# AIMETA 2005

## Atti del XVII Congresso dell'Associazione Italiana di Meccanica Teorica e Applicata

**Firenze, 11-15 settembre 2005**

*a cura di*

Claudio Borri
Luca Facchini
Giorgio Federici
Mario Primicerio

**Volume II**

Firenze University Press
2006

AIMETA 2005 : atti del XVII Congresso dell'Associazione italiana di meccanica teorica e applicata : Firenze, 11-15 settembre 2005 : volume II / a cura di Claudio Borri, Luca Facchini, Giorgio Federici, Mario Primicerio. - Firenze, Firenze university press, 2006
(Atti, 19)
http://digital.casalini.it/8884534593
Stampa a richiesta disponibile su http://epress.unifi.it

ISBN 10: 88-8453-459-3 (online)
ISBN 13: 978-88-8453-459-0 (online)
ISBN 10: 88-8453-460-7 (print)
ISBN 13: 978-88-8453-460-6 (print)

531 (ed. 20)
Meccanica-Congressi-Firenze-2005



Università degli Studi di Firenze
Firenze University Press
Borgo Albizi, 28, 50122 Firenze, Italy
http://epress.unifi.it/

*Printed in Italy*

**Leaky waves in spatial stability analysis**

**Jan O. Pralits**
***Dipartimento di Ingegneria Meccanica, Università di Salerno, Italy***
***E-mail: pralits@fd.dimec.unisa.it***

**Paolo Luchini**
***Dipartimento di Ingegneria Meccanica, Università di Salerno, Italy***
***E-mail: luchini@unisa.it***

***Keywords*: fluid mechanics, transition, stability analysis, Orr-Sommerfeld, leaky waves**

SUMMARY: Linear stability analysis of flat plate boundary layers implies, for wave like perturbations, to solve the so called Orr-Sommerfeld equations which solution can be expressed in terms of a continuous, and discrete spectrum. As the number of discrete modes change with the Reynolds number, and further seem to disappear behind the continuous spectrum at certain Reynolds numbers, it is of interest to investigate if an all-discrete representation of the solution is possible. This can be done solving the response of the flat plate boundary forced instantaneously in space. Since the solution of the forced and homogeneous Laplace transformed problem both depend on the free stream boundary conditions, it is shown here that an opportune change of variables can remove the branch cut in the complex eigen value plane. As a result integration of the inversed Laplace transform along the new path corresponding to the continuous spectrum, which is now given by a straight line, equals the summation of residues of additional discrete eigen values appearing to the left of it. It is further shown that these additional modes are computed accounting for solution which grow in the wall normal direction. A similar problem is found in the theory of optical waveguides, such as optical fibers, where so called *leaky waves* are attenuated in the direction of the wave-guide, while it grows unbounded in a direction perpendicular to it. The theory is here applied to the case of two-dimensional flat plate boundary layers, of incompressible flows, subject to a pressure gradient.

## 1. INTRODUCTION

Laminar-turbulent transition in a flat plate boundary layer subject to a low free stream turbulence level is usually caused by perturbations with infinitesimal amplitude which grow as they propagate downstream. These perturbations are commonly analysed using the Orr-Sommerfeld equations (OSE), either in a temporal or a spatial framework. In both cases, the mode structure of the OSE is composed of a finite number of discrete modes which decay at infinity in the wall normal direction ($y$), and a continuous spectrum of propagating modes behaving as $\exp(\pm\lambda\, y)$ at infinity, where $\lambda$ is a complex wave number in the wall normal direction, see [3]. The number of discrete modes changes with Reynolds number and following the trajectory of a certain discrete mode, it seems as if it disappears behind the continuous spectrum at a certain Reynolds number. It is therefore of interest to investigate if something in particular happens as these modes *disappear*. Further, since the representation of a given solution as a superposition of modes is not unique, i.e. an all-continuous representation always exist, it is of interest to investigate if it is possible to find an all-discrete representation. Such a representation has several possible applications. It might help the physical understanding and ordering of *higher* modes (discrete modes other than the least stable one such as the Tollmien-Schlichting waves). It can further be useful in applications where these higher modes are important, e.g. evaluating the first order correction of the eigen functions in the multiple scales method. An intrinsic problem with the discrete modes appearing in the vicinity of the continuous spectrum is that they are ill-conditioned. This is an additional problem in applications where higher modes are important. The analysis is here made looking at the response in a flat plate boundary layer forced instantaneously at a point in space and time. Since the solution of the forced and homogeneous Laplace transformed problem both depend on the free stream boundary conditions, it is shown here that an opportune change of variables can remove the branch cut in the complex eigen value plane. As a result integration of the inversed Laplace transform along the new path corresponding to the continuous spectrum, which is now given by a straight line, equals the summation of residues of additional discrete eigen values appearing to the left of it. It is further shown that these additional modes are computed accounting for solution which grow in the wall normal direction. A similar problem is found in optics, and especially in the theory of optical waveguides (such as optical fibers), where a general solution of the inhomogeneous Helmholtz equation are inhomogeneous plane waves. A solution can either be represented as a all-continuous spectrum or a sum of residues and branch cuts corresponding to the discrete and continuous spectrum respectively, depending on the integration contour made in respective complex wave number plane. In the latter case, non-parallel plane waves are accounted for and the integral of the continuous spectrum becomes almost negligible with respect to the discrete sum of residues (spectrum), viz. a complete solution is well approximated by the discrete spectrum which is a rapidly converging summation. In this particular case, non-parallel waves are introduced using complex wave numbers associated with the so called *leaky waves*. Associated in the sense that the non-parallel plane waves do not have total reflection, it *leaks* or refracts some energy in to the surrounding media, and therefore the name leaky waves (LW). A characteristics of LW is that while they are attenuated in one direction, such as the direction of the wave-guide, it grows unbounded in a direction perpendicular to it. The solution of leaky waves can be seen as e.g. a superposition of inhomogeneous plane waves, see [6], or e.g. as a resonant solution of a boundary value problem, see [5]. The theory is here applied to spatial linear stability analysis of two-dimensional flat plate boundary layer flows subject to a pressure gradient.

## 2. LINEAR STABILITY ANALYSIS

A two-dimensional flat plate boundary layer is considered where the flow conditions are such that the fluid is assumed incompressible. A Cartesian coordinate system is used where x, and y are the streamwise and wall-normal coordinates respectively. The corresponding dimensionless velocity components are $\tilde{u}$, and $\tilde{v}$, and $\tilde{p}$ is the pressure. The velocity components are made dimensionless using a reference velocity $u^\star_\infty$, the pressure using two times the dynamic pressure, and the coordinates using a reference scale $\delta^\star$. Using these reference quantities, the Reynolds number is given as $Re = u^\star_\infty \delta^\star/\nu$, where $\nu$ is the kinematic viscosity. We want to analyse the evolution of perturbations with infinitesimal amplitude inside the boundary layer, and consider therefore the flow decomposed into a steady mean flow $U$, and a perturbation $u'$ as $\tilde{u}(x, y, t) = U(y) + u'(x, y, t)$. If the flow decomposition is introduced into the Navier-Stokes equations for two-dimensional incompressible flow, the mean flow is subtracted, and omitting non-linear perturbation terms, the linearised Navier-Stokes equations are obtained for two-dimensional flows. It is further assumed wave-like solutions of the form

$$u'(x,y,t) = u(y)e^{\mathrm{i}(\alpha x - \omega t)}. \tag{1}$$

where $\alpha$ is the streamwise wave number, and $\omega$ is the angular frequency. If the ansats (1) is introduced into the linearised Navier-Stokes equations, the resulting equations can be written

$$\mathbf{A}\,\mathbf{q} + \mathbf{B}\frac{d\mathbf{q}}{dy} = 0. \tag{2}$$

The wall-normal derivative of equation (2) has been reduced to first order using $u1 = u_y$, $v1 = v_y$, and the continuity equation has then been used to obtain the state vector $\mathbf{q} = (u1, u, v, p)$. The coefficients of the four by four matrices **A**, and **B** are found in the appendix. Equation (2) is subject to the following boundary conditions $u(0) = v(0) = 0$, and $(u(y), p(y)) \to (0, 0)$ as $y \to \infty$. A spatial stability analysis is obtained with $\omega$ as a real valued parameter, and solving (2) as an eigen value problem for the complex wave number $\alpha$. The real part of $\alpha$ is the associated wave number and minus the imaginary part is the spatial growth rate.

### 2.1 Asymptotic behaviour in the free stream

The free stream boundary conditions of (2) are given as $y$ approaches infinity. In order to have a finite domain in the analysis, which is necessary in the case (2) is solved numerically, it is favourable to analyse its asymptotic behaviour when the mean flow becomes constant, and corresponding wall-normal derivatives are zero. In this case (2) becomes a system of equations with constant coefficients whose solution can be written

$$u(y) = \sum_i a_i\,\bar{u}_i\,e^{-\lambda_i y}, \tag{3}$$

where $a_i$ is a constant, and $\bar{u}_i$ is the normalised value of $u$ outside the boundary layer, for the ith component. If the solution $u(y) = a\,\bar{u}\,e^{-\lambda y}$ is introduced into (2), we obtain the following eigen value problem

$$[\mathbf{A} - \lambda\mathbf{B}]\,\bar{\mathbf{q}} = 0. \tag{4}$$

The above equation can be solved analytically from which the following four eigen values are obtained

$$\lambda_{1,2} \quad = \quad \pm\alpha, \tag{5}$$

$$\lambda_{3,4} \quad = \quad \pm\sqrt{\alpha^2 + \mathrm{i}Re(\alpha - \omega)}, \tag{6}$$

Inserting these eigen values into equation (3), it can be seen that there are two terms in the sum which grow as $y$ goes to infinity, and two terms which decay. Choosing the two eigen values which result in a decaying solution give the asymptotic behaviour of **q** in the free stream. An alternative way to impose the free stream boundary conditions is to use a condition which exclude solutions which grow at infinity. This can be obtained using the bi-orthogonality relation between the right, $\mathbf{q}$, and the left, $\mathbf{v}$, eigen vector of (4), where the left eigen vector is the solution of the equation

$$\mathbf{v}\cdot[\mathbf{A} - \lambda\mathbf{B}] = 0. \tag{7}$$

Using (4), and (7), the following equation is obtained

$$(\lambda_i - \lambda_j)\mathbf{v}_j \cdot \mathbf{B}\,\mathbf{q}_i = 0, \tag{8}$$

whose solutions can be written

$$\mathbf{v}_j \cdot \mathbf{B}\,\mathbf{q}_i = 0, \quad \text{if} \quad i \neq j, \tag{9}$$

$$\mathbf{v}_j \cdot \mathbf{B}\,\bar{\mathbf{q}}_i = \mathbf{a}_i, \quad \text{if} \quad i = j, \tag{10}$$

if a normalisation is used such that $\mathbf{p}_i \cdot \mathbf{B}\,\bar{\mathbf{q}}_i = 1$. It is now evident from equation (9), that cancelling one of the four terms in (3) is obtained by the scalar product between the corresponding left eigen vector and $\mathbf{B}\,\mathbf{q}$. The new free stream boundary conditions therefore become

$$\mathbf{v}_j \cdot \mathbf{B}\,\mathbf{q} = 0, \quad j = k, l, \tag{11}$$

where $j = k, l$ are the two undesired solutions.

## 3. MEAN FLOW

We consider a steady two-dimensional flat plate boundary layer subject to a pressure gradient where $x\ \ , y$ denote the streamwise, and wall-normal coordinates, respectively. Here $U$, and $V$ are the streamwise, and wall-normal velocity components respectively, $\star$ denote dimensional quantities, and the mean flow at the boundary layer edge is assumed to satisfy $U^\star_\infty = U^\star_0 (x^\star/x^\star_0)^{\beta_H/(2-\beta_H)}$. If we introduce the dimensionless coordinate $\eta = y^\star/\delta^\star$, with $\delta^\star = \sqrt{(2-\beta_H)\nu x^\star / U^\star_\infty}$ and a stream function $\psi^\star = \sqrt{(2-\beta_H) U^\star_\infty \nu x^\star}\ f(\eta)$ with $U^\star = \psi^\star_{y^\star}$ and $V^\star = -\psi^\star_{x^\star}$, then the boundary layer equations can be written as a function of a single similarity variable, $\eta$,

$$f''' + \beta_H(1 - f'^2) + f f'' = 0, \tag{12}$$

where prime denotes derivative with respect to $\eta$. The boundary conditions of equation (12) are given as $f'(0) = f(0) = 0$, and $f'(\eta) \to 1$, as $\eta \to \infty$. The solution of equation (12) is usually denoted the Falkner-Skan boundary layer and from the solution of $f$ we obtain the streamwise mean flow profile as $U(y) = f'(\eta)$.

## 4. LEAKY WAVES IN BOUNDARY LAYER FLOW

In this section a motivation is given for the appearance of leaky waves in flat plate boundary layer flow, considering the Orr-Sommerfeld equations as an initial value problem. The stability equation (2) is written using as primitive variables the perturbation velocities, wall normal derivative of the streamwise component, and the pressure. An often used alternative approach is the velocity-vorticity ($v - \eta$) formulation, see e.g. [4], in which the linearised Navier-Stokes equations are written as two equations; one for the wall-normal perturbation velocity, and one for the wall normal vorticity. If an ansatz as $u'(x, y, t) = u(y) \exp(\mathrm{i}\,\alpha\, x)$, which considers two-dimensional waves, is assumed then the two equations are uncoupled. The equation for the wall-normal perturbation velocity can, in the temporal case, be written

$$(\frac{\partial}{\partial t} + \mathrm{i}\alpha U)\Delta_2\, v + \mathrm{i}\alpha U'' v = \frac{1}{Re}\Delta_2\,\Delta_2 v \tag{13}$$

with $v = 0$, and $\mathcal{D}v = 0$ at the wall and in the free stream, $\mathcal{D}$ denotes the wall normal derivative, and $\Delta_2 = \mathcal{D}^2 - \alpha^2$. Equation (13) is, at time $t = 0$, given an initial condition $v(y, t = 0) = v_0(y)$. Problems of this type are commonly solved using a Laplace transform which for $v$ can be written

$$\tilde{v}(\alpha, y, \sigma) = \mathcal{L}(v) = \int_0^\infty v(\alpha, y, t)\, e^{-\sigma t}\, dt$$

Introducing the Laplace transform into equation (13) we obtain an inhomogeneous equation which solution can be written

$$v = \frac{1}{2\pi \mathrm{i}} \int \int G(y, y', \sigma, \alpha) \Delta_2\, v_0(y')\, dy' e^{\sigma t} d\sigma, \tag{14}$$

where $G$ is the Green's function. The above solution can be found by the method of variation of parameters, see [1]. Equation (13) is now written in terms of the Green's function as

$$(\sigma + \mathrm{i}\alpha U)\Delta_2\, G + \mathrm{i}\alpha U'' G - \frac{1}{Re} \Delta_2\, \Delta_2 G = \delta_D(y - y') \tag{15}$$

with boundary conditions $G(0) = G'(0) = 0$, and $G \to 0$ as $y \to \infty$. Here $\delta_D$ denotes the delta function. The resulting Green's function singularities is given by the free stream boundary conditions. We can note that equation (15) only has one solution, so does its homogeneous counterpart. Further, the behaviour as $y \to \infty$ for either case is given by the solution of the inviscid problem, viz. where $U = 1$, and $U'' = 0$. In this case equation (15) reduces to

$$\left(\sigma + \mathrm{i}\alpha - \frac{1}{Re} \Delta_2\right) \Delta_2 G = 0. \tag{16}$$

We can assume that $\Delta_2 G$ behaves as $\exp(-\beta y)$ as $y \to \infty$, and the solution of $\beta$ can be derived from equation (16) as

$$\beta^2 = \alpha^2 + Re\,(\sigma + \mathrm{i}\alpha). \tag{17}$$

As the wall normal wave number $\beta$ is solved from a second order equation, a general solution can be written $A_1\, \exp(\beta y) + A_2\, \exp(-\beta y)$. A combination of equation (15) and its homogeneous counterpart can be used to obtain a solution which decays as $y \to \infty$, provided that $A_1(\sigma) \neq 0$. The equation $A_1(\sigma) = 0$ on the other hand determines the pole singularities. It is further important to note that the square root relation between $\sigma$ and $\beta$ determines the branch point. The latter will be further investigated in the next section.

### 4.1 The $\sigma$-, and $\beta$ formulations

The representation commonly used of the initial value problem, which we denote the $\sigma$-formulation, is given by equation (15) with boundary conditions $G(0) = G'(0) = 0$ at $y = 0$, and for solutions decaying at $y \to \infty$ the solution is written

$$G = C\, e^{-\sqrt{\alpha^2 + Re\,(\sigma + \mathrm{i}\alpha)}},$$

where $C$ is a constant. The solution at $y \to \infty$ in this case is multi valued with its origin situated at the branch point. A different formulation, denoted the $\beta$-formulation, is therefore proposed in order to render the solution at infinity one-valued. From equation (17) we can write an expression for $\sigma$ as

$$\sigma = -\mathrm{i}\alpha + \frac{1}{Re}(\beta^2 - \alpha^2) \tag{18}$$

If equation (18) is substituted into equation (15) we obtain the equation

$$(-\mathrm{i}\alpha + \frac{1}{Re}(\beta^2 - \alpha^2) + \mathrm{i}\alpha U)\Delta_2\, G + \mathrm{i}\alpha U'' G - \frac{1}{Re}\Delta_2\, \Delta_2 G = \delta_D(y - y') \tag{19}$$

with boundary conditions $G(0) = G'(0) = 0$ at $y = 0$, and for solutions at $y \to \infty$ $G$ can be written

$$G = C\, e^{-\beta y} \tag{20}$$

which is now one valued. The result of this new formulation is given in the next section.

4.2 Appearance of leaky modes

The difference between the two formulations, $\beta$, and $\sigma$, can be seen comparing integration paths in respective complex plane. In figure 1 a sketch of the integration path's, $P_j$, are given in the $\sigma$-, and $\beta$ plane respectively. The branch point is given by the filled circle, and the singularities are given by open circles. If path $P_1$ is chosen in the $\sigma$-plane, figure 1(a), then the integration path can be closed

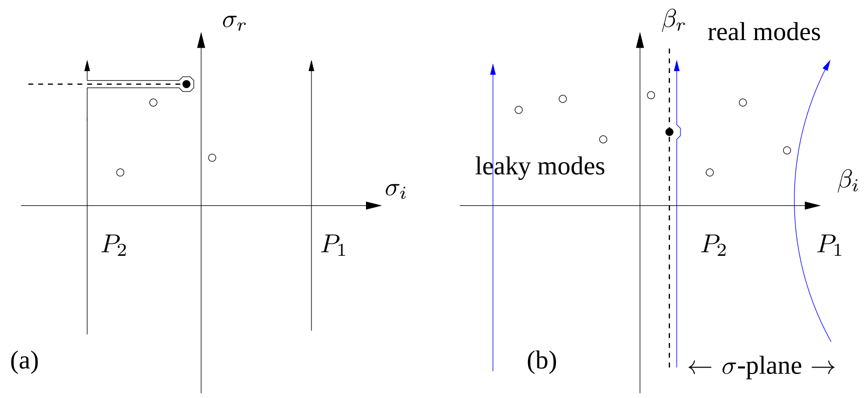


Figure 1: Sketch of integration path's in the complex (a) $\sigma$-plane, (b) $\beta$-plane.

in the counter clock-wise direction and thus satisfying the Cauchy integral theorem. The result is the sum of residues due to the pole singularities plus an integral along the sides of the branch cuts which is associated with the continuous spectrum. If we instead follow the path $P_2$, then it can be seen that the integral will be given only by the contribution due to the sides of the branch cut. A sketch of the spectrum when the $\beta$-formulation is used is found in figure 1(b). In this case $G$, see expression (20), is one valued function of $\beta$ and the unique solution is defined for $\pm\beta$, noting that $\beta$ is a two-valued function of $\sigma$. The branch cut in the $\sigma$-plane is now found as a straight line in the $\beta$-plane. To the right of it, the $\sigma$-plane is found containing its original poles. To the left, a number of new modes, the so called *leaky modes*, appear. Using the residue theorem, an integral along path $P_2$, associated with the continuous spectrum, equals the sum of the residues of the leaky modes.

## 5. NUMERICAL SOLUTION OF LEAKY WAVES

In section 4. it was discussed that the solution of the initial value problem is depending on the free stream boundary condition, and its general solution can be written $A_1 \exp \beta y + A_2 \exp -\beta y$. To compute the leaky modes we need to keep the term which grow as $y \to \infty$ while the decaying term must vanish. Such a problem is ill-conditioned why we have to pay some attention while solving it. Here, two different methods are shown.

### 5.1 Analytical continuation

One way to solve this problem is to think of the solution of equation (15) as an analytical solution and move the wall normal coordinate as a ray into the complex $y$-plane. The ray can be expressed using the imaginary part of $y$ as a function of the real part, $y_i = f(y_r)$, and the most simple function is a straight line between the origin $(0,0)$ to the free stream $(\max(y_r), \max(|y_i|))$. The idea is to choose the point in the complex $y$-plane which corresponds to the free stream, for which the required $\beta$ is dominant. Obviously, if the function is really analytical then the solution should not depend on the function $y_i = f(y_r)$ between the origin and the point in the free stream. There are however some limitations to the choice of the ray. As the mean flow appear as the coefficients of the Orr-Sommerfeld equations, it is of importance to see how the mean flow equations are effected by the introduction of a complex $y$-coordinate. Such an analysis was made by [2], where it is shown in what part of the complex $y$-plane the solution converges. Defining an angle $\varphi$ (in degrees) such that $\tan(\varphi) = \max(y_r)/\max(y_i)$, then a region where the mean flow converges is given roughly by $-30 \leq \varphi \leq 30$.

### 5.1 Impedance condition

An alternative approach is to directly impose a free stream boundary condition which contains a term which grows as $y \to \infty$. If we consider the Orr-Sommerfeld equation outlined in section 2. then the analytical solutions of the wall-normal wave number, $\lambda$, in the free stream are given by (5)–(6). As the solution in the free stream behaves as $e^{-\lambda y}$, it is clear that for $\lambda$ with the real part, $\lambda_r$, being positive the solution will decay at infinity. Since two solutions, one viscid and one inviscid, remain with the real part being negative it must be clarified which one to impose. We consider the perturbation (1) which in the free stream has a solution of the type (3). The wall normal wave number $\lambda$ is dependent on the input parameters, which for a given Reynolds number can be written $\lambda = \lambda(\omega)$. If the angular frequency is assumed complex then the time dependence can be written $\exp(-\mathrm{i}\omega t) = \exp(-\mathrm{i}[\omega_r + \mathrm{i}\omega_i]t) = \exp(-\mathrm{i}\omega_r t)\exp(\omega_i t)$. Setting the value of $\omega_i < 0$, and gradually increasing the negative value should, for correct boundary conditions imposed in the free stream, result in a damping of the perturbation. If a correct boundary condition is imposed or not will be found by evaluating $\lambda_r = \lambda_r(\omega_i)$, for different values of $\omega_i < 0$, at each Reynolds number. A $\lambda_r$ which decreases or even changes sign as the negative value of $\omega_i$ increases is not a damped perturbation which means that the boundary conditions are incorrect. Normally the discrete modes are solved imposing the two roots from equations (5)–(6) which decay in the free stream. Performing the analysis above using $\omega_i \neq 0$ for a certain discrete mode other than the least damped one, it turns out that the viscous root with positive real part is incorrect as free stream boundary condition for Reynolds numbers such that the discrete mode is close to or enters into the continuous spectrum. By changing the viscous root from positive to negative such that solutions growing in the wall-normal direction are allowed, it is possible to continue to follow the discrete mode at even lower Reynolds numbers. The free stream boundary conditions are here imposed using equation (11) which means that solution containing four terms, equation (3), is scalarly multiplied with the two left eigenvectors corresponding to the terms we do not wish to impose. In the case we impose as boundary condition the sum of the damped inviscid root and the growing viscous root it is possible to introduce errors, as we are computing the sum of a very large and a very small number. To avoid this the four by four matrices, equation (4), used to compute the eigen value solution in the free stream are derived directly from the discretised eigen value problem equation (2).

## 6. SOLVING THE EIGEN VALUE PROBLEM

To solve the non-linear eigen value problem we use an inverse iteration algorithm (IIA), which for an initial guess in the vicinity of the desired eigen value converges in a few iterations. In order to compute several eigen values, we have to make sure that the initial guess of the nth eigen value, $\mathbf{q}_n^0$, does not converge to an old one. This is obtained using the IIA, given a converged solution, by subtracting from $\mathbf{q}_n^0$ the previous solution using the bi-orthogonality between the right and left eigen vector.

## 7. NUMERICAL SOLUTION

The numerical solution is obtained discretising equation (2) in the wall-normal direction using a second order accurate central difference scheme on a uniform mesh. The stream-wise and wall-normal momentum equations, and the perturbation velocity components in respective directions are given at the node points, and the continuity equation, perturbation pressure, and the equation $u1 = u_y$ are staggered half a node point. The discretised version of (2) is a set of algebraic equations which are written in a block tri-diagonal form. Inversion of the operators, as shown in IIA, is made using a LU decomposition of the matrix. The similarity solution of the mean flow, equation (12), is discretised using a second order accurate central difference scheme, using the same node points as for the discretised form of equation (2). The equation is solved iteratively given an initial guess of $f''(0)$, and convergence is reached when the absolute value of the difference between two consecutive iterations of the streamwise velocity components wall-normal derivative at the wall, is less than $10^{-8}$. The convergence criteria, $err$, of the inverse iteration algorithm has a value of $err = 10^{-10}$.

## 8. RESULTS

The case studied here is a flat plate boundary layer subject to three different pressure gradients which are given by the Hartree parameters $\beta_H = -0.1, 0, 0.1$. The computations are, for each pressure gradient case, performed at three different Reynolds numbers for a fixed reduced frequency $F = \omega/Re = 25.10^{-6}$. The latter is chosen such that the amplification, $\ln(A/A_0)$, of the corre-

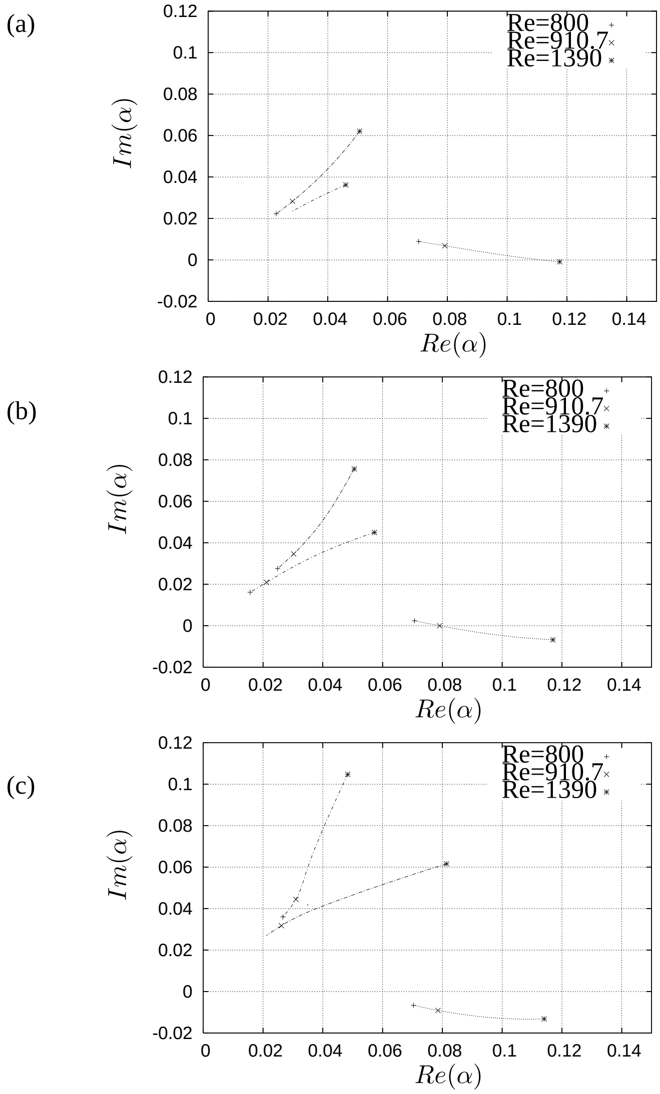


Figure 2: Spatial eigen value spectrum, $\alpha$, for three values of the Reynolds number including the trajectories of the discrete eigen values, given a reduced frequency $F = 25.10^{-6}$. The streamwise pressure gradient in the mean flow is given by the Hartree parameter (a) $\beta_H = 0.1$, (b) $\beta_H = 0$, (c) $\beta_H = -0.1$.

(a)

(b)

(c)

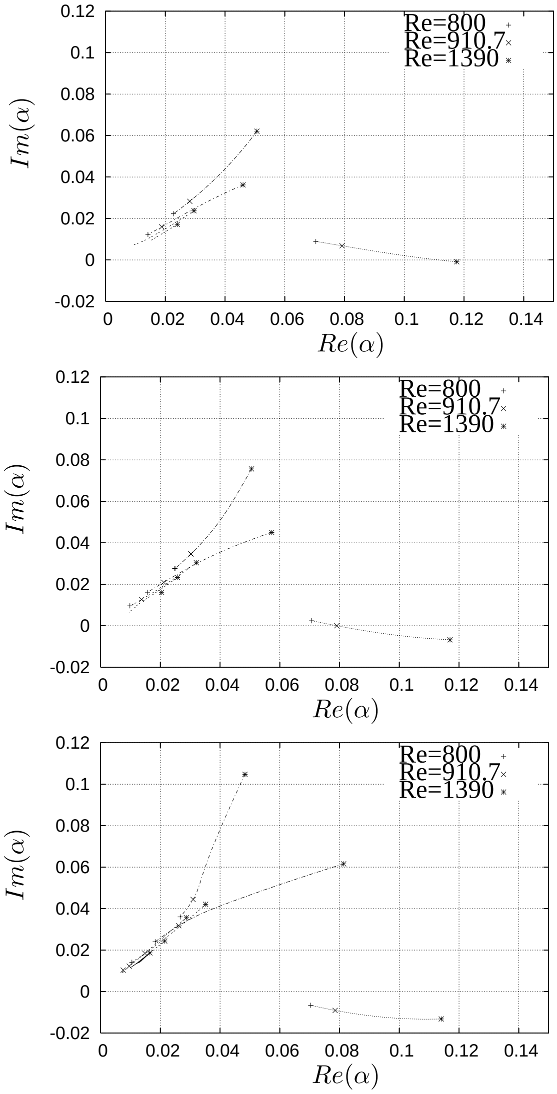


Figure 3: Spatial eigen value spectrum, $\alpha$, for three values of the Reynolds number including leaky modes and the trajectories, given a reduced frequency $F = 25.10^{-6}$. The streamwise pressure gradient in the mean flow is given by the Hartree parameter (a) $\beta_H = 0.1$, (b) $\beta_H = 0$, (c) $\beta_H = -0.1$.

sponding least stable wave for the case of $\beta_H = 0$, reaches a maximum value of about 9–10. Note that $A$ is the perturbation amplitude and subscript $0$ indicates the upstream neutral position. The *real* discrete modes of the eigen value spectrum $\alpha$ is found in figure 2 for the three different pressure gradients, and at three different Reynolds numbers respectively. The spectrum for each Reynolds number is computed by first using the inverse iteration algorithm outlined in section 6.1 with an initial guess of the streamwise wave number such that the phase velocity of $c_r \approx 0.3$. The additional higher modes are then computed using the algorithm given in section 6.2. The free stream boundary conditions are imposed using equation (11) such that only exponentially decaying solutions are allowed. The discrete modes at $Re = 800$ have further been used as a starting solution for the IIA to compute respective path changing the Reynolds number between $Re = [800, 1390]$, which are shown in the figure by the dotted lines. In order to render the graph more clear the discrete representation of the continuous spectrum, which for different Reynolds numbers is found at different values of $\alpha_r$, has not been plotted. The computation of the leaky modes has been made using both methods outlined in section 5. The analytical continuation was computed using a straight line in the complex $y$-plane with an angle $\varphi = \tan^{-1}(y_i/y_r) = -30$ degrees. Using the method to compute the global solution starting with the least stable mode, a number of new discrete modes are found at each Reynolds number. These modes, which are denoted leaky modes, all have the phase velocity $c_r > 1$ and are part of the discrete representation of the continuous spectrum. The same method as earlier described to compute the path changing the Reynolds number for each discrete mode, is again used now including the leaky modes. The results are found in figure 3 where the leaky modes appear, for each Reynolds number, at values of $\alpha_r$ smaller than for the real modes since the phase velocity is greater than one. It can be noted that no discontinuity is found following the path from the real modes to the leaky modes. Following each path for increasing phase velocity, decreasing $\alpha_r$, it can be noted that that the path of all modes tend towards a common path. The results shown in figure 3 have also been computed using the impedance condition. This method is in practice computationally more time consuming as, for each Reynolds number, the $\omega_i$-test has to be performed in order to impose the correct free stream boundary condition. The results in figure 3 are obtained computing one discrete mode, in the Reynolds number range $Re = [800, 1390]$, at a time and at each discrete Re perform the $\omega_i$-test. Since we know that changing the boundary condition becomes crucial when we come close to the continuous spectrum, it is actually enough to start using the test in its vicinity where $c_r < 1$.

## 9. CONCLUSIONS

The aim of the present work was to investigate if it is possible to have an all discrete representation of the eigen value spectrum of the Orr-Sommerfeld equations applied to flat plate boundary layer flows. In addition to investigate why some discrete modes disappear, or seem to disappear behind the continuous spectrum at certain Reynolds numbers. The problem is formulated to investigate the response in a flat plate boundary layer forced instantaneously at a point in space and time. Since the solution of the forced and homogeneous Laplace transformed problem both depend on the free stream boundary conditions, it is shown here that an opportune change of variables can remove the branch cut in the complex eigen value plane. As a result integration of the inversed Laplace transform along the new path corresponding to the continuous spectrum, which is now given by a straight line, and equals the summation of residues of additional discrete eigen values appearing to the left of it. It is further shown that these additional modes are computed accounting for solution which grow in the wall normal direction. A similar problem is found in optics, and especially in the theory of optical waveguides (such as optical fibers), where it exist solutions that are attenuated in one direction, such as the direction of the wave-guide, while it grows unbounded in a direction perpendicular to it. The solutions are so called *leaky waves*. The theory is here applied both to spatial linear stability analysis in the case of flat plate boundary layer flow subject to different pressure gradients. The analysis performed shows that the trajectory of the discrete modes in fact persist behind the continuous spectrum at each Reynolds number appearing as *leaky waves* with phase velocities ($c_r$) larger than one. These leaky waves are, as discussed above, part of the discrete representation of the continuous spectrum.

# References


[1] W. E. Boyce and R. C. DiPrima. *Elementary differential equations and boundary value problems*. John Wiley & Sons, 5 edition, 1992.

[2] J. P. Boyd. The Blasius function in the complex plane. *Experimental Mathematics*, 8(4):381–394, 1999.

[3] C. E. Grosch and H. Salwen. The continuous spectrum of the Orr-Sommerfeld equation. Part 1. The spectrum and the eigenfunctions. *J. Fluid Mech.*, 87:33–54, 1978.

[4] D. S. Henningson and P. J. Schmid. *Stability and transition in shear flows*. Springer, 1st edition, 2001.

[5] D. Marcuse. *Theory of dielectric optical waveguides*. Academic press, inc., 2nd edition, 1991.

[6] T. Tamir. Leaky waves in planar optical waveguides. *Nouv. Rev. Optique*, 6(5):273–284, 1975.


APPENDIX

The matrices **A**, and **B** in equation (2) are given as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & S & U_y & \mathrm{i}\alpha \\ 0 & \mathrm{i}\alpha & 0 & 0 \\ \mathrm{i}\alpha Re^{-1} & 0 & S & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -Re^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $S = \mathrm{i}(\alpha U - \omega) + \alpha^2 Re^{-1}$.