MINIMIZATION OF BOUNDARY LAYER KINETIC ENERGY BY MEANS OF LAGRANGIAN MULTIPLIERS APPROACH

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SUMMARY

The aim of this work is to develop a control model able to minimize the kinetic energy of a viscous boundary layer bounded between a steady and a moving wall. The physic of the problem is based upon the one dimensional homogenous viscous Burger's equation, which represents the motion of the fluid. The proposed method to evaluate the upper moving wall velocity able to minimize the kinetic energy is based on the Lagrangian multipliers approach. At first the problem is solved in an approximate way, that is to say solving the linearized Burgers' Equation. Successively the problem is solved by employing the non-linear homogenous Burgers' Equation. Numerical examples are presented to illustrate the effectiveness of the method.

NOMENCLATURE

a, b, c, d	Lagrangian multipliers
A	Matrix of the direct problem
B	Matrix of the adjoint problem
C_1, C_2	Constants of integration
J	Cost function
I	Identity matrix
$\mathcal{L}$	Lagrangian operator
L	Distance between the two flat walls
p	Search direction
q	Weight function
S_L, S_R	Slope of the left and right segment in the smoothing procedure
t	Time
T	Time of simulation
FT	Forcing term
U	Velocity W at $x = L$
u	Control function
w	Solution of the linearized direct problem
$\overline{w}_0$	Initial solution for the direct problem
$\overline{w}$	Solution for the direct problem
W	Solution of the stationary Burger's Equation
x, y	Coordinate system

Greeks

α	Exponential parameter
Δx	Spatial interval
Δt	Time interval
ε	Small perturbation coefficient
ν	Kinematic viscosity of the fluid
σ	Step length

Apex

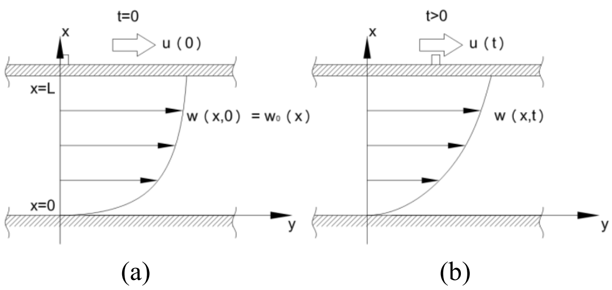
n	Time index
N	Upper bound of time index

Subscripts

i	Spatial index
M	Upper bound of spatial index

INTRODUCTION

We consider the problem of a boundary layer velocity profile bounded between two flat walls. The problem can be sketched as in Fig. 1:



(a) (b)

Figure 1 – Problem representation at initial condition (a) and at time t (b)

The lower wall is stuck, while the upper one is moving with a time dependent speed $u(t)$, which is assumed as the control function of the problem. The coordinate system x, y is setted as in Fig. 1, with the x direction bounded between 0 and L, and the y coordinate directed along the flat plane. The profile velocity is indicated with w, and it depends on both time and spatial coordinate x. The study of this kind of problem is motivated by flow control problems where the control action is located on the walls [1], [2], [3]. A similar approach, but based on the optimal feedback law derived from distributed parameters, has been developed in [4], while in this paper we will focus on the minimization of boundary layer kinetic energy through the Lagrangian multipliers approach [5].

MATHEMATICAL MODEL

In order to represent the motion inside boundary layer we consider the viscous Burgers' Equation:

$$\frac{\partial}{\partial t}\bar{w}(x,t) = \nu\frac{\partial^2}{\partial x^2}\bar{w}(x,t) - \frac{\partial}{\partial x}\frac{1}{2}\bar{w}(x,t)^2 \quad 0 < x < L \quad 0 < t \leq T \tag{1}$$

with homogeneous boundary condition at $x = 0$

$$\bar{w}(x=0,t) = 0 \tag{2}$$

and the Dirichlét boundary control at $x = L$:

$$\bar{w}(x=L,t) = u(t) \tag{3}$$

The initial condition is given by:

$$\bar{w}(x,t=0) = \bar{w}_0(x) \tag{4}$$

The constant parameter ν represents the kinematic viscosity of the fluid. As a first approach, in order to simplify the problem solution and to increase the rate of convergence and stability, the homogeneous viscous Burgers' Equation has been linearized. The linearization is made by applying a small perturbation in the neighbourhood of the stationary solution of the following problem:

$$\nu\frac{\partial^2}{\partial x^2}W(x) - \frac{\partial}{\partial x}\frac{W^2(x)}{2} = 0 \quad 0 < x < L \tag{5}$$

With the boundary conditions:

$$W(x=0) = 0 \tag{6}$$

$$W(x=L) = U \tag{7}$$

The general solution of Eq.(5) is:

$$W(x) = \sqrt{2C_1}\tanh\left(-\frac{\sqrt{C_1}x}{\nu\sqrt{2}} + C_2\right) \tag{8}$$

By imposing boundary conditions of Eq.(6) and Eq.(7) the two constants C_1 and C_2 are:

$$\tanh\left(-\frac{\sqrt{C_1}}{\sqrt{2}\nu}\right)\sqrt{2C_1} - U = 0 \tag{9}$$

$$C_2 = 0 \tag{10}$$

If we consider that:

$$\bar{w}(x,t) = W(x) + \varepsilon w(x,t) \quad \varepsilon \ll 1 \tag{11}$$

the linearization of problem (1) can be obtained as:

$$\frac{\partial}{\partial t}[W(x) + \varepsilon w(x,t)] = \nu\frac{\partial^2}{\partial x^2}[W(x) + \varepsilon w(x,t)] - \frac{\partial}{\partial x}\frac{1}{2}[W(x) + \varepsilon w(x,t)]^2 \tag{12}$$

And thus, neglecting the higher order terms, we obtain:

$$\frac{\partial}{\partial t}w(x,t) = \nu\frac{\partial^2}{\partial x^2}w(x,t) - \frac{\partial}{\partial x}[W(x)w(x,t)] \tag{13}$$

With the boundary conditions:

$$w(x=0,t) = 0 \tag{14}$$

$$w(x=L,t) = u(t) \tag{15}$$

And the following initial condition:

$$w(x,t=0) = w_0(x) \tag{16}$$

In order to minimize the kinetic energy of the boundary layer, the following cost function J is chosen [4]:

$$J = \int_0^T e^{\alpha t}\left[\int_0^L q(x)\,\bar{w}(x,t)^2 dx + u(t)^2\right] dt \tag{17}$$

That for the linearized equation becomes:

$$J = \int_0^T e^{\alpha t}\left[\int_0^L q(x)\,w(x,t)^2 dx + u(t)^2\right] dt \tag{18}$$

The constant α is a positive number, and q(x) is a user defined weight function. When α is a positive number, there is an additional performance requirement [6], [7], [8].

LAGRANGIAN APPROACH

If we consider Eq. (13), the Lagrangian approach leads to the expression:

$$\begin{aligned}\mathcal{L}(w,u,a,b,c,d) = \\ = J - \int_0^T\int_0^L a\left\{\frac{\partial}{\partial t}w(x,t) - \nu\frac{\partial^2}{\partial x^2}w(x,t) + \right. \\ \left. + \frac{\partial}{\partial x}[W(x)w(x,t)]\right\} dxdt + \\ - \int_0^L b[w(x,0) - w_0(x)]\,dx + \\ - \int_0^T c[w(0,t) - 0]\,dt + \\ - \int_0^T d[w(L,t) - u(t)]\,dt\end{aligned} \tag{19}$$

where a, b, c, d are the Lagrangian multipliers. Thus, setting the gradient of $\mathcal{L}$ equal to zero, we obtain the necessary conditions for the problem resolution:

- $\frac{\partial \mathcal{L}}{\partial a} = 0$

$$\frac{\partial}{\partial t}w(x,t) = \nu\frac{\partial^2}{\partial x^2}w(x,t) - \frac{\partial}{\partial x}[W(x)w(x,t)] \tag{20}$$

- $\frac{\partial \mathcal{L}}{\partial b} = 0$

$$w(x,0) = w_0(x) \tag{21}$$

- $\frac{\partial \mathcal{L}}{\partial c} = 0$

$$w(0,t) = 0 \tag{22}$$

■ $\dfrac{\partial \mathcal{L}}{\partial d} = 0$

$$w(L,t) = u(t) \tag{23}$$

■ $\dfrac{\partial \mathcal{L}}{\partial w} = 0$

$$-\frac{\partial}{\partial t} a(x,t) - W(x)\frac{\partial}{\partial x} a(x,t) + -\nu \frac{\partial^2}{\partial x^2} a(x,t) - e^{\alpha t}[2q(x)w(x,t)] = 0 \tag{24}$$

$$a(0,t) = 0 \tag{25}$$

$$a(L,t) = 0 \tag{26}$$

$$a(x,T) = 0 \tag{27}$$

$$b(x) = a(x,0) \tag{28}$$

$$c(t) = \nu \frac{\partial}{\partial x} a(x,t)\Big|_{x=0} \tag{29}$$

$$d(t) = -\nu \frac{\partial}{\partial x} a(x,t)\Big|_{x=L} \tag{30}$$

■ $\dfrac{\partial \mathcal{L}}{\partial u} = 0$

$$u(t) = \frac{\nu}{2e^{\alpha t}} \frac{\partial}{\partial x} a(x,t)\Big|_{x=L} \tag{31}$$

As one can note:

- the derivative with respect to a leads to the definition of the direct problem;
- the ones with respect to b, c and d leads to the definition of the direct problem boundary and initial conditions respectively;
- the one with respect to w leads to the definition of the adjoint problem [9], of its boundary and initial conditions and of the Lagrangian operator b, c and d;
- the one with respect to u leads to the definition of the optimal condition for control $u(t)$.

Furthermore, the adjoint equation Eq.(24) has to be integrated backward in time, because of the negative sign of the time derivative. Thus, the "initial" condition of the adjoint problem is intended at time $\mathrm{t} = \mathrm{T}$.

DISCRETIZATION METHOD

In this paragraph the discretization methods for the analysis of both original and linearized model are presented. The discretization approach employes for both cases an implicit finite difference scheme.

Non linearized model discretization

Starting from Eq. (1), if we choose an implicit finite difference scheme, with first order approach in time and second order approach in space, we have for the direct problem:

$$\frac{\overline{w}_i^{n+1} - \overline{w}_i^n}{\Delta t} = \nu \frac{\overline{w}_{i+1}^{n+1} - 2\overline{w}_i^{n+1} + \overline{w}_{i-1}^{n+1}}{\Delta x^2} + -\frac{\left(\overline{w}_{i+1}^{n+1}\right)^2 - \left(\overline{w}_{i-1}^{n+1}\right)^2}{4\Delta x} \tag{32}$$

This scheme leads to the system:

$$\overline{w}_1^{n+1} - 0 = 0$$

$$\overline{w}_i^{n+1} - \frac{\Delta t\, \nu}{\Delta x^2}\left(\overline{w}_{i+1}^{n+1} - 2\overline{w}_i^{n+1} + \overline{w}_{i-1}^{n+1}\right) + + \frac{\Delta t}{4\Delta x}\left[\left(\overline{w}_{i+1}^{n+1}\right)^2 - \left(\overline{w}_{i-1}^{n+1}\right)^2\right] - \overline{w}_i^n = 0 \tag{33}$$

$$i = 2, \dots M$$

$$\overline{w}_{M+1}^{n+1} - u^{n+1} = 0$$

Where:

$$n = 1, \dots N$$

This non-linear system can be solved for example through a Levenberg-Marquard method [10].

Linearized model discretization

Starting from Eq.(20), if we choose, as for the non-linearized model, an implicit finite difference scheme, with first order approach in time and second order approach in space, we have for the direct problem:

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} = \nu \frac{w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1}}{\Delta x^2} + + \frac{W_{i+1}w_{i+1}^{n+1} - W_{i-1}w_{i-1}^{n+1}}{2\Delta x} \tag{34}$$

This scheme leads to the system:

$$\frac{\underline{w}^{n+1} - \underline{w}^n}{\Delta t} + \underline{\underline{A}}\,\underline{w}^{n+1} - \underline{FT} = \underline{0} \tag{35}$$

where:

$$\underline{FT} = \left[w_1^{n+1}\left(\frac{\nu}{\Delta x^2} + \frac{W_1}{2\Delta x}\right) \quad 0 \dots 0 \quad w_{M+1}^{n+1}\left(\frac{\nu}{\Delta x^2} + \frac{W_{M+1}}{2\Delta x}\right)\right]^T$$

and the matrix $\underline{\underline{A}}$ is a three-diagonal matrix with the elements:

$$A_{i,j} = \frac{2\nu}{\Delta x^2} \qquad if\ i = j$$

$$A_{i,j} = -\frac{\nu}{\Delta x^2} - \frac{W_i}{2\Delta x} \qquad if\ i = j+1$$

$$A_{i,j} = -\frac{\nu}{\Delta x^2} + \frac{W_{i+2}}{2\Delta x} \qquad if\ i = j-1$$

where we have indicated:

$$i = 2, \dots, M$$

$$n = 1, \dots, N$$

$$\Delta x = \frac{L}{M}$$

$$\Delta t = \frac{T}{N}$$

Since the scheme is implicit, its resolution leads to:

$$\underline{w}^{n+1} = \left(\underline{\underline{I}} + \Delta t \underline{\underline{A}}\right)^{-1}\left(\underline{w}^n + \Delta t \underline{FT}\right) \quad n = 1, \dots, N \tag{36}$$

For the adjoint problem resolution, starting from Eq.(24), if we choose a coherent implicit finite difference scheme, with first order approach in time and second order approach in space, we have:

$$-\frac{a_i^{n+1}-a_i^n}{\Delta t}=\nu\frac{a_{i+1}^n-2a_i^n+a_{i-1}^n}{\Delta x^2}+ +W_i\frac{a_{i+1}^n-a_{i-1}^n}{2\Delta x}-\underline{FT}_{adj} \quad (37)$$

This scheme leads to the system:

$$-\frac{\underline{a}^{n+1}-\underline{a}^n}{\Delta t}+\underline{\underline{B}}\,\underline{a}^n-\underline{FT}_{adj}=\underline{0} \quad (38)$$

where:

$$\underline{FT}_{adj}= =\frac{\left[2q_2w_2^n \quad 2q_3w_3^n \quad \cdots \quad 2q_{M-2}w_{M-2}^n \quad 2q_{M-1}w_{M-1}^n\right]^T}{e^{-\alpha(n-1)\Delta t}}$$

and the matrix $\underline{\underline{B}}$ is a three-diagonal matrix with the elements:

$$B_{i,j}=\frac{2\nu}{\Delta x^2} \qquad if\ i=j$$

$$B_{i,j}=-\frac{\nu}{\Delta x^2}-\frac{W_i}{2\Delta x} \qquad if\ i=j+1$$

$$B_{i,j}=-\frac{\nu}{\Delta x^2}+\frac{W_i}{2\Delta x} \qquad if\ i=j-1$$

with the same spatial and time discretization of the direct problem. Since the scheme is implicit, its resolution leads to:

$$\underline{a}^n=\left(\underline{\underline{I}}+\Delta t\underline{\underline{B}}\right)^{-1}\left(\underline{a}^{n+1}+\Delta t\underline{FT}_{adj}\right) \quad n=N,\dots,1 \quad (39)$$

APPLICATIONS

The method previously shown has been written in an automatic calculation procedure in Matlab® environment [11]. It has been applied a smoothing procedure on the optimal condition of the control in order to keep the compatibility condition between the initial condition and the upper boundary condition. Such a algorithm is presented for case in which the first eight time step of the control are not calculated by the optimal condition. Since the first time step is given by the compatibility condition if one takes as unknowns the control points between the second and the eighth time step one can proceed as indicated in the following expressions.

$$u_5=-\frac{S_{R9}+S_{L9}}{2}4\Delta t+u_9$$

$$u_7=-\frac{S_{R5}+S_{R9}}{2}2\Delta t+u_9$$

$$u_3=-\frac{S_{L7}+S_{L5}}{2}2\Delta t+u_5$$

$$u_8=-\frac{S_{R9}+S_{R7}}{2}\Delta t+u_9$$

$$u_6=-\frac{S_{L8}+S_{L7}}{2}\Delta t+u_7$$

$$u_4=-\frac{S_{R3}+S_{L6}}{2}\Delta t+u_5$$

$$u_2=-\frac{S_{L4}+S_{L3}}{2}\Delta t+u_3$$

In Fig. 2 a graphical representation of the smoothing procedure is shown.

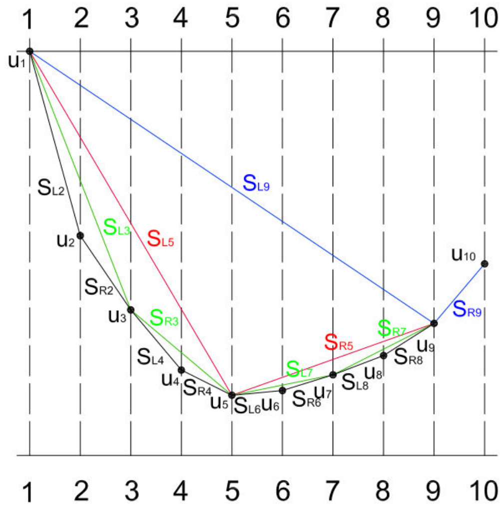


Figure 2 - Graphical representation of the smoothing procedure for the control determination

As a first application the linearized model has been employed to simulate the problem.
The values of α and $q(x)$ have been chosen as variable parameters. In particular:

$$\alpha=0,0.4,\dots,1.2 \qquad q(x)=k \text{ where } k=1,11,\dots,31 \quad (40)$$

As a first result the stability of the method has been investigated in order to define the proper discretization method (implicit or explicit). The Euler circle [12], which represents the stability zone of the scheme, is:

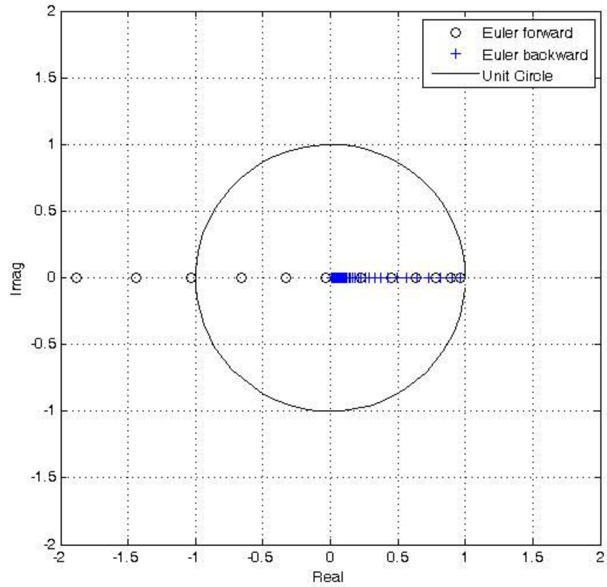
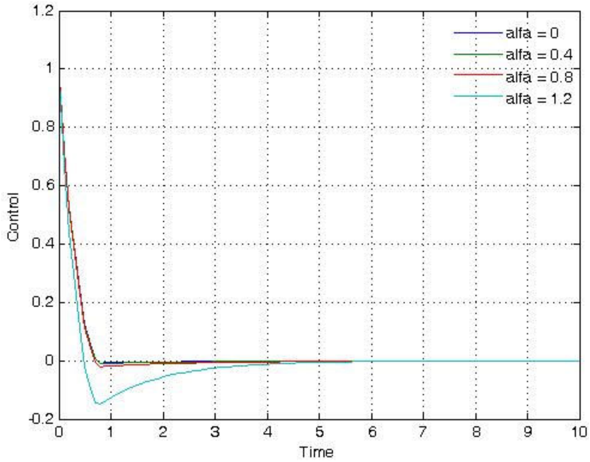
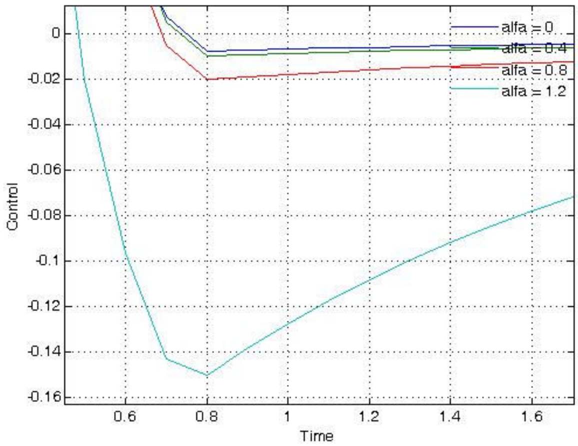


Figure 3 - Euler circle for the linearized model

Thus, an implicit scheme has been adopted. The calculated control law is presented. Firstly fixing $q(x)$ to one and letting α to vary as indicated in Eq.(40) [Figure 4 (a) and (b)], and then fixing α to zero and letting q(x) to vary as indicated in Eq.(40) [Figure 5 (a) and (b)].

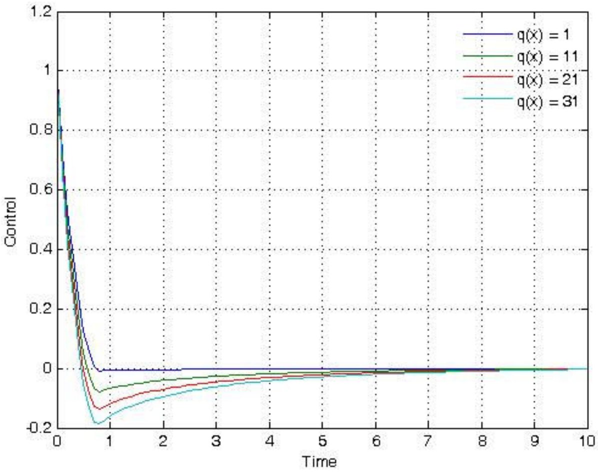


(a)

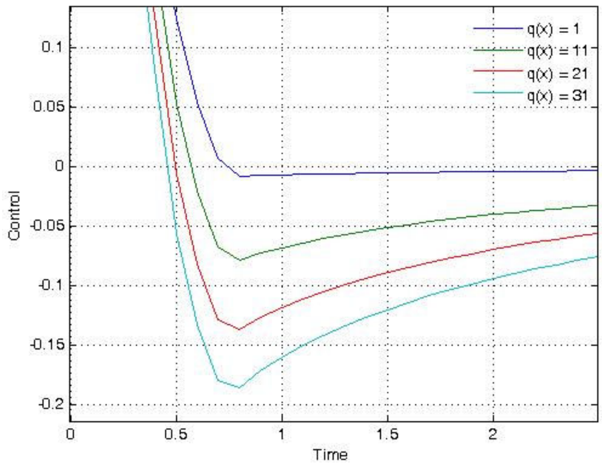


(b)

Figure 4 – Control law for the linearized problem letting α to vary: (a) global view, (b) zoomed view



(a)



(b)

Figure 5 – Control law for the linearized problem letting $q(x)$ to vary: (a) global view, (b) zoomed view

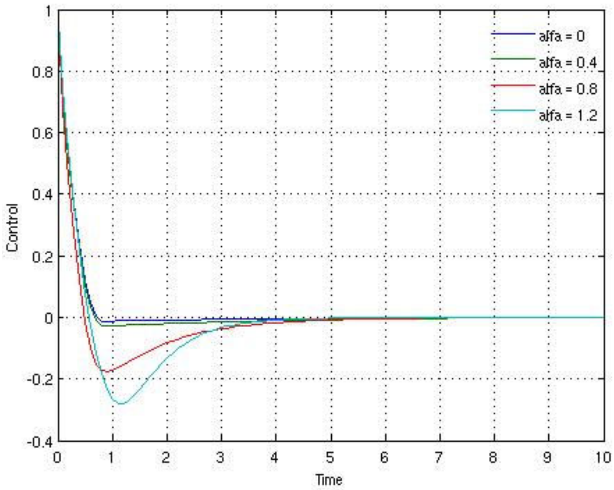
As a second application the non linearized model has been simulated. The adjoint problem resolution is determined using Eq. (41) instead of Eq. (37).

$$-\frac{a_i^{n+1}-a_i^n}{\Delta t} = \nu\frac{a_{i+1}^n - 2a_i^n + a_{i-1}^n}{\Delta x^2} + \\ +\overline{w}_i^n\frac{a_{i+1}^n - a_{i-1}^n}{2\Delta x} - \underline{FT}_{adj} \quad (41)$$

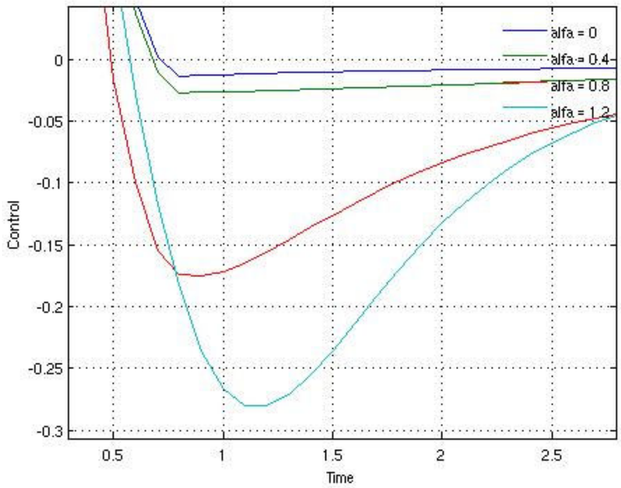
Eq.(41) is rigorous for the linearization of Eq. (32) in the neighborhood of the solution $\overline{\mathrm{w}}_{\mathrm{i}}^{\mathrm{n}}$ and not for the adopted Eq. (32). For this reason the optimal control $u_{opt}(t)$ from Eq. (31) is not used directly, but only to find out the descent direction and to calculate, through a proper choose of the step length [13], the new value of the control law in the iterative process.

$$u_{new}(t) = u_{old}(t) + \sigma p \\ u_{new}(t) = u_{old}(t) + \sigma(u_{opt}(t) - u_{old}(t)) \quad (42)$$

The values of $\boldsymbol{\alpha}$ and $\boldsymbol{q(x)}$ have been chosen as variable parameters taken as just written in Eq. (40). The calculated control law is presented. Firstly fixing $\boldsymbol{q(x)}$ to unity and letting $\boldsymbol{\alpha}$ to vary as indicated in Eq.(40), then fixing $\boldsymbol{\alpha}$ to zero and letting $\mathbf{q(x)}$ to vary as indicated in Eq.(40).

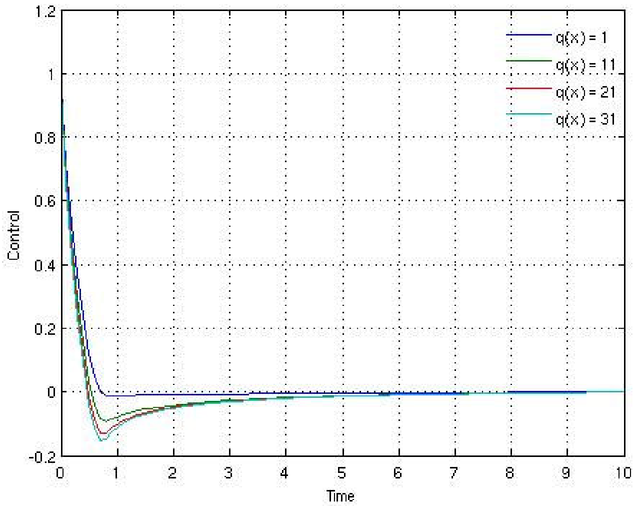


(a)

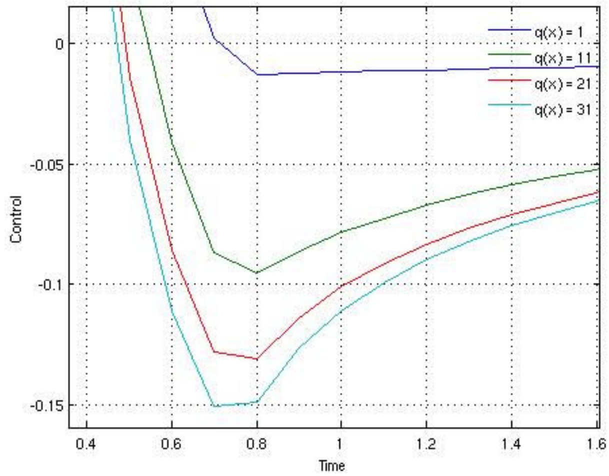


(b)

Figure 6 – Control law for the non-linearized problem letting α to vary: (a) global view, (b) zoomed view



(a)



(b)

Figure 7 – Control law for the non-linearized problem letting $q(x)$ to vary: (a) global view, (b) zoomed view

In the last two applications the step length has been chosen respectively equal to 0.03 and 0.2. Finally the solution $\overline{w}(x,t)$ resulting from the found control law is shown.

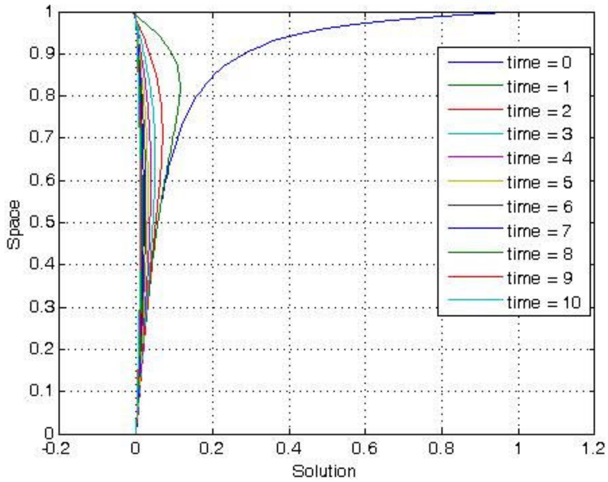


Figure 8 - Solution $\overline{w}(x,t)$: q(x) equal to 1 and α equal to 0

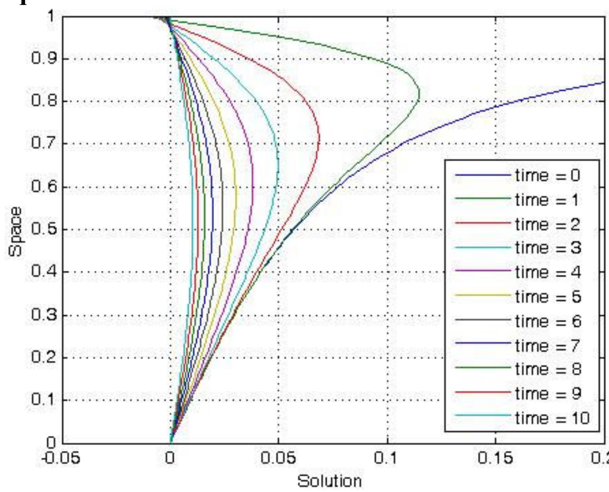


Figure 9 - Solution $\overline{w}(x,t)$: q(x) equal to 1 and α equal to 0, detailed view

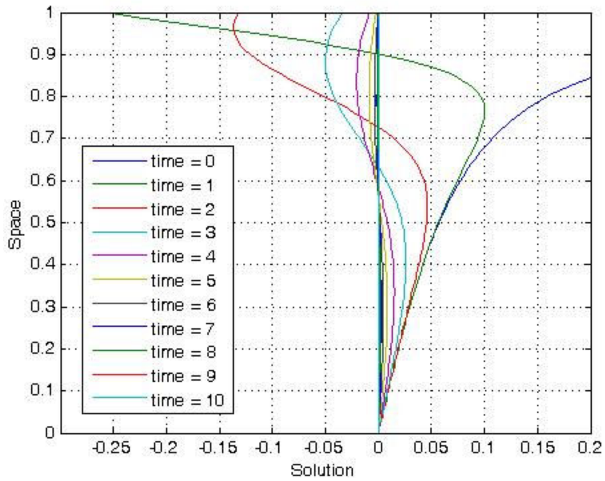


Figure 10 - Solution $\overline{w}(x,t)$: q(x) equal to 1 and α equal to 1.2, detailed view

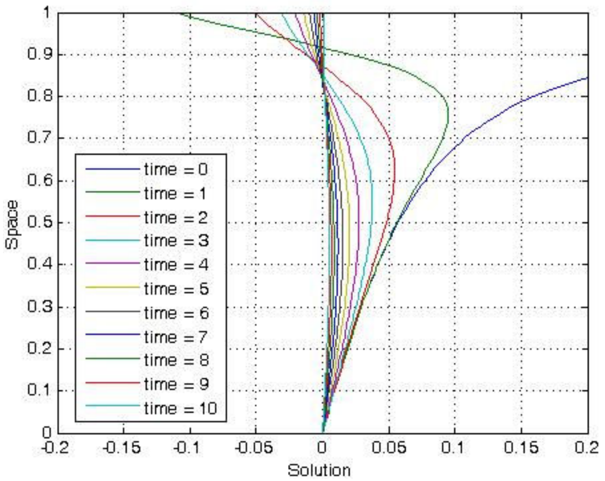


Figure 11 - Solution $\overline{w}(x,t)$: q(x) equal to 31 and α equal to 0, detailed view

CONCLUSIONS

A control model, able to minimize the kinetic energy of a viscous boundary layer, has been developed. The mathematical problem, based on the control of Burgers' Equation, has been solved through the employ of Lagrangian multipliers approach. The model, developed in Matlab environment, has been applied to the study of a boundary layer bounded between two walls. As a first application the Burgers' Equation has been linearized by applying a little perturbation in a neighborhood of a stationary solution in order to simplify the problem of minimizing the kinetic energy. In fact, for this case the kinetic energy is due to only the speed associated to the small perturbation. As one can notes the control begins to modify the solution for low values of the weight function $q(x)$, while the exponential coefficient α contribution is relevant on the control only for values greater then unity. As a second case the control model has been applied to the control of the non linear Burgers' Equation. For this case the sensitivity of the control with respect to the weight function is higher for lower values of $q(x)$, while the influence of coefficient α begin for lower values with respect to the first application. The smoothing procedure on control law (Fig. 2) allows to respect the compatibility condition, which is not a strictly imposed condition in the Lagrangian approach. Furthermore, the values of control law, as indicated in Figures from 4 to 7, strongly decreases from the initial condition till a negative peak, which amplitude depends on both exponential coefficient and weight function. This kind of behavior, unexpected in a first analysis of the problem, means that in order to strongly reduce the kinetic energy associated to the boundary layer velocity profile, it is necessary to have in the first instants of control, a reverse motion of the upper bounding wall, that successively start to increase till the value of zero. Thus, the only decrease of upper bounding wall speed till the value of zero should not be sufficient to minimize the kinetic energy of the problem.

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