Optimal control of a non-homogeneous convective wave equation in a mono-dimensional resonator: a variational approach.

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September 24, 2012

Abstract

Low level of pollutants can be achieved by a lean and premixed burning. Unfortunately, these are the conditions causing the undesirable phenomenon of self-excited thermo-acoustic oscillations, responsible for inefficient burning and structural stresses so intense that they can lead to engine and combustor failure. The phenomena is well described by the non-homogeneous convective wave equation that, in its simplest application, could be written in a one dimensional space domain. The article wants to let the reader gain sensitivity on the effect of the heat released from a source located in bounded flow. A variational analysis will be performed to show the optimal time-dependence of the heat source in order to minimize the oscillations inside the resonator.

1 Introduction

Thermo-acoustic instabilities may occur whenever combustion takes place inside a resonator. The phase difference between heat release oscillations and pressure waves at the injection holds responsibility for the phenomenon, as described by Lord Rayleigh [1]. Strong vibrations at low frequencies may establish inside the resonator causing the humming phenomenon that irremediably affects the functioning and the efficiency of the system. A simple analysis on a mono-dimensional problem based on a variational approach is performed to find out the optimal shape of the heat release. Step by step derivation of the math is explicitly given as well as main set up of the MATLAB code.

2 The physical model and the equations

The following problem can be easily inferred from a combination of linearised conservation principles of mass, momentum and energy. It differs from the well known D'Alembert Wave Equation due to the presence of the material derivative in place of the ordinary time derivative in order to take into account a non zero superimposed mean flow. The source term is the material derivative of the heat release $Q(x,t)$. Since such a kind of energy transfer is usually represented by a flame or, in experimental set up, by an heated grid, its space dependency could not be represented by continuous functions and piece-wise functions are needed (heaviside $H[(x-a)(b-x)]$ or Dirac Delta $\delta(x-f)$). For the sake of clarity, and for easier derivation, we choose the step-function H. Nonetheless, thank to this choice it is possible to modify the thickness of the flame shrinking it to a flat sheet when $a = b$. Boundary conditions are chosen in order to model an open-ended duct in both inlet and outlet. Initial conditions are chosen between the easiest harmonic function.

$$\begin{cases} \dfrac{D^2p(x,t)}{Dt^2} - c^2\dfrac{\partial^2 p(x,t)}{\partial x^2} = \dfrac{DQ(x,t)}{Dt} & t>0, \quad 0<x<L, \quad c>0 \\ \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} & \\ Q(x,t) = \hat{Q}q(t)H\left[(x-a)\cdot(b-x)\right] & 0<a\leq b<L \\ p(0,t) = p(L,t) = 0 & \\ p(x,0) = \tilde{p}(x) = sin\left(\dfrac{n\pi x}{L}\right) & \\ \dfrac{Dp(x,0)}{Dt} = \dot{\tilde{p}}(x) = 0 & \end{cases} \tag{1}$$

3 The direct system

3.1 Continuous form

In order to cast the above described problem as follows:

$$[\mathbb{C}]_c\frac{\partial \mathbf{\Phi}(x,t)}{\partial t} + [\mathbb{A}]_c\mathbf{\Phi}(x,t) = [\mathbb{B}]_c\mathbf{q}(t),$$

the hereinafter proposed definition of $\mathbf{\Phi}(x,t)$ is introduced:

$$\mathbf{\Phi}(x,t) = \begin{Bmatrix} p \\ \dot{p} \end{Bmatrix}$$

where $\dot{p} = \frac{Dp}{Dt}$. The expression of the source term is:

$$\frac{DQ}{Dt} = \dot{q}(t)H\left[(x-a)\cdot(b-x)\right] + u(b+a-2x)q(t)\delta\left[(x-a)\cdot(b-x)\right]$$

The resulting system of equations is:

$$\frac{\partial}{\partial t}\begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} + \begin{bmatrix} u\frac{\partial}{\partial x} & -1 \\ -c^2\frac{\partial^2}{\partial x^2} & u\frac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} p \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{DQ}{\partial t} \end{Bmatrix} \tag{2}$$

The Cost Function J to be minimized is defined as follows:

$$\begin{cases} J(\mathbf{\Phi}(x,t)) = \dfrac{\gamma_1}{2}\displaystyle\int_0^T \mathbf{\Phi}^T(x,t)[\mathbb{K}]_c\mathbf{\Phi}(x,t)dt + \dfrac{\gamma_2}{2}\displaystyle\int_0^T \mathbf{q}^2(t)dt \\ [\mathbb{K}]_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

3.2 Discrete form

The space-time domain $T-L$ is dived into N time steps indexed as n giving $dt = T/N$ as a time resolution and M space steps indexed as m giving $dx = L/M$ as a space resolution. To gain awareness on the stability of the scheme the space discretization is superimposed and the time step is given by the condition on the CFL number $c\cdot dt/dx$. Therefore the following discretization scheme (implicit-2^{nd}-order Crank-Nicolson) is given for each point contained in the discrete domain, boundary excluded.

$$\frac{\partial \mathbf{\Phi}}{\partial t} = \frac{\mathbf{\Phi}_j^{n+1} - \mathbf{\Phi}_j^n}{\Delta t} \qquad \frac{\partial \mathbf{\Phi}}{\partial x} = \frac{\mathbf{\Phi}_{j+1}^n - \mathbf{\Phi}_{j-1}^n}{2\Delta x} \qquad \frac{\partial^2 \mathbf{\Phi}}{\partial x^2} = \frac{\mathbf{\Phi}_{j+1}^n - 2\mathbf{\Phi}_j^n + \mathbf{\Phi}_{j-1}^n}{\Delta x^2}$$

A different graphical notation for the involved variables will be used to remind their new discrete nature. The system (2) is written in discrete terms as follows:

$$[\mathbb{C}]\frac{\mathtt{phi(:,n+1)} - \mathtt{phi(:,n)}}{\Delta t} + [\mathbb{A}]\frac{\mathtt{phi(:,n+1)} + \mathtt{phi(:,n)}}{2} = [\mathbb{B}]\mathtt{q(n,1)} \tag{3}$$

Where the matrices are just inferred with the proper boundary conditions in the first and last lines. For the sake of clarity, despite the size of Φ is $[2,1]$ there is only one boundary condition in $x=0$ and $x=L$ where $p=\Phi(1,1)=0$ and no condition on $Dp/Dt=\Phi(2,1)$ is provided. The answer is to be found in the definition of the state $\Phi(x,t)$ where $\Phi(2,1)$ is a function of $\Phi(1,1)$ and its value on the boundary is directly calculated from the neighbourhood with a different discretization of the spatial derivatives (2^{nd}-order as well as ones in the body of the matrix $[\mathbb{A}]$). The Cost Function J is inferred in discrete terms as follows and the adjoint system can be directly derived in discrete terms granting an exact adjoint solution for any chosen resolution.

$$\begin{cases} \mathtt{J} = \dfrac{\gamma_1}{2}\displaystyle\int_0^T \mathtt{q(n,1)}^T\mathtt{q(n,1)} + \dfrac{\gamma_2}{2}\displaystyle\int_0^T \mathtt{phi(:,n)}^T\,[\mathbb{K}]\,\mathtt{phi(:,n)}dt \\ \mathtt{k(i,j)} = 1 \quad \textit{when} \quad \mathtt{i=j=2m-1} \\ \mathtt{k(i,j)} = 0 \quad \textit{when} \quad \mathtt{i\,,\,j \neq 2m-1} \end{cases}$$

Before going on with the set up of the optimality system let us recap the definition and dimension of each matrix involved in the discrete formulation:

$$\mathbb{A} = \begin{bmatrix} -3u/2\Delta x & -1 & 2u/\Delta x & 0 & -u/2\Delta x & 0 & .. & .. \\ 1 & 0 & 0 & 0 & 0 & 0 & .. & .. \\ -u/2\Delta x & 0 & 0 & -1 & u/2\Delta x & 0 & .. & .. \\ -c^2/\Delta x^2 & -u/2\Delta x & 2c^2/\Delta x^2 & 0 & -c^2/\Delta x^2 & u/2\Delta x & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. & .. \\ .. & .. & -u/2\Delta x & 0 & 0 & -1 & u/2\Delta x & 0 \\ .. & .. & -c^2/\Delta x^2 & -u/2\Delta x & 2c^2/\Delta x^2 & 0 & -c^2/\Delta x^2 & u/2\Delta x \\ .. & .. & u/2\Delta x & 0 & -2u/\Delta x & 0 & 3u/2\Delta x & -1 \\ .. & .. & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (2M \times 2M)$$

$$\mathbb{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & -1/\Delta t & u(b+a-2x(m))+1/\Delta t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\Delta t & 1/\Delta t & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0t & -1/\Delta t & u(b+a-2x(m))+1/\Delta t & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \end{bmatrix} \quad (2M \times N)$$

$$\mathbb{C} = \begin{bmatrix} 1 & .. & .. & .. & .. & .. \\ .. & 0 & .. & .. & .. & .. \\ .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & 1 & .. \\ .. & .. & .. & .. & .. & 0 \end{bmatrix} \quad (2M \times 2M) \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ .. \\ q_n \\ .. \\ q_N \end{bmatrix} \quad (N \times 1)$$

$$\mathtt{phi} = \begin{bmatrix} \Phi(1,1) & .. & \Phi(1,n) & .. & \Phi(1,N) \\ \dot\Phi(1,1) & .. & \dot\Phi(1,n) & .. & \dot\Phi(1,N) \\ .. & .. & .. & .. & .. \\ \Phi(m,1) & .. & \Phi(m,n) & .. & \Phi(m,N) \\ \dot\Phi(m,1) & .. & \dot\Phi(m,n) & .. & \dot\Phi(m,N) \\ .. & .. & .. & .. & .. \\ \Phi(M,1) & .. & \Phi(M,n) & .. & \Phi(M,N) \\ \dot\Phi(M,1) & .. & \dot\Phi(M,n) & .. & \dot\Phi(M,N) \end{bmatrix} \quad (2M \times N)$$

Stability analysis The eigenvalues of the matrix $[\mathbb{C}]+\frac{dt}{2}[\mathbb{A}]$ has been eveluated in order to check the stability of the system. Results are shown in figure 1.

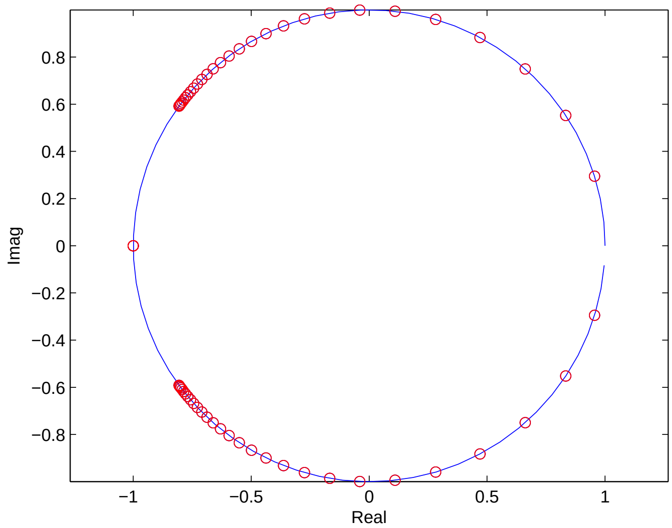

Figure 1: Eigenvalues of the matrix $[\mathbb{C}]+\frac{dt}{2}[\mathbb{A}]$ compared to the unit circle.

4 Optimization process

Let us write the Lagrange Operator to minimize the Cost Function J.

$$\mathcal{L}(\mathtt{phi},\mathtt{q},\mathtt{a},\mathtt{b}) = \frac{\gamma_1}{2}\int_0^T \mathtt{q}(\mathtt{n},1)^T\mathtt{q}(\mathtt{n},1) + \frac{\gamma_2}{2}\int_0^T \mathtt{phi}(:,\mathtt{n})^T\,[\mathbb{K}]\,\mathtt{phi}(:,\mathtt{n})dt+$$
$$-\int_0^T < \mathtt{a}\,,\left([\mathbb{C}]\frac{\partial\mathtt{phi}}{\partial t}+[\mathbb{A}]\mathtt{phi}-[\mathbb{B}]\mathtt{q}(\mathtt{n},1)\right) > dt - \mathtt{b}\,(\mathtt{phi}(:,0)-\Phi_0)$$

Differentiating the Lagrange Operator with respect to **a** and **b** trivially leads to the definition of the direct system and to the initial conditions on state $\Phi(x,0)$. On the other hand differentiating $\mathcal{L}$ with respect to $\Phi(x,t)$ and $\mathbf{q}(t)$ leads to the adjoint system and to the optimality condition respectively. The analytical details are shown hereinafter.

$$\frac{\partial\mathcal{L}(\mathtt{phi},\mathtt{q},\mathtt{a},\mathtt{b})}{\partial\mathtt{phi}} = \int_0^T \gamma_1[\mathbb{K}]\mathtt{phi}(:,\mathtt{n})\partial\delta\mathbf{\Phi}(x,t)dt$$
$$-\int_0^T < \mathtt{a}\,,\left([\mathbb{C}]\frac{\partial\delta\mathbf{\Phi}(x,t)}{\partial t}+[\mathbb{A}]\delta\mathbf{\Phi}(x,t)\right) > dt - \mathtt{b}\delta\mathbf{\Phi}(x,0) = 0$$

Integrating by parts the expression we get:

$$\int_0^T \gamma_1[\mathbb{K}]\mathtt{phi}(:,\mathtt{n})\partial\delta\mathbf{\Phi}(x,t)dt - \left[[\mathbb{C}]^T\mathtt{a}\delta\Phi(x,t)\right]_0^T +$$
$$+\int_0^T\left([\mathbb{C}]^T\frac{\partial\mathtt{a}}{\partial t}-[\mathbb{A}]^T\mathtt{a}\right)\delta\mathbf{\Phi}(x,t)dt - \mathtt{b}\delta\mathbf{\Phi}(x,0) = 0$$

$$\rightsquigarrow \quad \begin{cases} [\mathbb{C}]^T\frac{\partial}{\partial t}\mathtt{a}(:,\mathtt{n}) - [\mathbb{A}]^T\mathtt{a}(:,\mathtt{n}) + \gamma_1[\mathbb{K}]\mathtt{phi}(:,\mathtt{n}) = 0 \\ \mathtt{a}(:,N) = 0 \\ \mathtt{b}(:,1) = [\mathbb{C}]^T\mathtt{a}(:,1) \end{cases}$$

And finally the optimality condition:

$$\frac{\partial \mathcal{L}(\mathtt{phi},\mathtt{q},\mathtt{a},\mathtt{b})}{\partial \mathtt{q}} = \int_0^T \gamma_2 \mathtt{q}(\mathtt{n},\mathtt{1})dt - \int_0^T < \mathtt{a}\,, [\mathbb{B}] > dt = 0$$
$$\rightsquigarrow \qquad \mathtt{q}(\mathtt{n}) = \frac{\mathbb{B}(\,:\,,n)^T \mathtt{a}(\,:\,,\mathtt{n})}{\gamma_2}.$$

The adjoint system shows the same behaviour of the direct one; the same schemes will be applied:

$$[\mathbb{C}]^T \frac{\mathtt{a}(\,:\,,\mathtt{n+1}) - \mathtt{a}(\,:\,,\mathtt{n})}{\Delta t} - [\mathbb{A}]^T \frac{\mathtt{a}(\,:\,,\mathtt{n+1}) + \mathtt{a}(\,:\,,\mathtt{n})}{2} + \gamma_1 [\mathbb{K}]\mathtt{phi}(\,:\,,\mathtt{n}) = 0$$

Here follows the evaluation of the accuracy of the adjoint:

$$\int_0^T < \mathtt{a}\,, \left([\mathbb{C}]\frac{\partial \mathtt{phi}}{\partial t} + [\mathbb{A}]\mathtt{phi} - [\mathbb{B}]\mathtt{q}(\mathtt{n},\mathtt{1}) \right) > dt = [\mathtt{a}[\mathbb{C}]\mathtt{phi}]_0^T - \int_0^T \left([\mathbb{C}]^T \frac{\partial \mathtt{a}}{\partial t} - [\mathtt{A}]^T \mathtt{a} \right) \mathtt{phi} + [\mathbb{B}]^{\mathtt{T}}\mathtt{aq}dt$$
$$\rightsquigarrow \qquad [\mathtt{a}(\,:\,,\mathtt{N})[\mathbb{C}]\mathtt{phi}(\,:\,,\mathtt{N}) - \mathtt{a}(\,:\,,\mathtt{1})[\mathbb{C}]\mathtt{phi}(\,:\,,\mathtt{1})] = \int_0^T \left([\mathbb{B}]^T \mathtt{aq} - \gamma_1 \mathtt{phi}^T [\mathbb{K}]\mathtt{phi} \right) dt$$

In discrete terms the above expression stands for:

$$\mathtt{a}(\,:\,,\mathtt{n+1}) \cdot L\mathtt{phi}(\,:\,,\mathtt{n+1}) = \left(L^T \mathtt{a}(\,:\,,\mathtt{n+1}) \right) \mathtt{phi}(\,:\,,\mathtt{n})$$
$$\rightsquigarrow \qquad \mathtt{a}(\,:\,,\mathtt{n+1}) \left(\mathtt{phi}(\,:\,,\mathtt{n+1}) - [\mathbb{B}]q^n \right) = \mathtt{a}(\,:\,,\mathtt{n})\mathtt{phi}(\,:\,,\mathtt{n}) - \gamma_1 \mathtt{phi}(\,:\,,\mathtt{n})[\mathbb{K}]\mathtt{phi}(\,:\,,\mathtt{n})\Delta t$$

Integration over the whole time time domain leads to the condition:

$$\cancelto{0}{\mathtt{a}(\,:\,,\mathtt{N})\mathtt{phi}(\,:\,,\mathtt{N})} - \mathtt{a}(\,:\,,\mathtt{1})\mathtt{phi}(\,:\,,\mathtt{1}) =$$
$$\sum_{n=1}^{N} \left(\mathtt{a}(\,:\,,\mathtt{n+1})[\mathbb{B}]\mathtt{q}(\mathtt{n},\mathtt{1}) - \gamma_1 \mathtt{phi}(\,:\,,\mathtt{n+1})[\mathbb{K}]\mathtt{phi}(\,:\,,\mathtt{n+1}) \right) \Delta t.$$

The evaluation of the accuracy should lead to the machine precision thank to the discrete derivation of the adjoint system. In such a case this not occur and the accuracy never shrink beyond 10^{-3}. The reason of this unexpected behaviour is not clear and might be found in the strong gradients appearing in the adjoint solution that could produce relevant diffusion phenomena (figure 3).

5 Results

Here follows the parameters chosen for the optimization.

L	1 m	T	0.05 s
m	31	n	CFL·$\Delta x/c$
c	343 m/s	CFL	3
Mach	0.2	$\hat{Q}$	1
a	0.2 L	b	0.3 L
γ_1	1	γ_2	$10^{-}5$

Table 1: Parameters of the system.

In order to understand the reason of the little value of the ratio γ_2/γ_1 a brief sensitivity analysis of the Cost Function $J(\Phi(\mathbf{q}), \mathbf{q})$ with respect to the control $\mathbf{q}$) is performed.

$$\frac{\partial J}{\partial \mathbf{q}} = \frac{\partial}{\partial \mathbf{q}} \left(\frac{\gamma_1}{2} \int_0^T \Phi^T \Phi dt + \frac{\gamma_2}{2} \int_0^T \mathbf{q}^T \mathbf{q} dt \right) = \gamma_1 \int_0^T \Phi \frac{\partial \Phi}{\partial \mathbf{q}} + \gamma_2 \int_0^T \mathbf{q} dt$$

Following to the definition of the direct system, $\frac{\partial \Phi}{\partial \mathbf{q}}$ is something proportional to the matrix $\mathtt{inv}([\mathbb{C}] + \frac{dt}{2}[\mathbb{A}]) \cdot ([\mathbb{B}]dt\hat{Q})$. Given that $[\mathbb{B}]$ is a nearly empty matrix to be integrated over the whole time domain, the reason of the little value of γ_2/γ_1 is straightforwardly highlighted. Such an analysis is able to outline that the control $\mathbf{q}$ is able to have relevant effect on the solution only for large value of $\hat{Q}$. This is actually the case of gas-turbine thermo-acoustics where the dimension of $\hat{Q}$ is a power density (W/m^3) usually with the order of magnitude of 10^6.

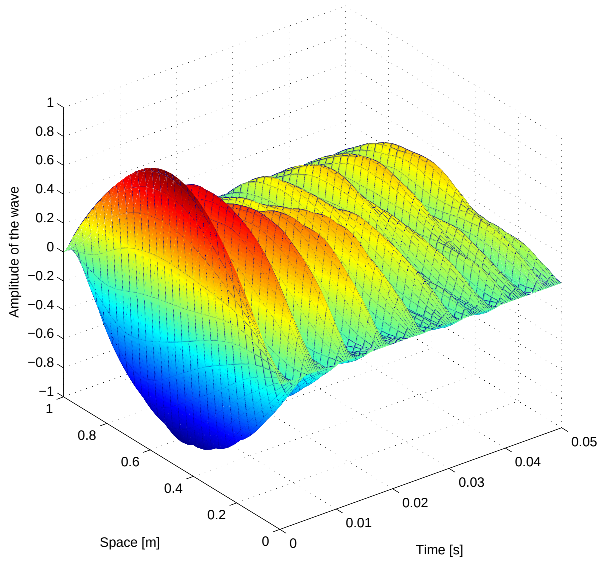

Figure 2: Screen shot after the optimization process showing the shape of the optimized state in a space-time representation.

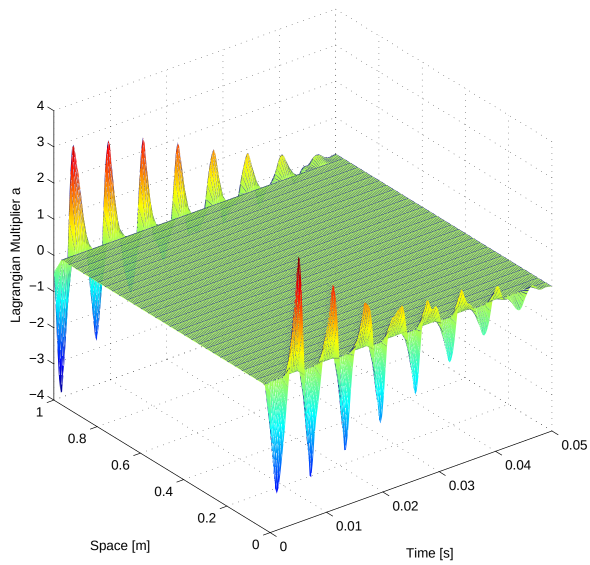

Figure 3: Screen shot after the optimization process showing the shape of an adjoint variable in a space-time representation.

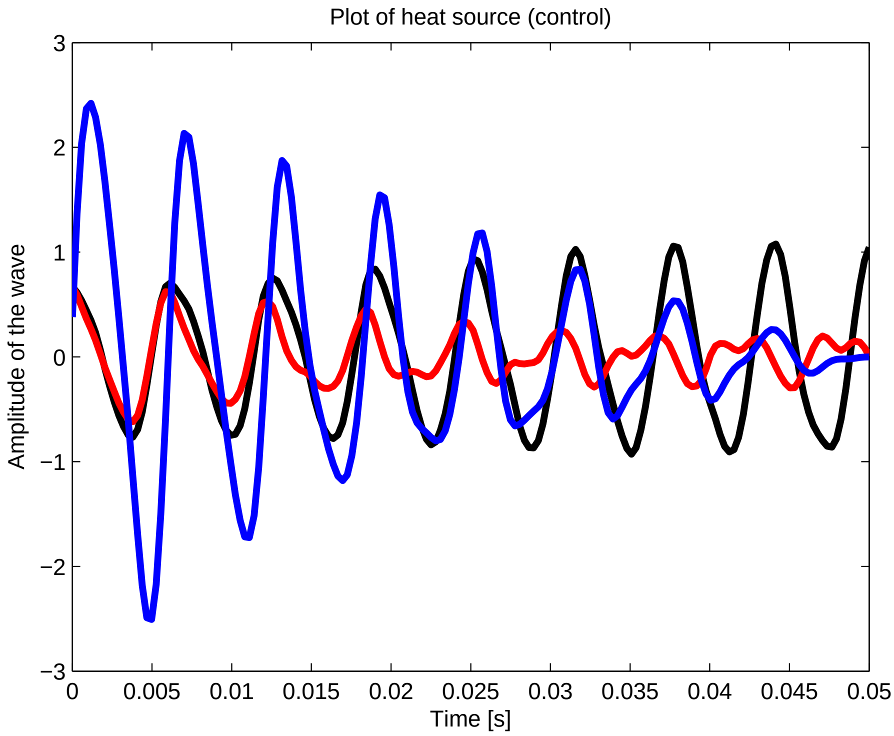

Figure 4: Plot of the state shape before (black) and after (red) the control in $x = L/4$ (center of the heat source) superimposed to the plot of the heat source (blue) in the same location (different scale).

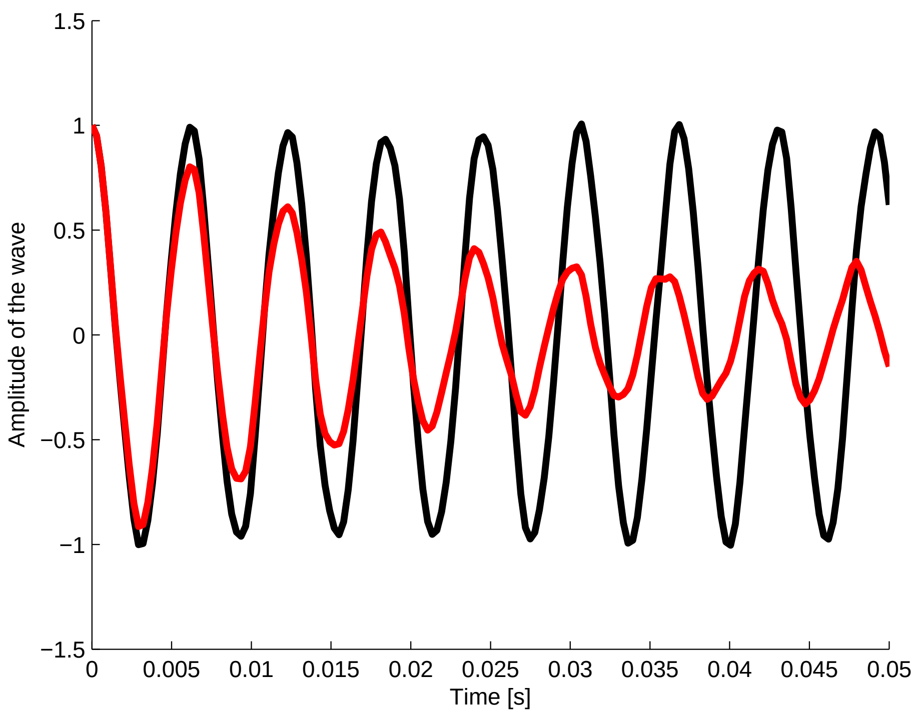

Figure 5: Plot of the state shape before (black) and after (red) the control in $x = L/2$.

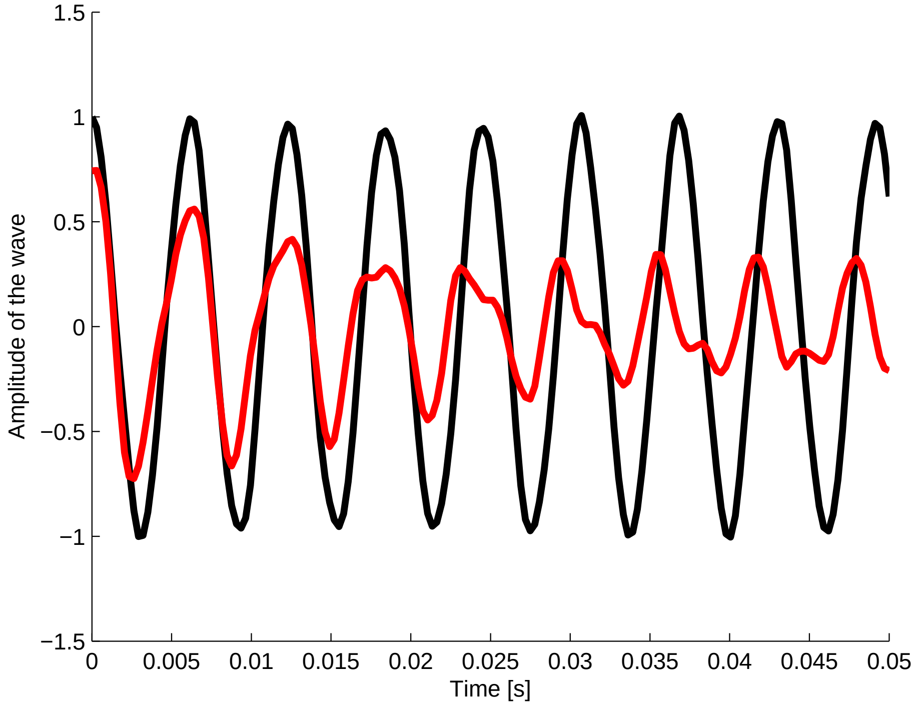

Figure 6: Plot of the state shape before (black) and after (red) the control in $x = 3L/4$.

6 Conclusion

An optimization tool based on a variational approach has been developed and tested for a simple hyperbolic equation. A sensitivity analysis of the state has been performed in order to grab the order of magnitude involved in the problem. Further development could be planned in order to get a more robust derivation of the numerical scheme. We claim this due to the fact that, at present, convergence seems to be too weak and strongly affected by lots of parameters negatively influencing the possible extents of such a tool.

A Listing of the main scripts

Main script

```
%% -------------------------------------------------------------------
% MAIN
% --------------------------------------------------------------------
%% -------------------------------------------------------------------
% GENERAL PARAMETERS
% --------------------------------------------------------------------
loadParameters;

%% -------------------------------------------------------------------
% DEFINITION OF THE INITIAL CONDITIONS ON THE DIRECT SYSTEM
% --------------------------------------------------------------------
defInitialValues;

%% -------------------------------------------------------------------
% DEFINITION OF PARAMETERS OF THE HEAT SOURCE (location, width)
% --------------------------------------------------------------------
```

```
heatSource;

%% ----------------------------------------------------------------------
% DEFINITION OF THE MATRIX A, C, and RELATED ONES....
% -----------------------------------------------------------------------
matrixA;
C=eye(2*M); C(2,2)=0; C(2*M,2*M)=0;
Aplus      = C +dt/2*A ;
Aminus     = C -dt/2*A ;
traspAplus = C'+dt/2*A';
traspAminus= C'-dt/2*A';

%% ----------------------------------------------------------------------
% DEFINITION OF OBJECTIVE FUNCTION PARAMETERS
% -----------------------------------------------------------------------
%             / T                     / T
%      gamma2 |                gamma1 |
% J = --------| q'(t)q(t) dt + -----  | phi'(x,T)[K] phi(x,T)
%        2    |                  2    |
%            / 0                     / 0

K=zeros(2*M,2*M); for m=1:M, K(2*m-1,2*m-1)=1; end
Jactual=10^10;
gamma1=1; gamma2=10^-5*gamma1;

%% ------------------------------------------------------------------------
% MAIN LOOP
%-------------------------------------------------------------------------
iter=1;dJrel=1;

while dJrel>10^-1

    directSystem; adjointSystem;

    Jold=Jactual; Jactual=Jiter; dJrel=abs((Jold-Jactual)/Jactual);
    iter=iter+1;

    plotResults;

    %% --------------------------------------------------------------------
    % ACCURACY OF THE ADJOINT
    % ---------------------------------------------------------------------
    adjointAccuracy(iter)=abs(a_in'*C*phi_out-a_out'*C*phi_in-errorSum)

end

disp(['Number of iterations: ',num2str(iter),'.'])
```

Definition of the main matrix $[\mathbb{A}]$.

```
% ----------------------------------------------------------------------
% DEFINITION OF THE MATRIX A
% ----------------------------------------------------------------------
% p1      dp1      p2       dp2     p3       dp3      p4      p4
% p j-1   dp j-1   p j      dp j    p j+1    dp j+1   p j+1   dp j+1
% p M-3   dp M-3   p M-2    dp M-2  p M-1    dp M-1   p M     dp M
```

```

% initialize the matrix
%A=zeros(2*M,2*M);

% -----------------------------------------------------------------------
% boundary condition in j=1 (second order accurate)
% -----------------------------------------------------------------------
A(1:2,1:6)=[
% def of material derivative
-1.5*u/dx  -1 2*u/dx  0  -0.5*u/dx  0;
% BOUNDARY CONDITION p(1)=0
1          0  0       0   0         0;
];

% -----------------------------------------------------------------------
% body of the matrix (second order accurate)
% -----------------------------------------------------------------------
subA= [
% def of material derivative
-u/2/dx    0         0            -1  u/2/dx    0     ;
% WAVE EQUATION
-c^2/dx^2  -u/2/dx  2*c^2/dx^2  0  -c^2/dx^2  u/2/dx;
];

for j=1:M-2
A(1+2*j:2+2*j,2*j-1:2*j+4)=subA;
end

% -----------------------------------------------------------------------
% boundary condition in j=M (second order accurate)
% -----------------------------------------------------------------------
A(2*M-1:2*M,2*M-5:2*M)=[
% def of material derivative
0.5*u/dx  0   -2*u/dx   0  1.5*u/dx   -1;
% BOUNDARY CONDITION p(M)=0
0         0   0         0  1           0;
```

Definition of the source term matrix $[\mathbb{B}]$.

```
%% --------------------------------------------------------------------
% DEFINITION OF THE MATRIX B
% ---------------------------------------------------------------------
B=zeros(2*M,N);
for n=2:N
   for m=1:M
      if (m-aGrid)*(bGrid-m)>=0
          if (m-aGrid)*(bGrid-m)==0
              B(2*m,n-1)=-1/dt;
              B(2*m,n)=u*(bFlame+aFlame-2*x(m))+1/dt;

          else
              B(2*m,n-1)=-1/dt;
              B(2*m,n)=1/dt;
          end

      end
```

```
    end
end
```

References

[1] Rayleigh, L., 1896. *The Theory of Sound*. McMillan.