Optimal perturbation and stability analysis of a spatial developing flow

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Abstract

Short-term instabilities play an important role in fluid dynamical stability theory, where the most common approach is dominated by the quest for the optimal initial condition that results in the maximum amplification of itself over a finite time span. In the present paper, both the optimal perturbation and the non-modal stability theory is applied to the one-dimensional linearized Ginzburg-Landau model, which describes the evolution of a perturbation in a spatially developing flow.

1 Introduction

The aim of the present paper is dual. First, we look for the optimal perturbation in a spatially developing flow, then the stability of the flow is determined applying tools from both the modal and non-modal stability analysis. The fluid dynamical system in object is described by the Ginzburg-Landau model, which is used to describe a wide variety of phenomena, from phase transition in thermodynamic systems to superconductivity. However, in our case, the Ginzburg-Landau model will be used to describe the wave amplitude in a bifurcating spatially developing flow.

After the declaration of all the quantities of the problem, both the adjoint equation and the optimality system is derived for the Ginzburg-Landau model in Section 2 and numerically discretized along with the direct equation in Section 3. Moreover, some optimal perturbations for different sets of parameters are shown in Section 4. The stability analysis will be discussed in Section 5 and 6 with respectively modal and non-modal theory. Finally, conclusions and future improvements are depicted in Section 7.

The linearized equation for the amplitude of a perturbation about the basic state is governed by the Ginzburg-Landau model:

$$\frac{\partial \phi}{\partial t} = (-U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma)\ \phi \tag{1}$$

where

- $\phi = \phi(x,t)$ is the wave amplitude of the perturbation,
- U is the velocity of the mean flow,
- γ is the diffusion coefficient,
- $\sigma(x)$ is the local bifurcation parameter ($\sigma(x) = \sigma_0 - \sigma_2\frac{x^2}{2}, \sigma_2 \geq 0$),
- $g = g(x)$ is the initial condition.

The above-written equation will be solved in a one-dimensional infinite domain $D = (-\infty, +\infty)$ from time 0 to T, optimizing g by means in order to maximize the following quantity

$$\frac{< \phi(t=T), \phi(t=T) >}{< g, g >}$$

which represents the ratio between some measure related to the energy of the system at final and at the initial time. where $< a, b >= \int_D a \cdot b \; dx$. Initial conditions are $\phi(x, t = 0) = g(x)$, whereas asymptotic boundary conditions are $\phi(x \to \pm\infty, t) \to 0$.

2 Adjoint equations

So far we have defined our *state equation* F as

$$F(\phi, g) = \frac{\partial \phi}{\partial t} + (U \frac{\partial}{\partial x} - \gamma \frac{\partial^2}{\partial x^2} - \sigma) \; \phi$$

in $0 < t < T$ with initial condition $\phi(x, t = 0) = g(x)$ and boundary conditions $\phi(x \to \pm\infty, t) \to 0$, along with the following *cost function* J

$$J = \frac{< g, g >}{< \phi(t = T), \phi(t = T) >} \tag{2}$$

In order to derive the optimal condition with equality contraints with the method of Lagrangian multipliers we have to find the stationary points of the Lagrangian function $\mathcal{L}$ with respect to its variables:

$$\mathcal{L}(\phi, g, a, b, c, d) = J(\phi, g) - \int_0^T < a, F(\phi, g) > dt - < b, \phi(x, t = 0) - g > +$$

$$- \int_0^T c[\phi(x \to +\infty, t) - 0] dt - \int_0^T d[\phi(x \to -\infty, t) - 0] dt$$

Whereas derivation with respect to a, b, c and d leads to the state equation, initial and boundary conditions

$$\frac{\partial \mathcal{L}}{\partial a} = 0 \Rightarrow \frac{\partial \phi}{\partial t} + (U \frac{\partial}{\partial x} - \gamma \frac{\partial^2}{\partial x^2} - \sigma) \; \phi = 0 \; ;$$

$$\frac{\partial \mathcal{L}}{\partial b} = 0 \Rightarrow \phi(x, t = 0) = 0 \; ;$$

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow \phi(x \to +\infty, t) \to 0 \quad ; \quad \frac{\partial \mathcal{L}}{\partial d} = 0 \Rightarrow \phi(x \to -\infty, t) \to 0 \; .$$

derivatives of $\mathcal{L}$ with respect to ϕ and g give the *adjoint equation*

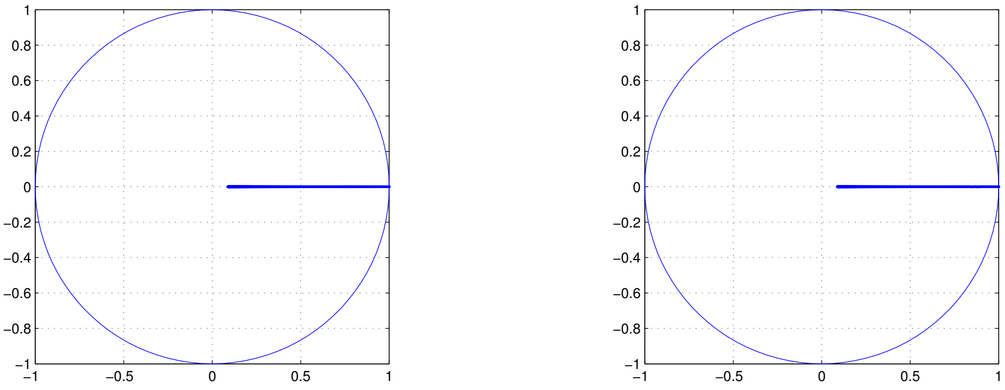
$$-\frac{\partial a}{\partial t} = (U \frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma) a$$

with its initial and boundary conditions

$$a(x, t = T) = \frac{2(\int_{-\infty}^{+\infty} g(\tilde{x}) g(\tilde{x}) \; d\tilde{x}) \; \phi_T}{< \phi(\tilde{x}, t = T), \phi(\tilde{x}, t = T) >^2} \quad ; \quad a(x = \pm\infty, t) = 0 \tag{3}$$

along with the following *optimality conditions* (for the full derivation see Appendix A):

$$g(x) = -\frac{a(x, t = 0)}{2} \int_{-\infty}^{+\infty} \phi(\tilde{x}, t = T) \; \phi(\tilde{x}, t = T) \; d\tilde{x} \tag{4}$$



(a)

(b)

Figure 1: Eigenvalues of implicit scheme used for the integration of direct (a) and adjoint (b) equations compared to the unity circle.

3 Numerical Scheme

A Matlab script has been written in order to solve numerically the optimization problem. The main steps are here briefly outlined:

- forward integration of the state equation;
- evaluation of the cost function;
- backward integration of the adjoint equation;
- assessment of a new control function via the optimality equation;

These steps have been embedded inside a loop stopping when the absolute difference between two consecutive values of J is lower than an imposed accuracy.

Both integrations of state and adjoint equations are performed using an implicit backward Euler finite difference scheme:

- State equation

$$\frac{\phi_i^{n+1}-\phi_i^n}{\Delta t} = -U\frac{\phi_{i+1}^{n+1}-\phi_{i-1}^{n+1}}{2\Delta x} + \gamma\frac{\phi_{i+1}^{n+1}-2\phi_i^{n+1}+\phi_{i-1}^{n+1}}{\Delta x^2} + \sigma_i\phi_i^{n+1} \Rightarrow$$

$$\phi_{i-1}^{n+1}[-\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] + \phi_i^{n+1}[1+\frac{2\gamma\Delta t}{\Delta x^2} - \sigma_i\Delta t] + \phi_{i+1}^{n+1}[\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] = \phi_i^n$$

- Adjoint equation

$$-\frac{a_i^n-a_i^{n-1}}{\Delta t} = U\frac{a_{i+1}^{n-1}-a_{i-1}^{n-1}}{2\Delta x} + \gamma\frac{a_{i+1}^{n-1}-2a_i^{n-1}+a_{i-1}^{n-1}}{\Delta x^2} + \sigma_i a_i^{n-1} \Rightarrow$$

$$a_{i-1}^{n-1}[\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] + a_i^{n-1}[1+\frac{2\gamma\Delta t}{\Delta x^2} - \sigma_i\Delta t] + a_{i+1}^{n-1}[-\frac{U\Delta t}{2\Delta x} - \frac{\gamma\Delta t}{\Delta x^2}] = a_i^n$$

Both methods have proved to be stable when investigated with the *absolute stability* condition due to the implicit method used (see Figure 4).

The accuracy of the adjoint has been checked using the *adjoint equality*

$$< a, L\phi > = < \phi, L^\dagger a > + B.T. \tag{5}$$

which in our case gives

$$\int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\ \phi]dtdx =$$

$$= \int_0^T \int_{-\infty}^{+\infty} \phi[\frac{\partial a}{\partial t} + (U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma)a]dtdx + [a\ \phi]_0^T$$

but since both state and adjoint equation does not have any source term,

$$[a\ \phi]_0^T = 0 \Rightarrow a(0)\phi(0) = a(T)\phi(T),$$

which in all our simulation has been less than 10^{-10}, next to the machine precision.

4 Optimal perturbation

The particular choice of the control as the initial condition and the cost function as the ratio between quantities proportional to the energy of initial and final perturbation defines g as the *optimal perturbation*.

$$J = \frac{< g, g >}{< \phi(t = T), \phi(t = T) >}$$

To prove the effectiviness of the code we present different optimal perturbations g at different values of the Ginzburg-Landau parameters (U, γ and σ_0/σ_2), trying to give them an interpretation form the physical point of view (see Figure 2). In all our simulation, we obtained different values of J_{min}, which are summarized in Table 1:

different U By increasing U, the optimal perturbation tends to move slighty backward. This result is because of our approximation of the infinite domain with a finite grid, since the boundary conditions are $\phi = 0$.

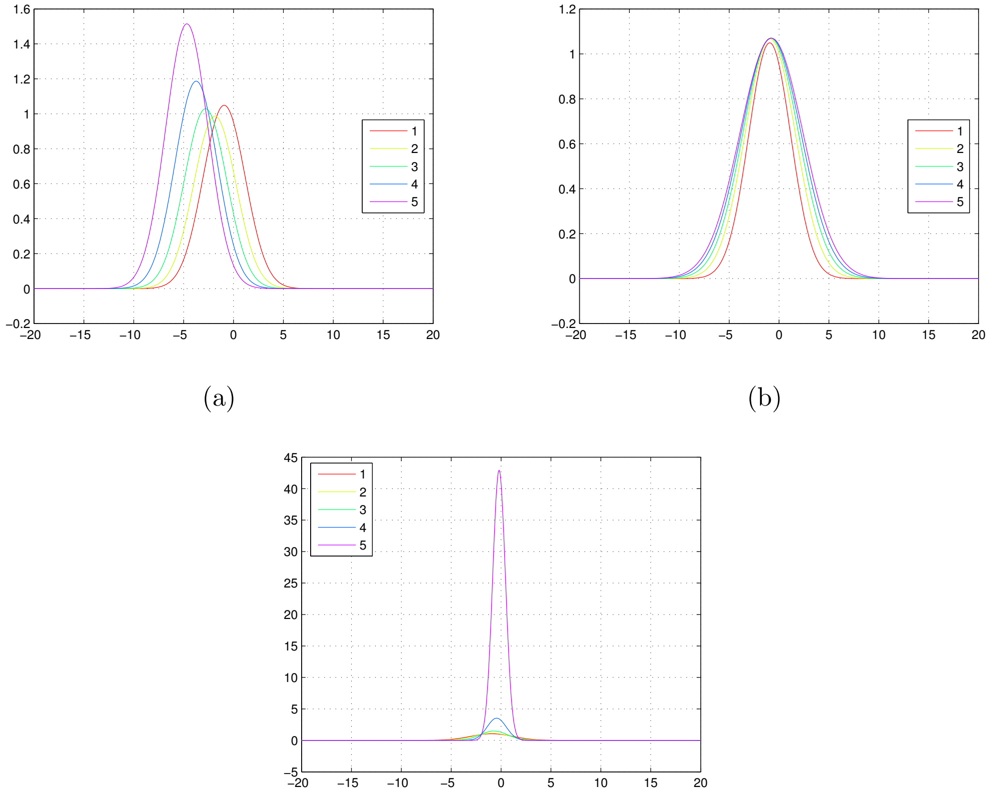
different γ As the diffusion parameter γ grows we notice that the peak of the optimal perturbation increases and the stiffness decreases, in order to minimize the diffusive effects.

different $\boldsymbol{\sigma_0/\sigma_2}$ Since σ_0 is constant positive and the bifurcation function $\sigma(x)$ is given as $\sigma_0 - (\sigma_2/2)x^2$, the increment of this ratio means a larger portion of domain in which $\sigma > 0$, i.e. where the solution exponentially grows. So, as the ratio increases the optimal perturbation does not have to be as energetic as the previous ones.

Moreover, we presents the evolution of the optimal perturbation with three different sets of parameters, whose discussion will be clarified in the section about non-modal stability analysis (Figure 3).

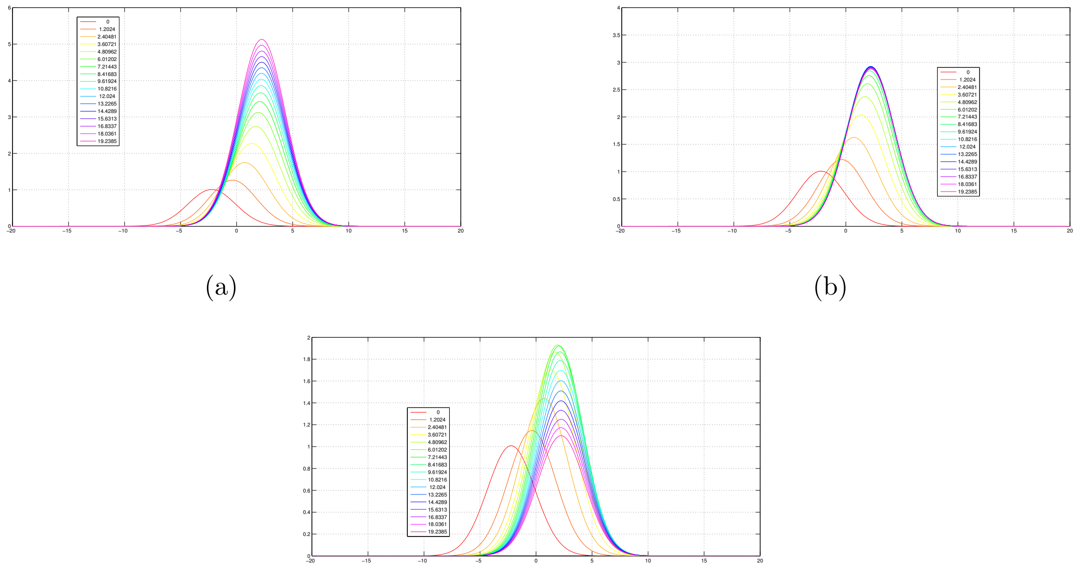
parameters	1	2	3	4	5
U varying, $\gamma = 1$, $\sigma_0 = 0.48$, $\sigma_0 = 0.1$	0.3817	0.4602	0.6274	0.9671	1.6850
$U = 1$, γ varying, $\sigma_0 = 0.48$, $\sigma_2 = 0.1$	0.3817	0.5504	0.7286	0.9229	1.1364
$U = 1$, $\gamma = 1$, $\sigma_0 = 0.48$, σ_2 varying	0.3817	0.5835	1.7186	20.1317	$5.3399{\cdot}10^3$

Table 1: Values of J_{min} obtained with different parameters configurations.



(a) (b) (c)

Figure 2: Optimal perturbation for systems with different values of parameters U (a), γ (b) equations compared to the unity circle and σ (c).



(a) (b) (c)

Figure 3: Different evolution of the initial perturbation in a (a) unstable ($\sigma_0 = 0.5$), (b) neutral ($\sigma_0 = 0.47$) and (c) stable ($\sigma_0 = 0.42$) sets of parameters. Note in the last figure the transient growth before the decaying of the perturbation.

5 Linear stability analysis

In order to investigate the behaviour of the solution with tools from linear stability analysis we perform the *normal mode decomposition*, substituting the solution written as

$$\phi(x,t) = \hat{\phi}(x)\ e^{\lambda t}$$

into the equation

$$\frac{\partial \phi}{\partial t} = \mathcal{A}\phi \quad (\mathcal{A} = -U\frac{\partial}{\partial x} + \gamma\frac{\partial^2}{\partial x^2} + \sigma).$$

This transforms the linear initial-value problem into a corresponding eigenvalue problem

$$\lambda\hat{\phi}(x) = \mathcal{A}\hat{\phi}(x)$$

$$(\mathcal{A} - \lambda\mathcal{I})\hat{\phi}(x) = 0$$

where $\mathcal{I}$ is the identity operator. If $\mathcal{A}$ has at least one eigenvalue $\lambda_i > 0$ the perturbation ϕ grows exponentially with time, meanwhile it will decay exponentially if all λs are minor than 0.

In the following we evaluate the behaviour of the most unstable eigenvalue of $\mathcal{A}$ with the parameter σ_0 (see Figure 4).

6 Transient Growth

As stated in [4], linear stability theory is concerned with a quantitative description of flow behavior involving infinitesimal disturbances superimposed on a base flow. However, for our case as for most wall-bounded shear flows the spectrum is a poor proxy for the disturbance behavior as it only describes the asymptotic ($t \to \infty$) fate of the perturbation and fails to capture short-term characteristics. To accurately describe the disturbance behavior for all times, it appears necessary to introduce a finite-time horizon over which an instability is observed.

As we are investigating the temporal evolution on an initial perturbation $g(t)$, we define the gain $G(t)$ as the ratio between some measure related to the energy of the current and initial perturbation

$$G(t) = \max_{g_0} \frac{\|\phi(x,t)\ \phi(x,t)\|}{\|g_0(x)\ g_0(x)\|} \tag{6}$$

but since the evolution of the system is described by

$$\phi(x,t) = g_0(x)exp(\mathcal{A}t)$$

equation (6) becomes

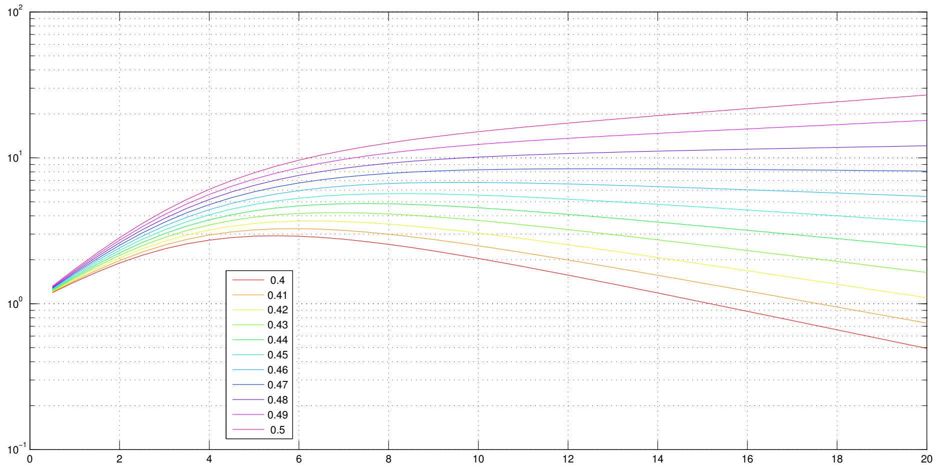
$$G(t) = \max_{g_0} \frac{\left\|g_0^2(x)\ exp(2\mathcal{A}t)\right\|}{\left\|g_0^2(x)\right\|} = \left\|\mathcal{S}exp(2\mathbf{\Lambda}t)\mathcal{S}^{-1}\right\|$$

It should become obvious that no information about the eigenvectors of $\mathcal{A}$, contained in $\mathcal{S}$, is considered when only the least stable mode is taken as a representation of the operator exponential.

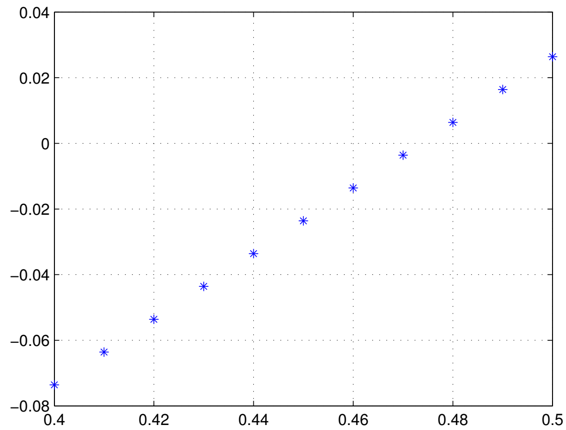
From the stability theory we know that the minimum growth-rate of the solution coincides at least with the most unstable eigenvalue, and because of the triangular disequality we can say that

$$exp(2\lambda_{max}t) \le G(t) \le \|\mathcal{S}\|\left\|\mathcal{S}^{-1}\right\| exp(2\lambda_{max}t)$$

The quantity $\|\mathcal{S}\|\left\|\mathcal{S}^{-1}\right\|$ represents the condition number of $\mathcal{S}$ ($k(\mathcal{S})$), a measure of the non-orthogonality of its columns. So if $k(\mathcal{S}) > 1$ (as in our case) the operator $\mathcal{A}$ is said to



(a)



(b)

Figure 4: (a) Semilogarithmic plot of the gain function $G(t)$ from which we can see the transient growth of the solution and (b) the most unstable eigenvalue of $\mathcal{A}$ for different values of σ_0 ($U = 1, \gamma = 1, \sigma_2 = 0.1$). Note that the long-term behaviour of the solution turns from stable to unstable as the most unstable eigenvalue of $\mathcal{A}$ becomes positive.

be non-normal, and systems governed by non-normal matrices can exhibit a large transient amplification of energy contained in the initial condition.

In our case we evaluated the evolution of the energy related to the perturbation for different values of σ_0, in that we can check the results from the accordance between modal and non-modal analysis. In Figure 4(a) we can see that, whereas for great times ($t \gtrsim 10$) the system undergoes a classical exponential behaviour ruled by the most unstable eigenvalue of the spatial operator $\mathcal{A}$, at lower times the system exhibits a *transient growth* explained by the non-hortogonality of the eigenvectors of $\mathcal{A}$. Results from numerical simultions agree qualitatively with the one shown in [2], given that ours refer to the energy of the perturbation and not to the perturbation itself.

7 Conclusions

In the present paper we have investigated the stability of an initial pertubation in a spatially developed flow described by the Ginzburg-Landau equation, with tools from both classic modal analysis and from recently-developed non-modal analysis.

Numerical simulations have shown that, whereas the long time behaviour is well-catched by the modal analysis, the solution exhibits a so-called transient growth on a finite-time horizon, explained by the non-normality of the spatial operator.

The results presented here are further borne out in [2], were it is stated that at the increase of σ_2 (i.e. when the flow is strongly non-parallel) the operator $\mathcal{A}$ becomes more and more non-normal.

A Derivation of adjoint equation and optimality condition

$$\mathcal{L}(\phi, g, a, b, c, d) = J(\phi, g) - \int_0^T < a, F(\phi) > dt - < b, \phi(x, t = 0) - g > +$$

$$- \int_0^T c[\phi(x \to +\infty, t) - 0]dt - \int_0^T d[\phi(x \to -\infty, t) - 0]dt =$$

$$= \frac{\int_{-\infty}^{+\infty} [g(x)\ g(x)]dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T]dx} - \int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \phi}{\partial t} + (U \frac{\partial}{\partial x} - \gamma \frac{\partial^2}{\partial x^2} - \sigma)\ \phi]dxdt +$$

$$- \int_{-\infty}^{+\infty} b[\phi(x, t = 0) - g]dx - \int_0^T c(\phi(x \to +\infty, t) - 0dt - \int_0^T d(\phi(x \to -\infty, t) - 0)dt$$

A.1 Derivation of $\mathcal{L}$ with respect to g

In the following we will use the notation $\phi_{\tilde{t}} = \phi(x, t = \tilde{t})$.

$$\frac{\partial \mathcal{L}}{\partial g} \delta g = \frac{\partial J}{\partial g} \delta g + \int_{-\infty}^{+\infty} b(x)\ \delta g\ dx = 0$$

but since

$$\frac{\partial J}{\partial g} \delta g = \lim_{\varepsilon \to 0} \frac{J(\phi, g + \varepsilon \delta g, a, b, c, d) - J(\phi, g, a, b, c, d)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{+\infty} (g + \varepsilon \delta g)\ (g + \varepsilon \delta g) dx - \int_{-\infty}^{+\infty} (g\ g) dx}{\varepsilon \int_{-\infty}^{+\infty} [\phi_T\ \phi_T] dx} =$$

$$= \lim_{\varepsilon \to 0} \frac{\int_{-\infty}^{+\infty} 2g\varepsilon \delta g dx}{\varepsilon \int_{-\infty}^{+\infty} [\phi_T\ \phi_T] dx} = \frac{\int_{-\infty}^{+\infty} 2g \delta g dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T] dx}$$

so

$$\frac{\int_{-\infty}^{+\infty} 2g \delta g dx}{\int_{-\infty}^{+\infty} [\phi_T\ \phi_T] dx} + \int_{-\infty}^{+\infty} b(x)\ \delta g\ dx = 0$$

$$\int_{-\infty}^{+\infty} 2g \delta g dx + \int_{-\infty}^{+\infty} [\int_{-\infty}^{+\infty} \phi_T\ \phi_T d\tilde{x}]\ b(x)\ \delta g\ dx = 0$$

$$\int_{-\infty}^{+\infty} \delta g[2g + \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}\ b(x)]dx = 0$$

Since the previous integral has to be zero $\forall g$,

$$2g + \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}\ b(x) = 0$$

$$g(x) = -\frac{b(x)}{2} \int_{-\infty}^{+\infty} \phi_T\ \phi_T\ d\tilde{x}$$

A.2 Derivation of $\mathcal{L}$ with respect to ϕ

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = \frac{\partial J}{\partial \phi}\delta\phi - \int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt +$$

$$- \int_{-\infty}^{+\infty} b\delta\phi_0 dx - \int_0^T c\delta\phi_{-\infty} dt - \int_0^T d\delta\phi_{-\infty} dt = 0$$

but since

$$\frac{\partial J}{\partial \phi} = \frac{\partial}{\partial \phi}(\frac{p(\phi)}{q(\phi)}) = \frac{p'q - pq'}{q^2}$$

where $p(\phi) =< g, g >$ and $q(\phi) =< \phi_T, \phi_T >$, so

$$\frac{\partial J}{\partial \phi}\delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2}$$

Since boundary conditions impose $\phi \to 0$ as $x \to \pm\infty$, also $\delta\phi_{\pm\infty} \to 0$, so

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2} +$$

$$- \int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt - \int_{-\infty}^{+\infty} b\delta\phi(x, t = 0)dx = 0$$

Now we have to develop the second integral of this relation:

$$\int_0^T \int_{-\infty}^{+\infty} a[\frac{\partial \delta\phi}{\partial t} + (U\frac{\partial}{\partial x} - \gamma\frac{\partial^2}{\partial x^2} - \sigma)\delta\phi]dxdt =$$

$$= \int_0^T \int_{-\infty}^{+\infty} a\frac{\partial \delta\phi}{\partial t}dxdt + \int_0^T \int_{-\infty}^{+\infty} aU\frac{\partial \delta\phi}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\gamma\frac{\partial^2 \delta\phi}{\partial x^2}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(a\delta\phi)}{\partial t}dxdt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(aU\delta\phi)}{\partial x}dxdt +$$

$$- \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}[a\gamma\frac{\partial \delta\phi}{\partial x}]dxdt + \int_0^T \int_{-\infty}^{+\infty} \frac{\partial(a\gamma)}{\partial x}\frac{\partial \delta\phi}{\partial x}dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T [aU\delta\phi]_{-\infty}^{+\infty} dt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T [a\gamma\frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt +$$

$$+ \int_0^T \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}(\frac{\partial(a\gamma)}{\partial x}\delta\phi)dxdt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2(a\gamma)}{\partial x^2}\delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial a}{\partial t}dxdt + \int_0^T [aU\delta\phi]_{-\infty}^{+\infty} dt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi\frac{\partial(aU)}{\partial x}dxdt - \int_0^T [a\gamma\frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt +$$

$$+ \int_0^T [\frac{\partial(a\gamma)}{\partial x}\delta\phi]_{-\infty}^{+\infty} dt - \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2(a\gamma)}{\partial x^2}\delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

but since $\delta\phi$ goes to zero for $x \to \pm\infty$

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi \frac{\partial a}{\partial t} dxdt - \int_0^T \int_{-\infty}^{+\infty} \delta\phi \frac{\partial (aU)}{\partial x} dxdt - \int_0^T [a\gamma \frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt +$$

$$- \int_0^T \int_{-\infty}^{+\infty} \frac{\partial^2 (a\gamma)}{\partial x^2} \delta\phi dxdt - \int_0^T \int_{-\infty}^{+\infty} a\sigma\delta\phi dxdt =$$

Rearranging members leads to

$$= \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_0^T \int_{-\infty}^{+\infty} \delta\phi [\frac{\partial a}{\partial t} + (U\frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma)]a \; dxdt - \int_0^T [a\gamma \frac{\partial \delta\phi}{\partial x}]_{-\infty}^{+\infty} dt$$

Given that this holds $\forall \; \delta\phi$, the previous equation gives the following *adjoint equation*

$$-\frac{\partial a}{\partial t} = (U\frac{\partial}{\partial x} + \gamma \frac{\partial^2}{\partial x^2} + \sigma)a$$

with the corresponding *boundary conditions* $a = 0$ for $x \to \pm\infty$. Inserting the previous one in the starting equation will lead to the initial condition for the adjoint equation:

$$\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi = -\frac{2 < g, g >< \phi_T, \delta\phi_T >}{< \phi_T, \phi_T >^2} - \int_{-\infty}^{+\infty} [a\delta\phi]_0^T dx - \int_{-\infty}^{+\infty} b\delta\phi_0 dx = 0$$

$$-2 \int_{-\infty}^{+\infty} < g, g > \phi_T \delta\phi_T \; dx - \int_{-\infty}^{+\infty} a_T < \phi_T, \phi_T >^2 \delta\phi_T \; dx +$$

$$+ \int_{-\infty}^{+\infty} a_0 < \phi_T, \phi_T >^2 \delta\phi_0 \; dx - \int_{-\infty}^{+\infty} b < \phi_T, \phi_T >^2 \delta\phi_0 dx = 0$$

Arranging in terms of $\delta\phi_T$ and $\delta\phi_0$,

$$\int_{-\infty}^{+\infty} \delta\phi_T [-2 < g, g > \phi_T - a_T < \phi_T, \phi_T >^2] \; dx +$$

$$+ \int_{-\infty}^{+\infty} \delta\phi_0 [a_0 < \phi_T, \phi_T >^2 - b < \phi_T, \phi_T >^2] \; dx = 0$$

And so, given that this holds $\forall \; \delta\phi_T$ and $\forall \; \delta\phi_0$

$$a_T = \frac{-2 < g, g > \phi_T}{< \phi_T, \phi_T >^2}, \quad b = a_0$$

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