Riccati-less optimal control of bluff-body wakes

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Abstract In this paper we propose a new method to solve the optimal control problem in which the feedback matrix K is computed in an efficient way for complex flows, with large number of degrees of freedom, using an approach similar to adjoint-based control optimization. The idea is to consider the direct-adjoint system as an input-output problem where the input is given by the current state and the output is the control. Since the control has much smaller dimension than the state, the feedback matrix K can be efficiently obtained from the solution of the adjoint of the direct-adjoint system. It can further be shown using the symplectic product that the direct-adjoint system is self adjoint. As a consequence the new adjoint system is equivalent to the direct-adjoint system with suitable initial and terminal conditions. With this method the optimal control problem can be solved efficiently for any value of the control penalty l^2. Results are presented of this novel technique as applied to suppressing the vortex shedding behind a circular cylinder, and compared to the minimal-energy feedback control presented in [4].

1 Background

Modern optimal control algorithms, based on the matrix Riccati equation, are usually difficult to apply to complex flows such as the wake behind a cylinder because of the large number of degrees of freedom originating from the discretized Navier-Stokes equations. An approximate method to overcome this problem, which has received attention in the literature, is to use reduced-order modeling. However, here we will present an exact method which does not rely on such modeling.

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The optimal control problem is to find the control $\mathbf{u}$ which minimizes the cost function

$$J = \frac{1}{2}\int_0^T \left[\mathbf{x}^H\mathbf{Q}\mathbf{x} + l^2\mathbf{u}^H\mathbf{R}\mathbf{u}\right] \mathrm{d}t, \tag{1}$$

where superscript H denotes conjugate transpose, and the state $\mathbf{x}$ and the control $\mathbf{u}$ are related via the state equation

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{on } 0 < t < T \qquad \text{with} \qquad \mathbf{x} = \mathbf{x}_0 \quad \text{at } t = 0. \tag{2}$$

The result depends on the initial state $\mathbf{x}_0$, the final time T, the choice of the matrices $\mathbf{Q}$ and $\mathbf{R}$, and the real valued parameter l. To increase the value of l means to ascribe a higher cost to the control, and vice versa. This problem can be solved using a gradient based method, and the gradient can be efficiently evaluated using the adjoint of (2). The adjoint equations are here derived using Lagrange multipliers. If we introduce the adjoint variable $\mathbf{p}$ then the cost function can be written

$$J = \frac{1}{2}\int_0^T \left[\mathbf{x}^H\mathbf{Q}\mathbf{x} + l^2\mathbf{u}^H\mathbf{R}\mathbf{u}\right] \mathrm{d}t + \int_0^T \mathbf{p}^H \left(\frac{\partial \mathbf{x}}{\partial t} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u}\right) \mathrm{d}t. \tag{3}$$

Integration by parts yields

$$J = \frac{1}{2}\int_0^T \left[\mathbf{x}^H\mathbf{Q}\mathbf{x} + l^2\mathbf{u}^H\mathbf{R}\mathbf{u}\right] \mathrm{d}t + \int_0^T \left[\mathbf{x}^H \left(-\frac{\partial \mathbf{p}}{\partial t} - \mathbf{A}^H\mathbf{p}\right) + \mathbf{u}^H\mathbf{B}^H\mathbf{p}\right] \mathrm{d}t + \left[\mathbf{p}^H\mathbf{x}\right]_0^T \tag{4}$$

Nullifying the functional derivative of J with respect to $\mathbf{x}$ gives

$$\frac{\partial \mathbf{p}}{\partial t} = -\mathbf{A}^H\mathbf{p} + \mathbf{Q}\mathbf{x} \quad \text{on } 0 < t < T \qquad \text{with} \qquad \mathbf{p} = 0 \quad \text{at } t = T. \tag{5}$$

Nullifying the functional derivative of J with respect to $\mathbf{u}$ gives

$$l^2\mathbf{R}\mathbf{u} - \mathbf{B}^H\mathbf{p} = 0. \tag{6}$$

At this point we can distinguish between two different approaches to solve the optimal control problem: in the first, the optimal control $\mathbf{u}$ corresponding to the state existing at each time step is computed in real time. This approach is generally combined with a finite horizon (value of T) to make it tractable. In the second, considering a feedback rule $\mathbf{u} = K\mathbf{x}$ and a system which is time invariant, the feedback matrix K is computed once and for all off-line. In this case we can rewrite equation (6) as

$$\mathbf{K}\mathbf{x} = \frac{1}{l^2}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{p}. \tag{7}$$

This problem is usually solved using a relation between the state vector $\mathbf{x} = \mathbf{x}(t)$ and adjoint vector $\mathbf{p} = \mathbf{p}(t)$ via a matrix $\mathbf{X} = \mathbf{X}(t)$ such that $\mathbf{p} = \mathbf{X}\mathbf{x}$ in order to write the two-point boundary value problem, given by the direct (2) and adjoint (5) equations, as one differential equation for $\mathbf{X}$. The resulting equation is usually denoted *differential Riccati equation*. However, such a frame work is not suitable for numerical calculations when considering flow problems involving a large number of degrees of freedom, given the size of the matrix $\mathbf{X}$.

There are, as shown by [1] and [4], at least some cases where a mathematically rigorous optimal control can become a reality. They considered a minimal-energy stabilizing feedback rule $\mathbf{u} = \mathbf{K}\mathbf{x}$ (using the problem definition above) in the limit as $l^2 \to \infty$. In this limit the eigenvalues of the closed-loop system $\mathbf{A}+\mathbf{B}\mathbf{K}$ are given by the union of the stable eigenvalues of A and the reflection of the unstable eigenvalues of $\mathbf{A}$ into the left-half plane. They showed, by considering the system in modal form, that the feedback gain matrix $\mathbf{K}$ is a function solely of the unstable eigenvalues and the corresponding left eigenvectors. It was further demonstrated that the feedback matrix $\mathbf{K}$, which is computed once and for all, works well even when applied to the complete nonlinear system.

So far *no* approach has been set forth in order to compute $\mathbf{K}$ for complex flows when the parameter l is allowed to take any value. A new approach to solve this problem is given in the next section.

2 Riccati-less optimal control

In this section the aim is to compute the feedback matrix $\mathbf{K}$ such that it is independent of the initial condition $\mathbf{x}_0$ and time invariant. This can in theory be done by investigating an number of initial conditions corresponding to the dimension of the state $\mathbf{x}$. However, this is often computationally expensive and it is therefore of interest to find an alternative method.

For any linear system where the dimension N_0 of the output is much smaller than the dimension N_i of the input the sensitivity can be computed efficiently from its adjoint. This can be understood by considering that N_0 computations of the adjoint completely replace N_i computations of the original system.

In the optimzation problem that leads from $\mathbf{x}_0$ to $\mathbf{u}_0$ the linear system is given by the direct and adjoint equations (2) and (5). Since the dimension of $\mathbf{u}_0$ is much smaller than $\mathbf{x}_0$ it is favorable to compute the sensitivity with respect to the initial condition using the *adjoint of the direct-adjoint system*. The new adjoint is solved using an initial condition of small dimension, $\mathbf{u}_0^+$, and its output, which is the sensitivity with respect to the initial condition, is of large dimension and corresponds to a row of the feedback matrix $\mathbf{K}$.

The adjoint of the direct-adjoint system is derived by introducing the adjoint variables $\mathbf{x}^+$ and $\mathbf{p}^+$ which are multiplied with the equations (2) and (5), respectively, and integrated in time from $t=0$ to $t=T$. This can be written

$$\int_0^T \mathbf{x}^{+H}\left(\frac{\partial \mathbf{x}}{\partial t} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{p}\right)\mathrm{d}t + \int_0^T \mathbf{p}^{+H}\left(\frac{\partial \mathbf{p}}{\partial t} + \mathbf{A}^H\mathbf{p} - \mathbf{Q}\mathbf{x}\right)\mathrm{d}t = 0. \quad (8)$$

Using integration by parts the differentiation operators are shifted from the direct to the adjoint equations. Considering that both **R** and **Q** are symmetric, we obtain

$$-\int_0^T \mathbf{p}^H \left(\frac{\partial \mathbf{p}^+}{\partial t} - \mathbf{A}\mathbf{p}^+ + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{x}^+ \right) \mathrm{d}t - \int_0^T \mathbf{x}^H \left(\frac{\partial \mathbf{x}^+}{\partial t} + \mathbf{A}^H\mathbf{x}^+ + \mathbf{Q}\mathbf{p}^+ \right) \mathrm{d}t + \left[\mathbf{p}^H\mathbf{p}^+\right]_0^T + \left[\mathbf{x}^H\mathbf{x}^+\right]_0^T = 0. \tag{9}$$

If we now define the new adjoint equations as

$$\frac{\partial \mathbf{p}^+}{\partial t} = \mathbf{A}\mathbf{p}^+ - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{x}^+, \tag{10}$$

$$\frac{\partial \mathbf{x}^+}{\partial t} = -\mathbf{A}^H\mathbf{x}^+ - \mathbf{Q}\mathbf{p}^+, \tag{11}$$

with $\mathbf{x}^+(t = T) = 0$ in equation (11), and use the terminal condition $\mathbf{p}(t = T) = 0$, the remaining terms in expression (9) are written

$$\mathbf{p}^{+H}(0)\mathbf{p}(0) + \mathbf{x}^{+H}(0)\mathbf{x}(0) = 0. \tag{12}$$

The optimality condition (7) can now be imposed one at a time by comparing each of its rows with the general identity (12). In particular, setting $\mathbf{p}^{+H}(t = 0)$ equal to one row of $\mathbf{R}^{-1}\mathbf{B}^H$ we shall obtain that $-\mathbf{x}^{+H}(t = 0)$ shall equal the corresponding row of **K**.

In order to compute **K** we now need to solve the coupled system of linear equations (10)-(11) with the initial and terminal conditions $\mathbf{x}^+(T) = 0$ and $\mathbf{p}^{+H}(t = 0)$ equals one row of $\mathbf{R}^{-1}\mathbf{B}^H$, respectively. However, if let $\mathbf{x}^+ \to -\mathbf{p}$ and $\mathbf{p}^+ \to \mathbf{x}$ then these equations become the same as the direct-adjoint system (2) and (5). In other words, with respect to the symplectic product, the Hamiltonian direct-adjoint system is self-adjoint.

This means that the adjoint of the direct-adjoint system can be obtained by solving the coupled system of linear equations (2) and (5) with an initial condition given by one row of $\mathbf{R}^{-1}\mathbf{B}^H$. With the "optimal control" **u** so obtained directly gives one row of **K**.

In order to obtain the minimum of (3) for $t \to \infty$ we iteratively search for a sufficiently large T. The possible storage problems posed by the need for storing $\mathbf{x}(t)$ on $[0, T]$ during the forward march in order to reuse it during the adjoint can be avoided using a *checkpointing* algorithm, see [2] and [3], which saves $\mathbf{x}(t)$ occasionally on the forward march and then recomputes $\mathbf{x}(t)$ as necessary from these checkpoints during the backward march of the adjoint.

It is important to note that the derivation shown in this section is made for a control of dimension one. This means that the initial condition for $\mathbf{x}^H(t=0)$ is given by the a row of $= \mathbf{R}^{-1}\mathbf{B}^H$. Further, one row of the feedback matrix is obtained as $\mathbf{p}^H(t=0)$. In a general case, however, the above solution procedure must be performed for each control variable.

3 Application

Results obtained from the new "Riccati-less" approach to compute the feedback matrix $\mathbf{K}$ are shown here in comparison with $\mathbf{K}$ computed assuming $l^2 \to \infty$, as in [4], when the flow past a cylinder at $Re = 55$ is considered, and angular oscillation of the cylinder is used as the control variable. The Reynolds number is based on the free-stream velocity and cylinder diameter. Both the base flow and the linearized equations are discretized using second-order finite differences over a staggered, stretched, Cartesian mesh. An immersed-boundary technique is used to enforce the boundary conditions on the cylinder. Both the nonlinear and linearized Navier-Stokes equations are solved using the Adams-Bashforth/Crank-Nicholson scheme. Further, the adjoint equations are derived from the discretized form of (2) and are exact to machine precision. In figures 1 and 2 the u and v components of $\mathbf{K}$ are shown for the cases in which $l^2 = 1$ and $l^2 \to \infty$, respectively. In both cases the u and v components are, respectively, anti symmetric and symmetric with respect to the horizontal axis, and the maximum values of both sensitivity components is situated close to the cylinder. Note that the maximum value for the case in which $l^2 = 1$ is larger compared to the $l^2 \to \infty$ case. The effect of the control on the lift and drag forces on the cylinder is shown in figure 3 in comparison with the forces in a stationary flow and in the fully developed unstationary flow in the absence of control. It can be seen that the drag force approaches the values for stationary flow as the control is applied. This is of course obtained more quickly in the case in which $l^2 = 1$.

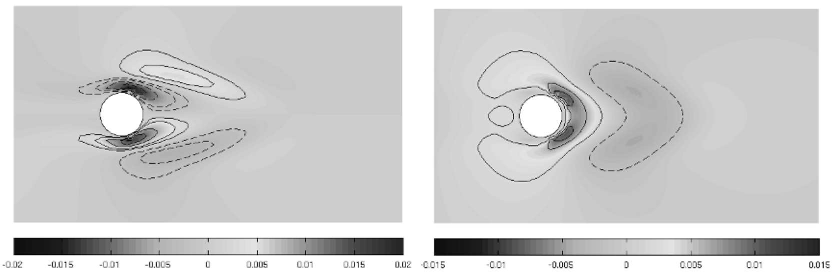

Fig. 1 $\mathbf{K}$ for $l = 1$, $Re = 55$; (left) K_u, (right) K_v. Solid contours inidicate positive values and dashed negative values.

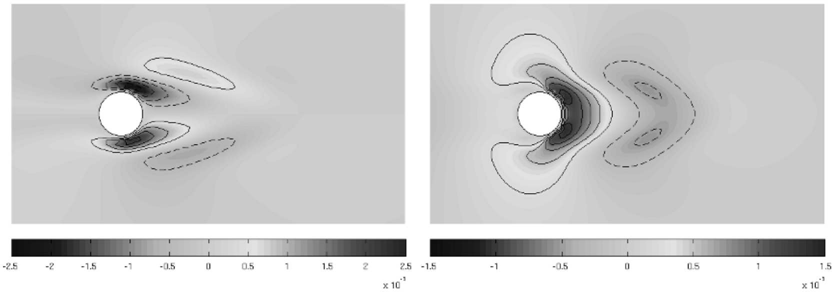

Fig. 2 K for $l \rightarrow \infty$ (Minimum energy control), $Re = 55$; (left) K_u, (right) K_v. Solid contours inidicate positive values and dashed negative values.

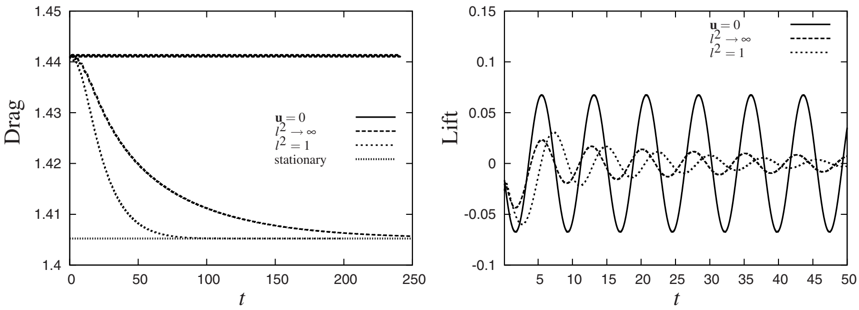

Fig. 3 Time trace of forces; (left) horizontal, (right) vertical

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