Reduction of the instability in an aeroelasticity problem using an optimization method

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SUMMARY

The aim of this work is to apply an optimization process, based on the Lagrange multipliers method, to reduce the oscillations and the instabilities of a wing due to aeroelasticity effects. The problem is governed by a linear dynamical system; an *objective* (or *cost*) function is defined, taking in account both the state of the system and the control term. The goal is to find the optimal control law in order to minimize the objective function in a specified time range, where the final time is a parameter of the problem.

Keywords: Aeroelasticity; Optimization; Adjoint method.

INTRODUCTION

The aeroelasticity problem of this work consist of a wing, with a fixed wing root and a free wing tip, provided with an aileron. The latter represents the control term of the problem. The aeroelastic model of the wing is not here reported, detailed aspects of the physics of this problem can be found in [1] and [2], here only a mention is done. In Fig. 5 (see pag. 8) is represented the model of the wing section; one can see the wing has two d.o.f., a vertical displacement $w(y)$ and a rotation $\theta(y)$ around the c.g. of the airfoil, where *y* is the coordinate in the spanwise direction. There are also stiffness and damping factors. The equation governing this system can be written as

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}\,\underline{x}(t) + \underline{B}\delta(t)$$

where $\underline{x}(t) = \left[w(t); \theta(t); \dot{w}(t); \dot{\theta}(t)\right]$ and A, B depend on the parameters of the problem. The time domain is $t \in [0, T]$ and the i.c. of the equation is $\underline{x}(0) = \underline{x}_0$. The objective function is defined as

$$J = \frac{1}{2}\gamma_1 \int_0^T \underline{x}^T(t)\underline{x}(t)\,dt + \frac{1}{2}\gamma_2 \int_0^T [\delta(t)]^2\,dt + \frac{1}{2}\gamma_3 \underline{x}^T(T)\underline{x}(T)$$

The first term of the function is associated to the evolution of the system, the second is associated to the energy of the control and the last term is associated to the state of the system at the time instant *T*. Each term is properly weighted with three coefficients γ_1, γ_2 and γ_3. Varying properly one of these weights more or less importance can be associated to each term.

DERIVATION OF THE OPTIMALITY SYSTEM

The Lagrangian function is defined as:

$$\mathcal{L}\big(\underline{x}(t), \delta(t), \underline{a}(t), \underline{b}\big) = J\left(\underline{x}(t), \delta(t)\right) - \int_0^T \underline{a}^T(t)\, \underline{F}\left(\underline{x}(t), \delta(t)\right) dt - \underline{b}^T\left[\underline{x}(0) - \underline{x}_0\right]$$

where the vectors *a* and *b* are the Lagrange multipliers, *J* is the cost function and

$$\underline{F}\left(\underline{x}(t), \delta(t)\right) = \frac{d\underline{x}(t)}{dt} - \underline{\underline{A}}\, \underline{x}(t) - \underline{B}\delta(t) = \underline{0}\,, \quad t \in [0, T]$$

that is another way to write the state equation.

Now the optimality conditions are obtained by setting the first variation of the Lagrangian function equal to zero, that means:

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial x}\delta x + \frac{\partial \mathcal{L}}{\partial \delta}\delta\delta + \frac{\partial \mathcal{L}}{\partial a}\delta a + \frac{\partial \mathcal{L}}{\partial b}\delta b = 0$$

hence $\nabla\mathcal{L} = \underline{0}$.

Performing all the derivatives one obtains:

- $\frac{\partial \mathcal{L}}{\partial b} = 0$ $$\delta\underline{b}^T\left[\underline{x}(0) - \underline{x}_0\right] = \underline{0} \Longrightarrow \boxed{\underline{x}(0) = \underline{x}_0} \quad \forall\, \delta\underline{b}$$

- $\frac{\partial \mathcal{L}}{\partial \delta} = 0$

$$\gamma_2 \int_0^T \delta(t)\delta\delta(t)\, dt + \int_0^T \underline{a}^T(t)\underline{B}\delta\delta(t)\, dt = 0$$

$$\int_0^T \left[\gamma_2\delta(t) + \underline{a}^T(t)\underline{B}\right]\delta\delta(t)\, dt = 0 \Longrightarrow \boxed{\delta(t) = -\frac{\underline{a}^T(t)\underline{B}}{\gamma_2}} \quad \forall\, \delta\delta(t),\ \ t \in [0, T]$$

- $\frac{\partial \mathcal{L}}{\partial a} = 0$

$$\int_0^T \delta\underline{a}^T(t)\underline{F}\left(\underline{x}(t), \delta(t)\right) dt = 0 \Longrightarrow$$

$$\Longrightarrow \boxed{\underline{F}\left(\underline{x}(t), \delta(t)\right) = \frac{d\underline{x}(t)}{dt} - \underline{\underline{A}}\, \underline{x}(t) - \underline{B}\delta(t) = \underline{0}} \quad \forall\, \delta\underline{a}(t),\ \ t \in [0, T]$$

- $\frac{\partial \mathcal{L}}{\partial x} = 0$

Using the integration by parts leads to the following expression:

$$\int_0^T \left[\frac{d\underline{a}(t)}{dt} + \underline{\underline{A}}^T \underline{a}(t) + \gamma_1 \underline{x}(t)\right]^T \delta\underline{x}(t)\, dt + \left[\gamma_3\underline{x}(T) - \underline{a}(T)\right]^T \delta\underline{x}(T) + \left[\underline{a}(0) - \underline{b}\right]^T \delta\underline{x}(0) = \underline{0}$$

Hence the adjoint system is:

$$\boxed{\begin{array}{ll} \dfrac{d\underline{a}(t)}{dt} + \underline{\underline{A}}^T \underline{a}(t) + \gamma_1 \underline{x}(t) = 0 & \forall\, \delta\underline{x}(t),\ \ t \in [0, T] \\ \underline{a}(T) = \gamma_3 \underline{x}(T) & \forall\, \delta\underline{x}(T) \\ \underline{b} = \underline{a}(0) & \forall\, \delta\underline{x}(0) \end{array}}$$

DISCRETE OPTIMALITY SYSTEM

In order to implement an algorithm to solve the problem numerically the equations have to be discretized in the time domain. Regarding the state equation a *Backward Euler* method is used.

$$\frac{\underline{x}^{i+1} - \underline{x}^i}{\Delta t} = \underline{\underline{A}}\,\underline{x}^{i+1} + \underline{B}\delta^i \Longrightarrow \underline{x}^{i+1} = \underline{\underline{L}}\left(\underline{x}^i + \underline{B}\delta^i \Delta t\right) \quad \text{for } i = 0,1,\dots,N-1$$

where $\underline{\underline{L}} = \left(\underline{\underline{I}} - \underline{\underline{A}}\Delta t\right)^{-1}$ and $N = T/\Delta t$. The initial condition is $\underline{x}^0 = \underline{x}_0$.

Using the discrete adjoint approach and starting from the adjoint identity one gets

$$\left(\underline{a}^{i+1}\right)^T \left(\underline{\underline{L}}\,\underline{x}^i\right) = \left(\underline{\underline{L}}^T \underline{a}^{i+1}\right)^T \underline{x}^i \quad \text{for } i = 0,1,\dots,N-1$$

By imposing

$$\underline{\underline{L}}^T \underline{a}^{i+1} = \left(\underline{a}^i - \tfrac{1}{2}\gamma_1 \underline{x}^i \Delta t\right) \quad \text{for } i = 0,1,\dots,N-1$$

and substituting in the previous identity one gets

$$\left(\underline{a}^{i+1}\right)^T \left(\underline{x}^{i+1} - \underline{\underline{L}}\,\underline{B}\delta^i \Delta t\right) = \left(\underline{a}^i - \frac{1}{2}\gamma_1 \underline{x}^i \Delta t\right)^T \underline{x}^i$$

$$\left(\underline{a}^{i+1}\right)^T \underline{x}^{i+1} - \left(\underline{a}^i\right)^T \underline{x}^i - \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\,\underline{B}\delta^i \Delta t + \frac{1}{2}\gamma_1 \left(\underline{x}^i\right)^T \underline{x}^i \Delta t = 0$$

Since the last relation has to be valid for any $i = 0,1,\dots,N-1$ one obtains an identity for evaluate the accuracy of the adjoint. Using the discrete approach the error has to be equal to the machine precision.

$$error = \left| \left(\underline{a}^N\right)^T \underline{x}^N - \left(\underline{a}^0\right)^T \underline{x}^0 - \sum_{i=0}^{N-1} \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\,\underline{B}\delta^i \Delta t + \sum_{i=0}^{N-1} \frac{1}{2}\gamma_1 \left(\underline{x}^i\right)^T \underline{x}^i \Delta t \right|$$

The optimality condition is found using both the adjoint identity and the definition of the objective function; the latter has to be linearized before proceeding since is not linear with respect to x and δ.

$$\delta J = \sum_{i=0}^{N-1} \gamma_1 \left(\underline{x}^i\right)^T \delta\underline{x}^i \Delta t + \gamma_2 \sum_{i=0}^{N-1} \delta^i \cdot \delta\delta^i \Delta t + \gamma_3 \left(\underline{x}^N\right)^T \delta\underline{x}^N = 0$$

$$\left(\underline{a}^N\right)^T \delta \underline{x}^N - \left(\underline{a}^0\right)^T \delta \underline{x}^0 - \sum_{i=0}^{N-1} \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\, \underline{B} \delta \delta^i \Delta t + \sum_{i=0}^{N-1} \gamma_1 \left(\underline{x}^i\right)^T \Delta t \delta \underline{x}^i = 0$$

Since the initial condition is fixed then $\delta \underline{x}^0 = 0$.

$$\sum_{i=0}^{N-1} \left[\gamma_2 \delta^i + \left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\, \underline{B}\right] \delta \delta^i \Delta t + \left[\gamma_3 \underline{x}^N - \underline{a}^N\right]^T \delta \underline{x}^N = 0$$

The optimality condition on the control is

$$\delta^i = -\frac{\left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\, \underline{B}}{\gamma_2} \quad \text{for } \; i = 0,1,\dots,N-1$$

Furthermore the terminal condition on the adjoint vector is

$$\underline{a}^N = \gamma_3 \underline{x}^N$$

Finally the discrete optimality system is

$$\begin{cases} \underline{x}^{i+1} = \underline{\underline{L}}\left(\underline{x}^i + \underline{B} \delta^i \Delta t\right) \\ \underline{x}^0 = \underline{x}_0 \\ \underline{a}^i = \underline{\underline{L}}^T \underline{a}^{i+1} + \dfrac{1}{2} \gamma_1 \underline{x}^i \Delta t \\ \underline{a}^N = \gamma_3 \underline{x}^N \\ \delta^i = -\dfrac{\left(\underline{a}^{i+1}\right)^T \underline{\underline{L}}\, \underline{B}}{\gamma_2} \end{cases}$$

The resolution scheme is the following:

- Resolution of the state equation from *t=0* to *t=T*, given the i.c. $\underline{x}^0 = \underline{x}_0$
- Resolution of the adjoint equation proceeding backward in time, from t=T to t=0, given the terminal condition $\underline{a}^N = \gamma_3 \underline{x}^N$
- Check of the accuracy and evaluation of the control δ^i using the optimality condition.

This scheme is repeated until convergence is reached, starting with an arbitrary control δ^i, e.g. constant in time or null.

RESULTS

The previous scheme has been implemented in an algorithm using the Matlab environment. Before proceeding it is useful to recall the state equation, i.e. the dynamical system governing the problem, to understand the mathematical aspects and define the problems we want to optimize.

$$\frac{d\underline{x}(t)}{dt} = \underline{\underline{A}}\, \underline{x}(t) + \underline{B} \delta(t)\,, \quad t \in [0, T]$$

Let us consider the related homogeneous equation, i.e. without the forcing term. As one can see it is a linear, autonomous dynamical system, because the matrix A is constant in time. The point $\underline{x} = \underline{0}$ clearly is a stationary solution of the system. The stability of this point, as known, can be evaluated by inspecting the eigenvalues of the coefficients matrix A; if for all eigenvalues $\lambda_i \in \mathbb{C}$, $Re(\lambda_i) < 0 \ (1 \leq i \leq dim(A))$ the point is said to be *stable*, otherwise, if at least one eigenvalue has positive real part, the point is said to be *unstable*.

The goal now is to find the control law in order to minimize the objective function *J*, starting from two different initial conditions, both in the neighborhood of the stationary point $\underline{x} = \underline{0}$. The first is to study and control the evolution of the system starting with the wing slightly bended, without twist, the second with the wing slightly twisted, without bending. For each case two further conditions are considered; since the matrix A (and so the eigenvalues) depends on the problem properties, varying wind speed the *stable* and the *unstable* conditions can be made. For the first case a freestream velocity $U = 12{,}5$ is used, in Fig.1 (a) is shown that all the eigenvalues of the coefficients matrix are in the negative real side of the complex plane. In Fig. 1 (b), with a freestream velocity $U = 12{,}7$, an eigenvalue has positive real part, that causes exponential growth and therefore the instability of the stationary solution.

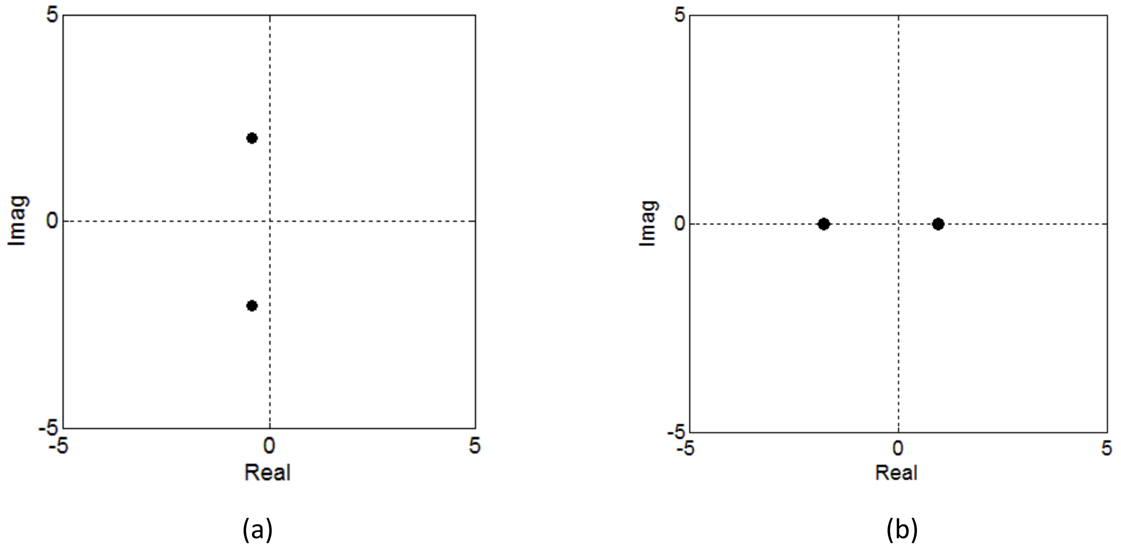


Fig. 1 *– Eigenvalues of the matrix A on the complex plane for the (a) stable case and (b) unstable case.*

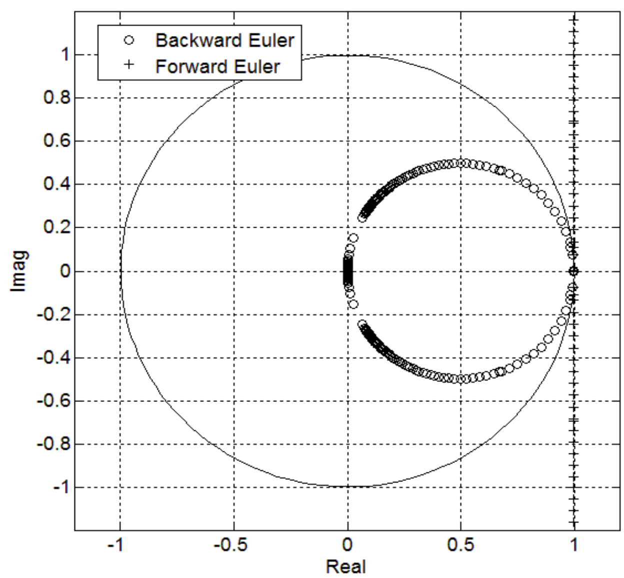
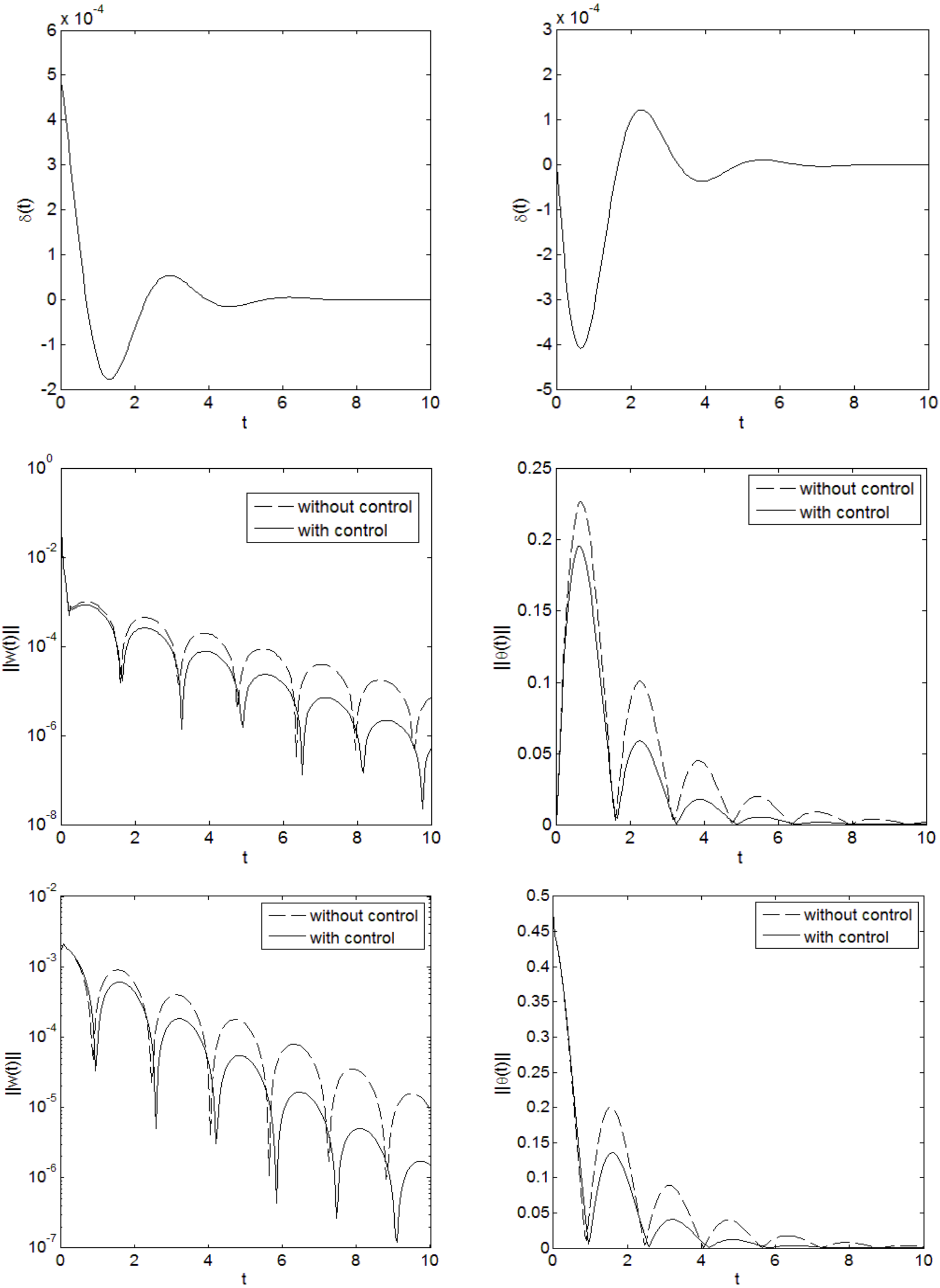


Fig. 2 *– Numerical stability of the discretized problem. For the backward Euler scheme all the eigenvalues modulus is less than one, therefore the numerical scheme is stable.*



(a) (b)

Fig. 3 *– Control law, $\|w(t)\|$ and $\|\theta(t)\|$ in the stable case for two different initial conditions:*

(a) $w_0(y) = 0{,}01 \cos\left(\frac{\pi}{2}\frac{y}{L}\right) - 0{,}01;\ \theta_0(y) = 0;\ T = 10, \gamma_1 = 1, \gamma_2 = 2 \cdot 10^5, \gamma_3 = 0$

(b) $w_0(y) = 0;\ \theta_0(y) = 0{,}1 \cos\left(\frac{\pi}{2}\frac{y}{L}\right) - 0{,}1;\ T = 10, \gamma_1 = 1, \gamma_2 = 2 \cdot 10^5, \gamma_3 = 0$

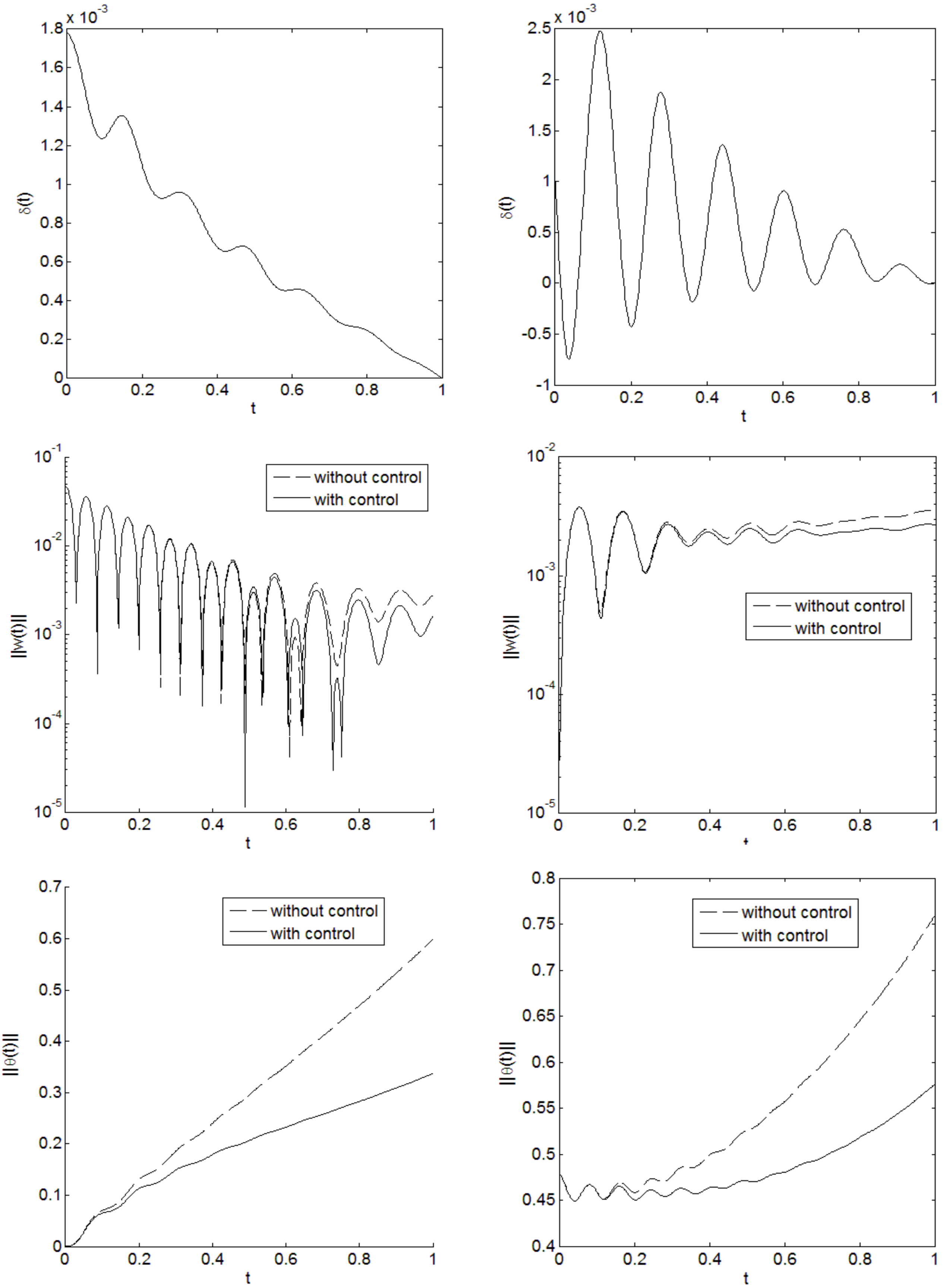


Fig. 4 *– Control law, $\|w(t)\|$ and $\|\theta(t)\|$ for the unstable case with the (a) and (b) initial conditions.*

$$(T = 1, \gamma_1 = 1, \gamma_2 = 0{,}8 \cdot 10^5, \gamma_3 = 0)$$

CONCLUSIONS

In this paper it has been developed a control model able to reduce the oscillations of a wing due to aerodynamic effects. The mathematical model used for the description of the problem is a linear dynamical system and the cost function chosen takes in account the evolution of the system, the energy of the control and the final state of the system itself; the algorithm, implemented in the Matlab environment, has been tested in a neighborhood of the stationary point in the stable and unstable cases. As one can see from the results the control is able to minimize the cost function for all the cases. In particular, in the stable case the control acts in order to reach in less time the stationary condition. In the unstable case the control is not able to stabilize the system, nevertheless this unstable behavior is delayed. This fact is probably due to the definition of the cost function, defined over a finite time range and containing both a system evolution term and an energy control term, and then the optimal condition of the control acts in order to have a compromise between this two terms.

Each term of the cost function is properly weighted through three coefficients; it is seen that, in order to have convergence, the weight of the energy control term has to be several order of magnitude greater than the system evolution term. Simulations not here reported, with the cost function taking in account only the final state of the system and the energy control term, has been performed and no appreciable differences with the previous cases were noticed.

REFERENCES

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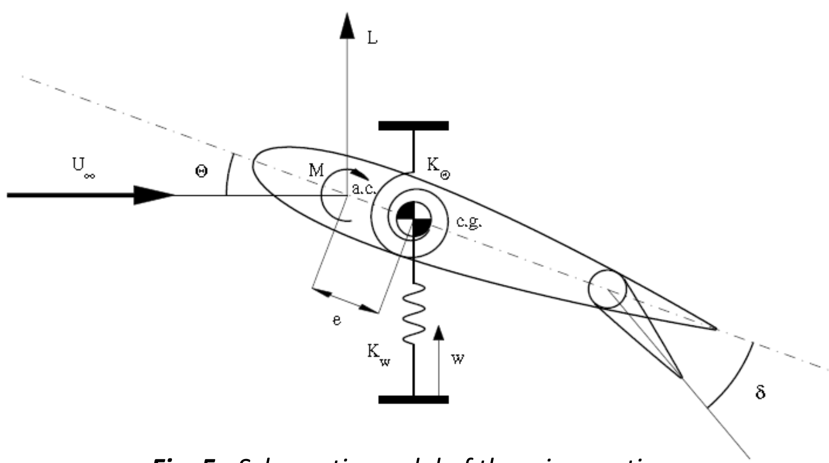

Fig. 5 *–Schematic model of the wing section.*