COURSE OF ADVANCED FLUID DYNAMICS

Optimal perturbation and stability analysis of a spatial developing flow

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INTRODUCTION

In the present project we will search for the optimal perturbation in a spatial developing flow. Using the modal and non-modal stability analysis we will calculate the stability of the fluid. The Ginzburg-Landau model will be used to describe the wave amplitude in a bifurcating spatially developing flow.

This project is divided in six parts, these consist in:

Part 1. Adjoint equations: we will use the Ginzburg-Landau model to derivate the optimality system and the adjoint equation with the established values in the beginning of the problem.

Part 2. Numerical solution: The optimization of the problem is solved numerically using the Finite Differences Method.

Part 3. Optimal perturbation: The optimal perturbation will be calculated varying the Ginzburg-Landau parameters.

Part 4. Linear stability analysis: The stability analysis is made through the modal theory.

Part 5. Transient Growth: The stability analysis is made through the non-modal theory.

Part 6 . Conclusion.

Here, we have the linearized equation for the amplitude of a perturbation governed by the Ginzburg-Landau model:

$$\frac{\partial \phi}{\partial t} = \sigma(x)\phi - U\left(\frac{\partial \phi}{\partial x}\right) + \mu\left(\frac{\partial^2 \phi}{\partial x^2}\right)$$

Knowing that:

- $\phi = \phi(x,t)$ is the wave amplitude of the perturbation
- U is the velocity of the main flow
- μ is the diffusion coefficient
- $\sigma(x)$ is the local bifurcation parameter $\sigma(x) = \sigma_0 - \sigma_2\left(\frac{x^2}{2}\right) \quad \sigma_2 \geq 0$
- g=g(x) is the initial condition

The above-written equation will be solved in a one-dimensional domain D=(α, β) from time t=(0,T), optimizing g in order to minimize de following quantity.

$$J = \frac{\langle g(x), g(x)\rangle}{\langle \phi(t=T), \phi(t=T)\rangle}$$

This equation is called COST FUNCTION.

We also have the boundary and initial conditions of the problem:

$$\phi(\alpha, t) = 0$$

$$\phi(\beta, t) = 0$$

$$\phi(x, 0) = g(x) = \phi_0$$

In the next point, we will calculate the adjoint equations.

1.- ADJOINT EQUATIONS

We have defined our STATE EQUATION as:

$$\frac{\partial \phi}{\partial t} = \sigma(x)\phi - U\left(\frac{\partial \phi}{\partial x}\right) + \mu\left(\frac{\partial^2 \phi}{\partial x^2}\right) = G\phi$$

In order to derive the optimality condition with equality constraints with the method of Lagrange multipliers we have to find the stationary points of the Lagrangian with respect to the variables (ϕ, g, a, b, c, d).

[1]

$$\mathcal{L}(\phi, g, a, b, c, d) = \mathcal{J} - \int_0^T < a, \frac{\partial \phi}{\partial t} - G\phi > dt - < b, \phi(x, 0) - \phi_0 > - \int_0^T c\{\phi(\beta, t) - 0\}dt - \int_0^T d\{\phi(\alpha, t) - 0\}dt$$

The derivation respects a,b,c and d leads to the state equation, initial and boundary conditions:

[1] *Taken from "Lecture Notes from the University of Genova" Chapter I: Constrained Optimization, example 5.2- Optimal growth for the non parallel Ginzburg-Landau operator in an unbounded domain.*

$$\frac{\partial\mathcal{L}}{\partial a} = 0 \;\rightarrow\; \frac{\partial\phi}{\partial t} = \;\sigma(x)\phi - U\left(\frac{\partial\phi}{\partial x}\right) + \mu\left(\frac{\partial^2\phi}{\partial x^2}\right)$$

$$\frac{\partial\mathcal{L}}{\partial b} = 0 \;\rightarrow\; \phi(x,0) = g(x) = \phi_0$$

$$\frac{\partial\mathcal{L}}{\partial c} = 0 \;\rightarrow \phi(\beta, t) = 0$$

$$\frac{\partial\mathcal{L}}{\partial d} = 0 \;\rightarrow\; \phi(\alpha, t) = 0$$

Derivation respects (ϕ, g) give us the ADJOINT EQUATION with boundary and optimality conditions (full derivation in the Appendix)

$$\frac{\partial\mathcal{L}}{\partial g} = 0 \qquad \mathrm{g(x)} = -\frac{\mathrm{b(x)}}{2} < \phi(x,T), \phi(x,T) >$$

$$\frac{\partial\mathcal{L}}{\partial \phi} = 0 \qquad \frac{\partial \mathrm{a}}{\partial t} = \; -\sigma(x)a - U\left(\frac{\partial a}{\partial x}\right) - \mu\left(\frac{\partial^2 a}{\partial x^2}\right) \;\; ; \quad a_T = \frac{-2\langle g(x), g(x)\rangle\phi_T}{\langle\phi_T,\phi_T\rangle\langle\phi_T,\phi_T\rangle} \;\; ; \quad b = a_0$$

2.- NUMERICAL SOLUTION

A Matlab script has been written in order to solve numerically the optimization problem. The main steps are briefly outlined here:

- Forward integration of the state equation.
- Evaluation of the cost function.
- Backward integration of the adjoint equation.
- Assessment of a new control function (optimality equation).

We will set all these steps inside a loop so that the difference between two consecutive values is not higher than an imposed “tolerance”.

The integrations of the state and adjoint equations are performed using an implicit backward Euler finite difference scheme:

STATE EQUATION [2]

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \; -U\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} + \mu\frac{\phi_{i+1}^{n+1} - 2\phi_i^{n+1} + \phi_{i-1}^{n+1}}{\Delta x^2} + \sigma_i\phi_i^{n+1}$$

[2] *Taken from “Lecture Notes from the University of Genova” Chapter I: Sensitivity analysis”.*

$$\boldsymbol{\phi_i^n = \phi_{i-1}^{n+1}\left[-\frac{U\Delta t}{2\Delta x} - \frac{\mu\Delta t}{\Delta x^2}\right] + \phi_i^{n+1}\left[1 + \frac{2\mu\Delta t}{\Delta x^2} - \sigma_i\Delta t\right] + \phi_{i+1}^{n+1}\left[\frac{U\Delta t}{2\Delta x} - \frac{\mu\Delta t}{\Delta x^2}\right]}$$

ADJOINT EQUATION[3]

$$-\frac{a_i^n - a_i^{n-1}}{\Delta t} = U\frac{a_{i+1}^{n-1} - a_{i-1}^{n-1}}{2\Delta x} + \mu\frac{a_{i+1}^{n-1} - 2a_i^{n-1} + a_{i-1}^{n-1}}{\Delta x^2} + \sigma_i a_i^{n-1}$$

$$\boldsymbol{a_i^n = a_{i-1}^{n-1}\left[\frac{U\Delta t}{2\Delta x} - \frac{\mu\Delta t}{\Delta x^2}\right] + a_i^{n-1}\left[1 + \frac{2\mu\Delta t}{\Delta x^2} - \sigma_i\Delta t\right] + a_{i+1}^{n-1}\left[-\frac{U\Delta t}{2\Delta x} - \frac{\mu\Delta t}{\Delta x^2}\right]}$$

The accuracy of the adjoint has been checked using the "adjoint equality" which in our case gives:

$$\iint_{0\,\alpha}^{T\,\beta} a\left[\frac{\partial \phi}{\partial t} - \sigma(x)\phi + U\left(\frac{\partial \phi}{\partial x}\right) - \mu\left(\frac{\partial^2 \phi}{\partial x^2}\right)\right] dtdx = \iint_{0\,\alpha}^{T\,\beta} \phi\left[\frac{\partial a}{\partial t} + \sigma(x)\phi + U\left(\frac{\partial \phi}{\partial x}\right) + \mu\left(\frac{\partial^2 \phi}{\partial x^2}\right)\right] dtdx + [a\,\phi]_0^T$$

$$[a\,\phi]_0^T = 0 \longrightarrow a(0)\phi(0) = a(T)\phi(T)$$

which in all our simulation has been less than 10^{-10}, next to the machine precision.

3.- OPTIMAL PERTURBATION

The initial condition g that maximises the ratio between the final and the initial disturbance kinetic energy, usually denoted G, is defined as the OPTIMAL PERTURBATION. This can be recast as a minimization problem where the aim is to minimize the inverse of G, ie $J = \frac{1}{G}$

$$J = \frac{\langle g(x), g(x)\rangle}{\langle \phi(t=T), \phi(t=T)\rangle}$$

So, we present the evolution of the OPTIMAL PERTURBATION analyzing the evolution of $G = \frac{1}{J}$ with time (0,20). In the next table, we can see that, varying σ_0 [0.4-0.5] the behavior of **G** values is as follows:

[3] *Taken from "Lecture Notes from the University of Genova" Chapter I: Sensitivity analysis".*

Sigma0	T=2	T=4	T=6	T=8	T=10	T=12	T=14	T=16	T=18	T=20
0.40	1,8857	2,6788	2,8090	2,4384	1,9294	1,4663	1,0957	0,8129	0,6014	0,4444
0.41	1,9623	2,9019	3,1666	2,8612	2,3568	1,8639	1,4497	1,1194	0,8621	0,6631
0.42	2,0425	3,1437	3,5702	3,3568	2,8777	2,3697	1,9179	1,5418	1,2356	0,9892
0.43	2,1259	3,4048	4,0258	3,9386	3,5149	3,0120	2,5374	2,1231	1,7712	1,4758
0.44	2,2124	3,6887	4,5372	4,6232	4,2937	3,8285	3,3568	2,9231	2,5381	2,2017
0.45	2,3026	3,9952	5,1151	5,4230	5,2438	4,8662	4,4405	4,0258	3,6377	3,2841
0.46	2,3964	4,3271	5,7670	6,3654	6,4020	6,1843	5,8754	5,5432	5,2138	4,8972
0.47	2,4944	4,6882	6,5020	7,4683	7,8186	7,8616	7,7700	7,6336	7,4683	7,3046
0.48	2,5961	5,0761	7,3314	8,7642	9,5511	9,9900	10,2775	10,5042	10,7066	10,8932
0.49	2,7020	5,5006	8,2645	10,2775	11,6550	12,7065	13,6054	14,4718	15,3374	16,2602
0.50	2,8121	5,9559	9,3197	12,0627	14,2450	16,1290	17,9856	19,9203	21,9780	24,2131

Table 1. Values of G obtained varying σ_0 and T.

According to the behavior of G, varying σ_0 we can come up with the next conclusions:

0.40<σ_0<0.44 The perturbation first increase, and when T<6 it decreases.

σ_0=0.44 The function/perturbation is STABLE, because the initial and final perturbation are almost the same.

0.44<σ_0<0.475 The perturbation has the same behavior than before, but in these cases the maximum perturbation occurs later and later. And our J value at T=20 (final perturbation) becomes more and more neutral.

σ_0=0.475 The function is NEUTRAL.

0.475<σ_0<0.5 The perturbation grows with time.

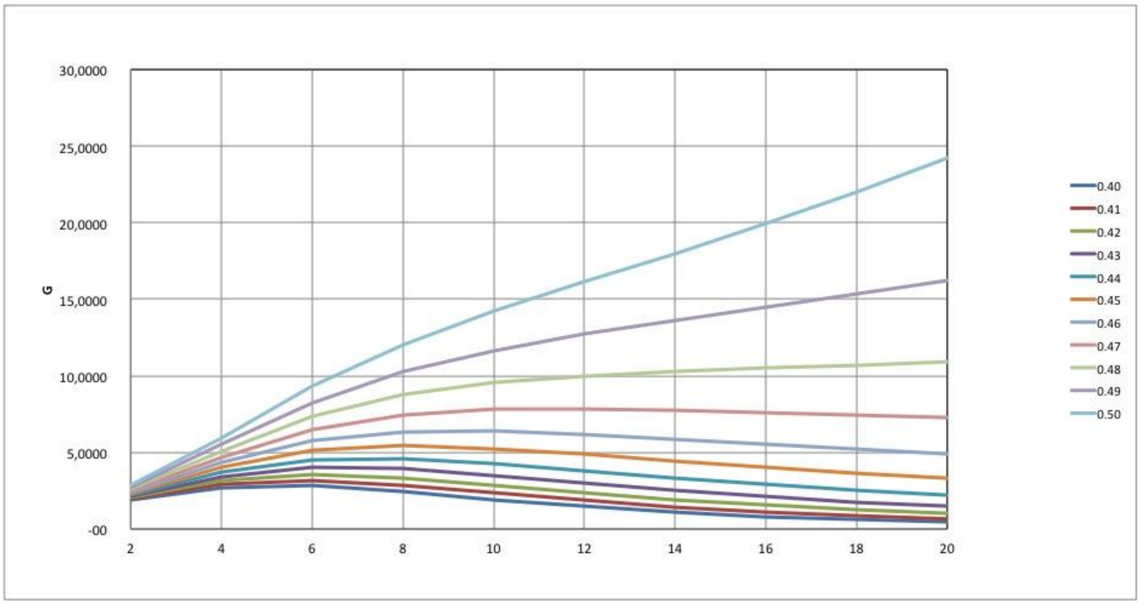

Figure 1. Evolution of G varying σ_0 and T.

4.- LINEAR STABILITY ANALYSIS

In order to test the performance of the solution with tools from linear stability analysis we work with NORMAL MODE DECOMPOSITION, substituting the solution written as:

$\phi(x,t) = \hat{\phi}(x)e^{\lambda t}$

Into the equation: $\frac{\partial \phi}{\partial t} = \mathcal{A}\phi \qquad (\mathcal{A} = -U\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial x^2} + \sigma)$

This transforms the linear initial-value problem into a corresponding eigenvalue problem where $\mathcal{J}$ is the identity operator

$$\lambda\phi(x) = \mathcal{A}\hat{\phi}(x)$$

$$(\mathcal{A} - \lambda\mathcal{J})\hat{\phi}(x) = 0$$

Doing this we convert our problem of "linear initial-value" in a problem of eigenvalues. A positive eigenvalue in $\mathcal{A}$ makes the function ϕ grows exponentially while for negative values it decreases exponentially. Therefore, we must find the highest eigenvalues because these will be the most unstable and will be used in the next part.

sigma0	0,4	0,41	0,42	0,43	0,44	0,45	0,46	0,47	0,48	0,49	0,5
eigenv.	-0,0758	-0,0658	-0,0558	-0,0458	-0,0358	-0,0258	-0,0158	-0,0058	0,0042	0,0142	0,0242

Table 2. The most unstable eigenvalues of $\mathcal{A}$ according to σ_0(0,4-0,5).

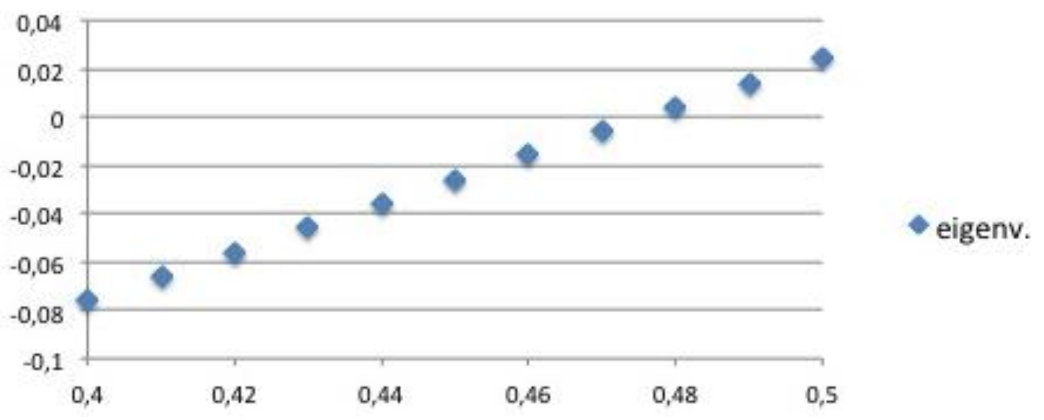

Figure 2. The most unstable eigenvalues of $\mathcal{A}$ according to σ_0(0,4-0,5).

5.- TRANSIENT GROWTH[4]

We define the gain G(t) as the ratio between some measure related to the energy of the current and initial perturbation:

$$G(t) = \max_{g_0} \frac{\| \phi(x,t)\, \phi(x,t) \|}{\| g_0(x) g_0(x) \|}$$

But since the evolution of the system is described by

$$\phi(x,t) = g_0(x) \exp{(\mathcal{A}t)}$$

The equation becomes

$$G(t) = \max_{g_0} \frac{\| g_0^2(x) \exp{(2\mathcal{A}t)} \|}{\| g_0(x) g_0(x) \|} = \| \mathcal{S} \exp{(2\Lambda t \mathcal{S}^{-1}} \|$$

It should become obvious that no information about the eigenvectors of $\mathcal{A}$, contained in $\mathcal{S}$, are considered only when the least stable mode is taken as a representation of the operator exponential.

From the stability theory we know that the minimum growth-rate of the solution matches at least with the most unstable eigenvalue, so we can say that

$$\exp{(2\lambda_{max} t)} \leq G(t) \leq \| \mathcal{S} \| \| \mathcal{S}^{-1} \| \exp{(2\lambda_{max} t)}$$

The quantity $\| \mathcal{S} \| \| \mathcal{S}^{-1} \|$ represents the condition number of $\mathcal{S}$ (k($\mathcal{S}$)), a measure of the non-orthogonality of this columns. So, if k($\mathcal{S}$)>1 (as in our case) the operator $\mathcal{A}$ is said to be non-normal, and systems governed by non-normal matrices can exhibit a large transient amplification of energy contained in the initial condition.

In our case, we have evaluated the evolution of energy present in a perturbation to different σ_0 values. In it we could verify the results since the match between optimal perturbations and no modal analysis, and we have observed that they actually match. In figure 2 we can observe that for lower values of "t" (t ≤ 6) the function represents a transient growth explained by the non-orthogonality of the eigenvectors of $\mathcal{A}$. For values higher than t (t > 6) the function behaves depending on the instability eigenvalues.

[4] *Taken from "Non-modal stability analysis. Part of the course Advanced Fluid Dynamics"*

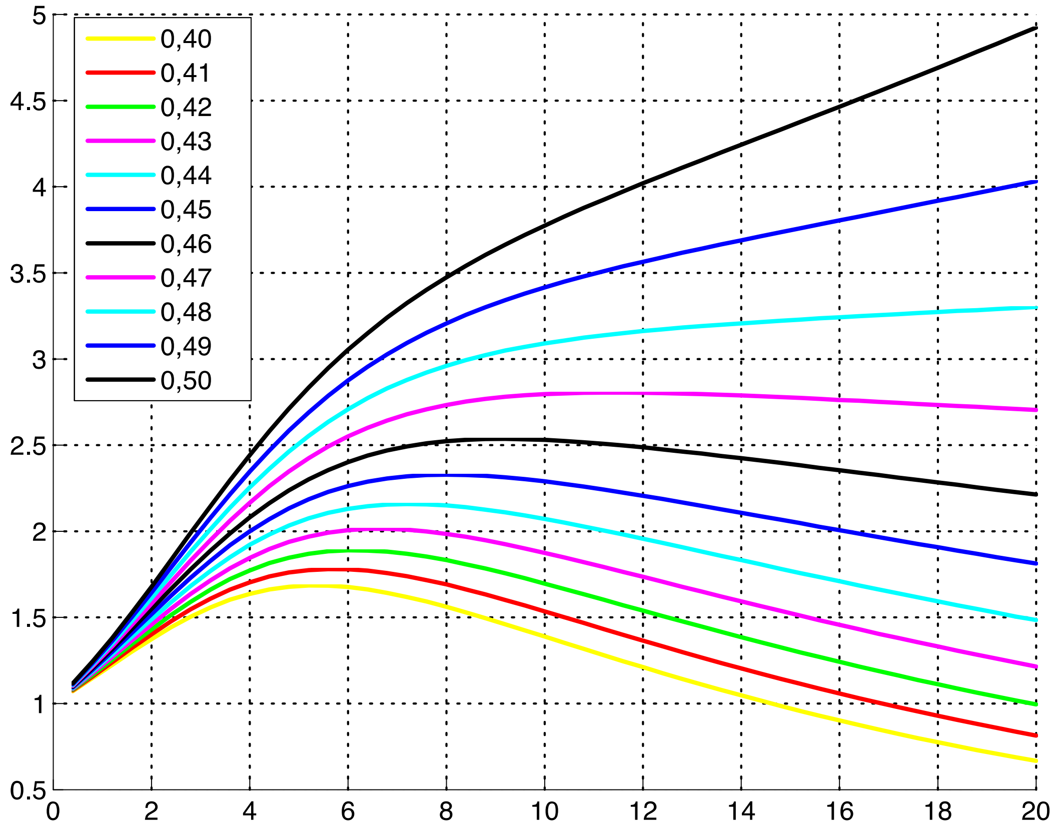

Figure 3. Semilogarithmic plot of the gain function G(t) from which we can we see the transient growth of the solution for different values of σ_0.

6.- CONCLUSIONS

In this project we have observed the stability of an initial perturbation in a spatially developed flow described by the Ginzburg-Landau equation, with classic modal analysis and from recently-developed non-modal analysis.

The results have shown that, we can come up with the same conclusion for both methods. The solution exhibits a so-called transient growth on a finite time and we can see that, by increasing σ_0 (strongly and non-parallel flow) the perturbation becomes more and more unstable (non-normal).

7.- APPENDIX

$$\mathcal{L}(\phi, g, a, b, c, d) = \mathcal{I} - \int_0^T < a, \frac{\partial \phi}{\partial t} - G\phi > dt - < b, \phi(x,0) - \phi_0 > - \int_0^T c\{\phi(\alpha, t) - 0\} dt$$
$$- \int_0^T d\{\phi(\beta, t) - 0\} dt$$

Stazionary point:

$$\partial \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\partial \phi + \frac{\partial \mathcal{L}}{\partial g}\partial g + \frac{\partial \mathcal{L}}{\partial a}\partial a + \frac{\partial \mathcal{L}}{\partial b}\partial b + \frac{\partial \mathcal{L}}{\partial c}\partial c + \frac{\partial \mathcal{L}}{\partial d}\partial d$$

$$Find \quad \partial \mathcal{L} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 0; \frac{\partial \mathcal{L}}{\partial g} = 0; \frac{\partial \mathcal{L}}{\partial a} = 0; \frac{\partial \mathcal{L}}{\partial b} = 0; \frac{\partial \mathcal{L}}{\partial c} = 0; \frac{\partial \mathcal{L}}{\partial d} = 0$$

Introduce: $g \leftarrow g + \delta g$; $\phi \leftarrow \phi + \delta \phi$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -2 < \phi(x,T), \delta\phi(x,T) > \frac{< g(x), g(x) >}{[< \phi(x,t), \phi(x,t) >]^2} + - \int_0^T < a, \frac{\partial \delta\phi}{\partial t} - G\delta\phi > dt -$$
$$< b, \delta\phi(x,0) > - \int_0^T c(\delta\phi(\alpha,t)) dt - \int_0^T d(\delta\phi(\beta,t)) dt = 0$$

Now, I have to resolve the integration by parts in space and time:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -2\langle \phi(x,T), \delta\phi(x,T)\rangle \frac{< g(x), g(x) >}{< \phi(x,t), \phi(x,t) >^2} - \langle a(x,T), \delta\phi(x,T)\rangle + \langle a(x,0), \delta\phi(x,0)\rangle$$
$$- \int_0^T \left[aU\delta\phi + \mu a \frac{\partial \delta\phi}{\partial x} - \mu \frac{\partial a}{\partial x}\delta\phi \right]_\alpha^\beta dt$$
$$- \int_0^T \langle \delta\phi, -\frac{\partial a}{\partial t} - G^* a\rangle dt - \langle b, \delta\phi(x,0)\rangle - \int_0^T [c\delta\phi(\alpha,t) + d\delta\phi(\beta,t)] dt = 0$$

Collect terms:

$$< -2\phi(x,T) > \frac{< g(x), g(x) >}{< \phi(x,t), \phi(x,t) >^2} - \langle a(x,T), \delta\phi(x,T)\rangle + \langle a(x,0) - b(x), \delta\phi(x,0)\rangle$$
$$+ \int_0^T \langle \delta\phi, -\frac{\partial a}{\partial t} - G^* a\rangle dt$$
$$+ \int_0^T \left[-Ua\delta\phi + \mu a \frac{\partial \delta\phi}{\partial x} - \mu \frac{\partial a}{\partial x}\delta\phi \right]_\alpha^\beta dt - \int_0^T [c\delta\phi(\alpha,t) + d\delta\phi(\beta,t)] dt = 0$$

Enforcing the boundary and initial conditions yields:

1. $a(x,T) = -2\phi(x,T)\frac{<g(x),g(x)>}{<\phi(x,T),\phi(x,T)>^2}$
2. $a(x,0) = b(x)$
3. $\frac{\partial a}{\partial t} = -G^* a = -a\sigma - U\frac{\partial a}{\partial x} - \mu\frac{\partial^2 a}{\partial x^2}$
4. $c(t) = \mu\frac{\partial a}{\partial x}(\alpha,t)$
5. $d(t) = -\mu\frac{\partial a}{\partial x}(\beta,t)$
6. $a(\alpha,t) = a(\beta,t) = 0$

Derivation of the optimality conditions:

$$\frac{\partial \mathcal{L}}{\partial g} = 0 \quad \longrightarrow \quad \frac{\partial \mathcal{L}}{\partial g} = \frac{2g(x)}{<\phi(x,t),\phi(x,t)>} + b = 0$$

$$\mathrm{g(x)} = -\frac{\mathrm{b}}{2} < \phi(x,t),\phi(x,t) >$$

Now I have the complete optimality system:

State equations:

- $\frac{\partial \phi}{\partial \mathrm{t}} = \mathrm{G}\phi$
- $\Phi(\mathrm{x},0) = \phi_0 = \mathrm{g(x)}$
- $\Phi(\alpha,\mathrm{t}) = 0$
- $\Phi(\beta,\mathrm{t}) = 0$

Adjoint equations:

- $\frac{\partial a}{\partial t} = -G^* a = -a\sigma - U\frac{\partial a}{\partial x} - \mu\frac{\partial^2 a}{\partial x^2}$
- $\mathrm{a}(\alpha,\mathrm{t}) = 0$
- $\mathrm{a}(\beta,\mathrm{t}) = 0$
- $a(x,T) = -2\phi(x,T)\frac{<g(x),g(x)>}{<\phi(x,T),\phi(x,T)>^2}$

Optimality Condition:

- $\mathrm{g(x)} = -\frac{\mathrm{b}}{2} < \phi(x,t),\phi(x,t) >$

8.- REFERENCES

PRALITS, Jan Oscar. Lecture Notes from the University of Genova. Academic publication. CHAPTER I: Control Optimization. April 8, 2013. P. 21.

PRALITS, Jan Oscar. Lecture Notes from the University of Genova. Academic publication. CHAPTER I: Sensitivity analysis. March 14, 2013.

PRALITS, Jan Oscar. Non-modal stability analysis. Part of the course Advanced Fluid Dynamics. 2013. P.22.