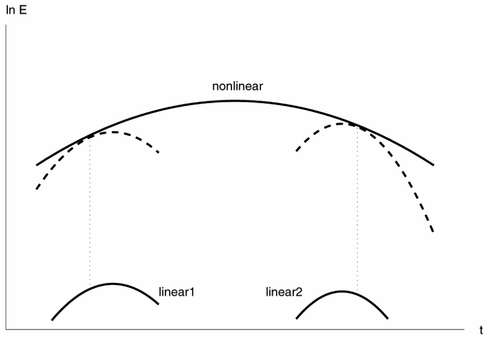
$$
\begin{aligned}
u_i \frac{\partial u_i}{\partial t} &= -u_i u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{Re}\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} \\
&+ \frac{\partial}{\partial x_j}\left[-\frac{1}{2}u_i u_i U_j - \frac{1}{2}u_i u_i u_j - u_i p \delta_{ij} + \frac{1}{Re} u_i \frac{\partial u_i}{\partial x_j}\right] \\
&\Rightarrow \\
\frac{dE}{dt} &= \int_\Omega -u_i u_j \frac{\partial U_i}{\partial x_j}\, d\Omega - \frac{1}{Re}\int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\, d\Omega
\end{aligned}
$$

Theorem : Linear mechanisms required for energy growth

Proof : $\frac{1}{E}\frac{dE}{dt}$ independent of disturbance amplitude

Linear growth mechanisms

$$\frac{1}{E}\frac{dE}{dt} = \frac{d}{dt}\ln E$$



Inviscid Analysis

Parallel shear flows : $U_i = U(y)\delta_{1i}$ I

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \quad &= -\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \quad &= -\frac{\partial p}{\partial z} \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

Initial conditions :

$$\{u, v, w\}(x, y, z, t = 0) = \{u_0, v_0, w_0\}(x, y, z)$$

Boundary conditions :

$$\mathbf{v}(x, y = y_1, z, t) \cdot \mathbf{n} = 0 \quad \textbf{solid boundary 1}$$

$$\mathbf{v}(x, y = y_2, z, t) \cdot \mathbf{n} = 0 \quad \textbf{solid boundary 2}$$

Parallel shear flows : $U_i = U(y)\delta_{1i}$ II

We can reduce the original **4 eq's & 4 unknowns** to a system of **2 eq's and 2 unknowns**
This is in two steps

1. Take the divergence of the momentum equations. This yields

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}.$$

2. The new pressure equation is introduced in the momentum equation for v. This yields

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} \right] v = 0.$$

The three-dimensional flow is then analyzed introducing the normal vorticity

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right] \eta = -U' \frac{\partial v}{\partial z}.$$

with the boundary conditions

$$v = \eta = 0 \qquad \text{at a solid wall and in the far field (or second solid wall)}$$

The Rayleigh equation I

Assume **wave-like solutions**:

$$v(x, y, z, t) = \tilde{v}(y) \exp\, i(\alpha x + \beta z - \omega t)$$

Introduce the ansatz in the v equation.
We limit ourselves to study the v-equation. This yields

$$(-i\omega + i\alpha U)(D^2 - k^2)\tilde{v} - i\alpha U''\tilde{v} \quad = \quad 0$$

$$\text{substitute} \quad \omega = \alpha c \qquad \Rightarrow$$

$$\left(D^2 - k^2 - \frac{U''}{U - c}\right)\tilde{v} \quad = \quad 0$$

Here, $k^2 = \alpha^2 + beta^2$ and $D^i = \partial^i / dy^i$, and the boundary conditions are

$$\tilde{v}(y = y_1) = \tilde{v}(y = y_2) = 0 \qquad \text{solid boundaries}$$

The Rayleigh equation II

- The Rayleigh equation poses an **eigenvalue problem** of second order with c as the complex eigenvalue. The coefficients of the operator are real. Any complex eigenvalue will therefore appear as complex conjugate pairs. So, if c is an eigenvalue, so is c^*.
- It has a **regular singular point** at $U(y_c) = c$, a condition where the order of the equation is reduced (**critical layer**).
- **Analytical solution** for the eigenfunctions exists (*Tollmien, 1928*)

Instability must depend on $U(y)$ (only parameter). U can be **any** base flow

- We look for base flows where the perturbations become unstable
- By definition perturbations in time behave as $\sim \exp(-i\alpha c_r t)\exp(\alpha \mathbf{c_i t})$
- Take $\alpha > 0$. If $\alpha c_i > 0$ the corresponding mode **grows exponentially in time**

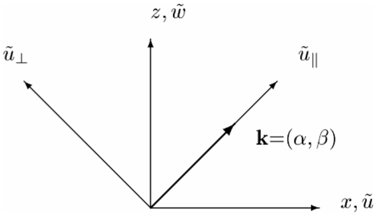
Interpretation of modal results I

$$\omega = \alpha c$$

$$\begin{aligned} v &= \text{Real}\{|\tilde{v}(y)| \exp i\phi(y) \exp i[\alpha x + \beta z - \alpha(c_r + ic_i)t]\} \\ &= |\tilde{v}(y)| \exp \alpha c_i t \cos[\alpha(x - c_r t) + \beta z + \phi(y)] \end{aligned}$$

ω	angular frequency
c_r	phase speed
c_i	temporal growth rate
α	streamwise wave number
β	spanwise wave number

Interpretation of modal results II



$$\tilde{u}_{||} = \frac{1}{k}(\alpha,\beta)\cdot\begin{pmatrix}\tilde{u}\\ \tilde{w}\end{pmatrix} = \frac{1}{k}(\alpha\tilde{u}+\beta\tilde{w}) = -\frac{1}{ik}\frac{d\tilde{v}}{dy}$$

$$\tilde{u}_\perp = \frac{1}{k}(-\beta,\alpha)\cdot\begin{pmatrix}\tilde{u}\\ \tilde{w}\end{pmatrix} = \frac{1}{k}(\alpha\tilde{u}-\beta\tilde{w}) = -\frac{1}{ik}\tilde{\eta}$$

Rayleigh's inflection point criterion (1887) I

Here we consider a parallel shear flow in a domain $y \in (-1, 1)$ and prove a **necessary condition** for instability.

THEOREM : If there exist perturbations with $c_i > 0$, then $U''(y)$ must vanish for some $y_s \in [-1, 1]$

PROOF :
The proof is given by multiplying the Rayleigh equation by $\tilde{v}^*$ and integrating y from -1 to 1. This yields

$$
\begin{aligned}
-\int_{-1}^{1} \tilde{v}^* \left(D^2\tilde{v} - k^2\tilde{v} - \frac{U''}{U-c}\tilde{v} \right) dy &= \\
\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2 \right) dy + \int_{-1}^{1} \frac{U''}{U-c}|\tilde{v}|^2 dy &= 0
\end{aligned}
$$

The first integral is positive definite. The equation equals zero if the second integrand of the second equation changes sign.

Rayleigh's inflection point criterion (1887) II

This is analyzed by multiplying and dividing the second integral with $U - c^*$. This yields

$$\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy + \int_{-1}^{1} \frac{U''(U-c^*)}{(U-c)(U-c^*)}|\tilde{v}|^2 dy \quad = \quad 0$$

The real part is

$$\int_{-1}^{1} \frac{U''(U-c_r)}{|U-c|^2}|\tilde{v}|^2 dy = -\int_{-1}^{1} \left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy,$$

the imaginary part states : U'' **must change sign to render the integral equal to zero if** $c \neq 0$.

$$\int_{-1}^{1} \frac{U'' c_i}{|U-c|^2}|\tilde{v}|^2 dy = 0.$$

Fjørtoft's criterion (1950) I

Here we consider the same flow as in the Rayleigh's criterion.

THEOREM : Given a monotonic mean velocity profile $U(y)$*, a necessary condition for instability is that* $U''(U-U_s)<0$ *for some* $y \in [-1,1]$*, with* $U_s = U(y_s)$ *as the mean velocity at the inflection point, i.e.* $U''(y_s)=0$

PROOF : Consider again the real part

$$\int_{-1}^{1} \frac{U''(U-c_r)}{|U-c|^2}|\tilde{v}|^2 dy = -\int_{-1}^{1}\left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy,$$

We add to the left side the following integral which is identically 0

$$(c_r - U_s)\int_{-1}^{1} \frac{U''}{|U-c|^2}|\tilde{v}|^2 dy = 0.$$

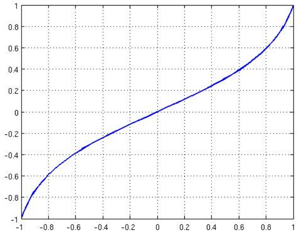
We then get

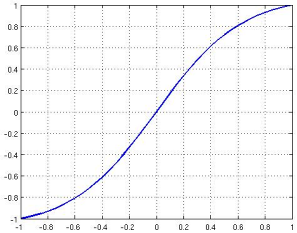
$$\int_{-1}^{1} \frac{U''(U-U_s)}{|U-c|^2}|\tilde{v}|^2 dy = -\int_{-1}^{1}\left(|D\tilde{v}|^2 + k^2|\tilde{v}|^2\right) dy,$$

For the integral on the LHS to be negative the value of $U''(U-U_s)$ must be negative somewhere in the flow.

Fjørtoft's criterion (1950) II

Here are two examples of parallel monotonic shear flow.





Both profiles lead to unstable solutions according to Rayleigh's criterion; however the inflection point has to be a maximum of the spanwise vorticity (not a minimum).

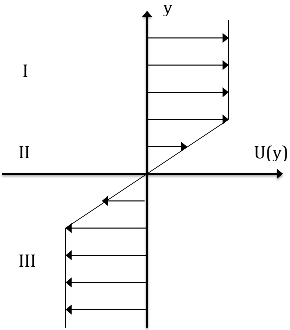
LEFT : unstable according to Fjørtoft **RIGHT :** stable according to Fjørtoft

Solutions to piecewise linear velocity profiles I

Before computers were available to researchers in the field of hydrodynamic stability theory, a common technique to solve inviscid stability problems was to **approximate continuous mean velocity profiles by piecewise linear profiles**. It allows to find **analytical expression for the dispersion relation** $c(\alpha, \beta)$ and the eigenfunctions.

General considerations:

- $U'' = 0$ which simplifies the Rayleigh equation (except at the connecting points)
- **Matching conditions** must be imposed where U is continuous but U'' is discontinuous



Solutions to piecewise linear velocity profiles II

Matching condition

We can rewrite the Rayleigh equation as

$$D[(U-c)D\tilde{v} - U'\tilde{v}] = (U-c)k^2\tilde{v}$$

and integrating over the discontinuity in U and/or U' located at y_D we get

$$[(U-c)D\tilde{v} - U'\tilde{v}]_{y_D-\epsilon}^{y_D+\epsilon} = k^2 \int_{y_D-\epsilon}^{y_D+\epsilon} (U-c)\tilde{v}dy$$

As $\epsilon \to 0$ the RHS $\to 0$ which gives the **first** matching condition

$$[\![(U-c)D\tilde{v} - U'\tilde{v}]\!] = 0, \qquad \textbf{Condition 1}$$

which is equivalent to **matching the pressure** across the discontinuity which in Fourier-transformed form reads

$$\tilde{p} = \frac{i\alpha}{k^2}(U'\tilde{v} - (U-c)D\tilde{v}).$$

Solutions to piecewise linear velocity profiles III

A second condition is derived by dividing the pressure $\tilde{p}$ by $i\alpha(U-c)/k^2$. This yields

$$-\frac{k^2\tilde{p}}{i\alpha(U-c)^2} = \frac{D\tilde{v}}{U-c} - \frac{U'\tilde{v}}{(U-c)^2} = D\left[\frac{\tilde{v}}{U-c}\right]$$

Integrating across the discontinuity in the velocity profile gives

$$\left[\frac{\tilde{v}}{U-c}\right]_{y_D-\epsilon}^{y_D+\epsilon} = -\frac{k^2}{i\alpha}\int_{y_D-\epsilon}^{y_D+\epsilon}\frac{\tilde{p}}{(U-c)^2}dy$$

Again, as $\epsilon \to 0$ we obtain the second matching condition

$$\left[\!\left[\frac{\tilde{v}}{U-c}\right]\!\right] = 0, \qquad \textbf{Condition 2}$$

which, for continuous U, corresponds to matching $\tilde{v}$.

Solutions to piecewise linear velocity profiles IV

Summary :
To solve the Rayleigh equation for a piecewise linear velocity profile we need to solve

$$(D^2 - k^2)\tilde{v} = 0$$

in **each subdomain** and impose **boundary** and **matching** conditions

$$\begin{aligned} [\![(U-c)D\tilde{v} - U'\tilde{v}]\!] &= 0, \\ \left[\!\!\left[\frac{\tilde{v}}{U-c} \right]\!\!\right] &= 0, \end{aligned}$$

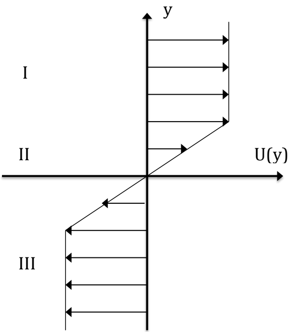
to determine the coefficients of the fundamental solution and finally the **dispersion relation** $c(k)$.

Solutions to piecewise linear velocity profiles V

Exercise : piecewise linear mixing layer

Velocity profile

$$U(y) = \begin{cases} 1 & \text{for} \quad y > 1 \\ y & \text{for} \quad -1 \le y \le 1 \\ -1 & \text{for} \quad y < -1 \end{cases}$$



Boundary conditions

$$\tilde{v} \to 0 \quad \text{as} \quad y \to \pm\infty$$

A general solution can be written

$$\begin{aligned} \tilde{v}_I &= A\exp(-ky) & \text{for} \quad & y > 1 \\ \tilde{v}_{II} &= \ldots & \text{for} \quad & -1 \le y \le 1 \\ \tilde{v}_{III} &= \ldots & \text{for} \quad & y < -1 \end{aligned}$$

Derive

$$c = c(k)$$

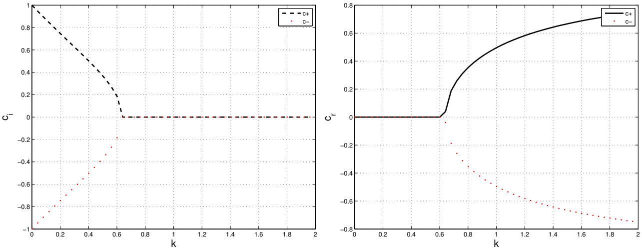
Make a plot of $c(k)$ for $k \in [0, 2]$ and discuss the results.

Solutions to piecewise linear velocity profiles VI

Results : Piecewise mixing layer

$$c = \pm\sqrt{\left(1 - \frac{1}{2k}\right)^2 - \left(\frac{1}{4k^2}\right)\exp(-4k)}$$

- For $0 \leq k \leq 0.6392$ the expression under the square root is negative resulting in purely imaginary eigenvalues
- For $k > 0.6392$ the eigenvalues are real, and all disturbances are neutral
- As the wave number goes to zero, the wavelength associated with the disturbances is much larger than the length scale associated with $U(y)$. The limit of small k is equivalent to the limit of zero thickness of region II.



Viscous Analysis

- Only **linear** or **parabolic** velocity profiles satisfy the steady viscous equations (Couette, Poiseuille)
- Inviscid criteria state that Poiseuille flow is stable
- Common sense would suggest that viscosity acts as a **damping**

However, viscous Poiseuille flow undergoes transition: **viscosity destabilizes the flow**

Parallel shear flows : $U_i = U(y)\delta_{1i}$ I

$$
\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + vU' &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\nabla^2 u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\nabla^2 v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + &= -\frac{\partial p}{\partial z} + \frac{1}{Re}\nabla^2 w \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0
\end{aligned}
$$

Initial conditions :

$$\{u, v, w\}(x, y, z, t = 0) = \{u_0, v_0, w_0\}(x, y, z)$$

Boundary conditions : **depend on flow case**

$$\{u, v, w\}(x, y = y_1, z, t) = 0$$ **solid boundaries**

Semi-infinite domain :

$$\{u, v, w\}(x, y \to \infty, z, t) \to 0$$ **free stream**

Closed domain :

$$\{u, v, w\}(x, y = y_2, z, t) = 0$$ **solid boundary 2**

Parallel shear flows : $U_i = U(y)\delta_{1i}$ II

We can reduce the original **4 eq's & 4 unknowns** to a system of **2 eq's and 2 unknowns**
This is in two steps

1. Take the divergence of the momentum equations. This yields

$$\nabla^2 p = -2U' \frac{\partial v}{\partial x}.$$

2. The new pressure equation is introduced in the momentum equation for v. This yields

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - U'' \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^4 \right] v = 0.$$

The three-dimensional flow is then analyzed introducing the normal vorticity

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},$$

where η satisfies

$$\left[\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{1}{Re} \nabla^2 \right] \eta = -U' \frac{\partial v}{\partial z}.$$

with the boundary conditions

$$v = v' = \eta = 0 \qquad \text{at a solid wall and in the far field (or second solid wall)}$$

Orr-Sommerfeld and Squire equations

Assume **wave-like solutions**:

$$v(x, y, z, t) = \tilde{v}(y) \exp\, i(\alpha x + \beta z - \omega t)$$

Introduce the ansatz in the equations for $\{v, \eta\}$. This yields

$$\begin{aligned} \left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right] \tilde{v} &= 0 \\ \left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right] \eta &= -i\beta U' \tilde{v} \end{aligned}$$

Here, $k^2 = \alpha^2 + \beta^2$ and $D^i = \partial^i / dy^i$.

Orr-Sommerfeld modes : $\{\tilde{v}_n, \tilde{\eta}_n^p, \omega_n\}_{n=1}^N$

Squire modes : $\{\tilde{v} = 0, \tilde{\eta}_m, \omega_m\}_{m=1}^M$

Squire modes I

THEOREM : **Squire modes are always damped**, *i.e.* $c_i < 0 \ \forall \alpha, \beta, Re$

Rewriting the homogeneous Squire equation we get

$$(U - c)\tilde{\eta} = \frac{1}{i\alpha Re}(D^2 - k^2)\tilde{\eta}$$

Multiplying by $\tilde{\eta}^*$ and integrating

$$c \int_{-1}^{1} |\tilde{\eta}|^2 dy = \int_{-1}^{1} U|\tilde{\eta}|^2 dy - \frac{1}{i\alpha Re} \int_{-1}^{1} \tilde{\eta}^* (D^2 - k^2)\tilde{\eta} dy$$

Taking the imaginary part and integrating by parts yields

$$c_i \int_{-1}^{1} |\tilde{\eta}|^2 dy = -\frac{1}{\alpha Re} \left(k^2 |\tilde{v}|^2 + |\frac{\partial \tilde{v}}{\partial y}|^2 \right) < 0$$

Squire's transformation and theorem I

Let's consider 3D and 2D Orr-Sommerfeld equation with $\omega = \alpha c$

$$
\begin{aligned}
(U-c)(D^2-k^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha Re}(D^2-k^2)^2\tilde{v} &= 0 \\
(U-c)(D^2-\alpha_{2D}^2)\tilde{v} - U''\tilde{v} - \frac{1}{i\alpha_{2D} Re_{2D}}(D^2-\alpha_{2D}^2)^2\tilde{v} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{2D} &= k = \sqrt{\alpha^2+\beta^2} \\
\alpha_{2D} Re_{2D} &= \alpha Re \\
&\Rightarrow \\
Re_{2D} &= Re\frac{\alpha}{k} < Re
\end{aligned}
$$

Squire's transformation and theorem II

Each 3D Orr-Sommerfeld mode corresponds to a 2D Orr-Sommerfeld mode at a **lower** *Re, i.e.*

$$Re_{2D} = Re\frac{\alpha}{k} < Re$$

We can therefore define a **critical Reynolds number** for parallel shear flows as

$$Re_c \equiv \min_{\alpha,\beta} Re_L(\alpha, \beta) = \min_{\alpha} Re_L(\alpha, 0)$$

since the growth rate increases with the Reynolds number.

Discretization of the equations in y

The Orr-Sommerfeld equations

$$\begin{aligned}\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right]\tilde{v} &= 0\\ \left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right]\eta &= -i\beta U'\tilde{v}\end{aligned}$$

including boundary conditions $\tilde{v} = D\tilde{v} = \eta = 0$ $y = \pm 1$, can, after suitable discretization (Chebyshev polynomials, finite-differences), be written on the following compact form

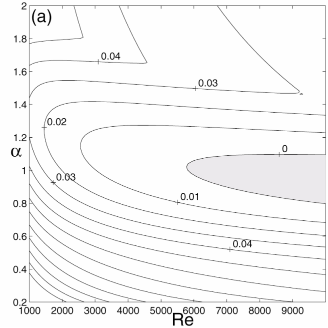
$$\omega\tilde{q} = A\tilde{q} \quad \text{with} \quad \tilde{q} = (\tilde{v}, \tilde{\eta})$$

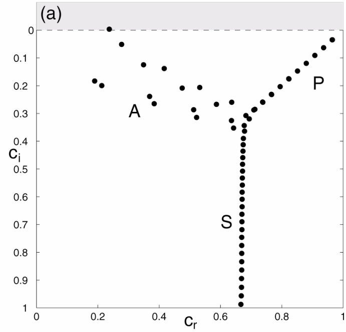
where A is a matrix $\in \mathbb{C}^{2N\times 2N}$. This is an eigenvalue problem from which a solution is obtained for the **eigenvalue** ω_n and **eigenvector** $\tilde{q}_n$. Note that N is the number of discrete points in the wall-normal direction.

Solutions of Eigenvalue analysis I

Plane Poiseuille flow

Neutral curve & spectrum ($Re = 10.000$, $\alpha = 1$, $\beta = 0$)



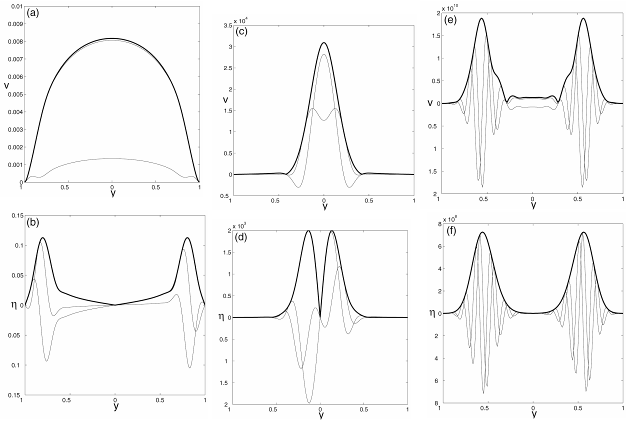


A ($c_r \to 0$), P ($c_r \to 1$), S ($c_r = 2/3$), *Mack (1976)*

Solutions of Eigenvalue analysis II

A, P, S- Eigenfunctions for PPF

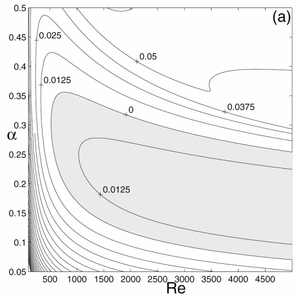
$Re = 5000$, $\alpha = 1$, $\beta = ?$

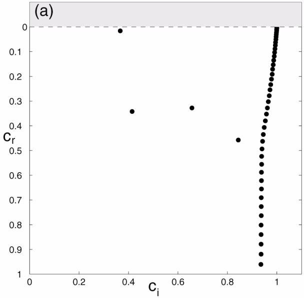


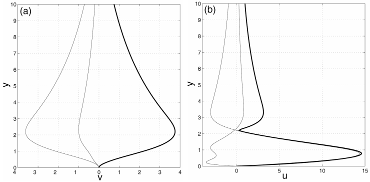
Solutions of Eigenvalue analysis III

Blasius boundary layer

Neutral curve & spectrum ($Re = 500$, $\alpha = 0.2$, $\beta = 0$)



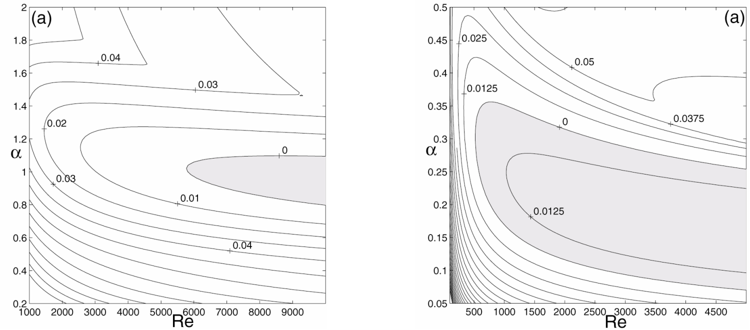




Critical Reynolds numbers

Flow	α_{crit}	Re_{crit}	$c_{r_{crit}}$
Plane Poiseulle	1.02	5772	0.264
Blasius boundary layer flow	0.303	519.4	0.397

Plane Poiseuille Flow & Blasius boundary layer



Continuous spectrum

As $y \to \infty$ the OSE reduces to

$$(D^2 - k^2)^2 \tilde{v} = i\alpha Re[(U_\infty - c)(D^2 - k^2)]\tilde{v}$$

If we assume that

$$\tilde{v}(y) = \hat{v} \exp(\lambda_n y)$$

then the solution is analytical with eigenvalues

$$\lambda_{1,2} = \pm\sqrt{i\alpha Re(U_\infty - c) + k^2}, \quad \lambda_{3,4} = \pm k$$

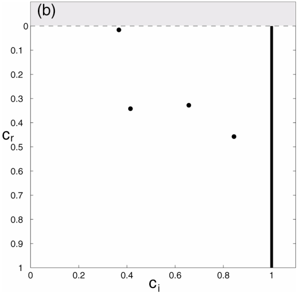
Assuming that $i\alpha Re(U_\infty - c) + k^2$ is **real and negative** which means that $\tilde{v}$ and $D\tilde{v}$ are bounded, $\lambda_{1,2} = \pm iC$

$$\Rightarrow \quad \alpha Re c_i + k^2 < 0, \quad \alpha Re(U_\infty - c_r) = 0$$

From which we can derive analytically $c(k, Re)$

$$c = U_\infty - i(1 + \xi^2)\frac{k^2}{\alpha Re}$$

Example : Blasius boundary layer



Numerical solution of the Orr-Sommerfeld equations I

The Orr-Sommerfeld equations

$$\begin{aligned} -i\omega\tilde{v} &= -(D^2-k^2)^{-1}\left[i\alpha U(D^2-k^2) - i\alpha U'' - \frac{1}{Re}(D^2-k^2)^2\right]\tilde{v} \\ -i\omega\eta &= -\left[i\alpha U - \frac{1}{Re}(D^2-k^2)\right]\eta - i\beta U'\tilde{v} \end{aligned}$$

including boundary conditions $\tilde{v} = D\tilde{v} = \eta = 0$ $y = \pm 1$, can, after suitable discretization, be written on the following compact form

$$-i\omega\tilde{q} = A\tilde{q} \quad \text{with} \quad \tilde{q} = (\tilde{v}, \tilde{\eta})$$

Once we have the discrete problem on this form any available solver can be used to compute the corresponding **eigenvalues** ω_n and **eigenvectors** $\tilde{q}_n$.

Exercise: Solve numerically for the Plane Poiseuille flow

- Start by plotting the eigenvalue spectrum and one mode from each branch (A,P,S)
- Verify Squire's theorem
- The neutral curve $c_i(\alpha, \beta = 0, Re) = 0$
- Find the critical Reynolds number

A matlab program is available in which the discrete A has been discretized using Chebyschev polynomials.

A matlab script

```
%%%% parameters
Re=1000; %reynolds number (based on channel half width)
N=50;%number of collocation points in wall normal direction
kx=1;%streamwise wave number
kz=0;%spanwise wave number

%%%% differentiation matrices
[yvecT,DM] = chebdif(N+2,2);
yvec=yvecT(2:end-1);

%%%% the velocity profile
U.u  = 1-yvec.^2;
U.P = -2*yvec;
U.PP= -2*ones(size(yvec));

% implement homogeneous boundary conditions
D2=DM(2:N+1,2:N+1,2);

% fourth derivative with clamped conditions
[y,D4]=cheb4c(N+2);

%%%% laplacian
I=eye(N);
k2=kx^2+kz^2;
delta=(D2-k2*I);
delta2=(D4-2*k2*D2+k2*k2*I); % laplacian  squared

%%%% compute dynamic matrix
LOS  =  i*kx*diag(U.u)*delta -i*kx*diag(U.PP) -delta2/Re ;
LC   = -i*kz*diag(U.P) ;
LSQ  = -i*kx*diag(U.u) + delta/Re;
A = [-delta\LOS  zeros(N,N); LC  LSQ ];
```

Some hints

- recall that the eigenvalue solution is $-i\omega$, so if you want to plot c you must first...
- compute eigenvalues using **[V,D]=eig(A)**. D is a diagonal matrix of eigenvalues and V is a full matrix where the columns correspond to the eigenvalues in D.
- Only the least stable solution is needed. Note that it is not necessarily unstable.
- the function **sort** can be used to find the least stable eigenvalue
- make it automatic by setting up a double loop (over α and Re). For each combination (α, Re) use **eig** and **sort** to find the least stable mode.
- use the function **contour** to plot the neutral curve.