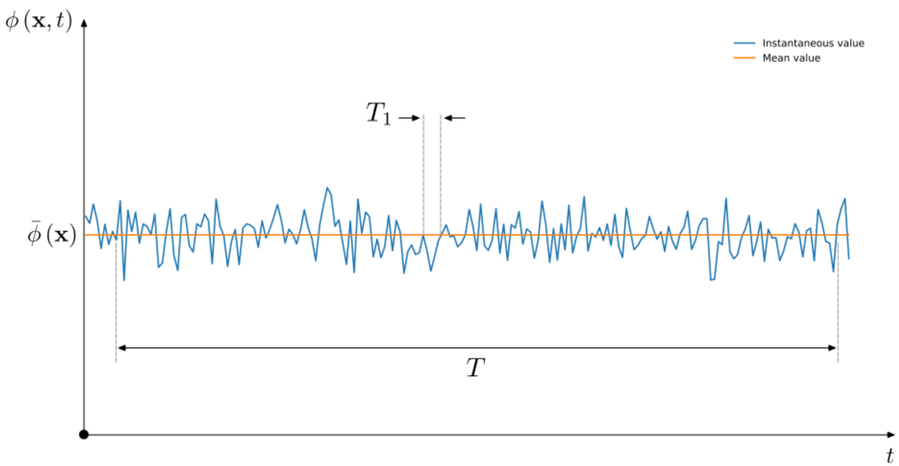
Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. **RANS equations – Reynolds averaging**
3. The Boussinesq hypothesis
4. The gradient diffusion hypothesis
5. Sample turbulence models

Governing equations – Reynolds averaging

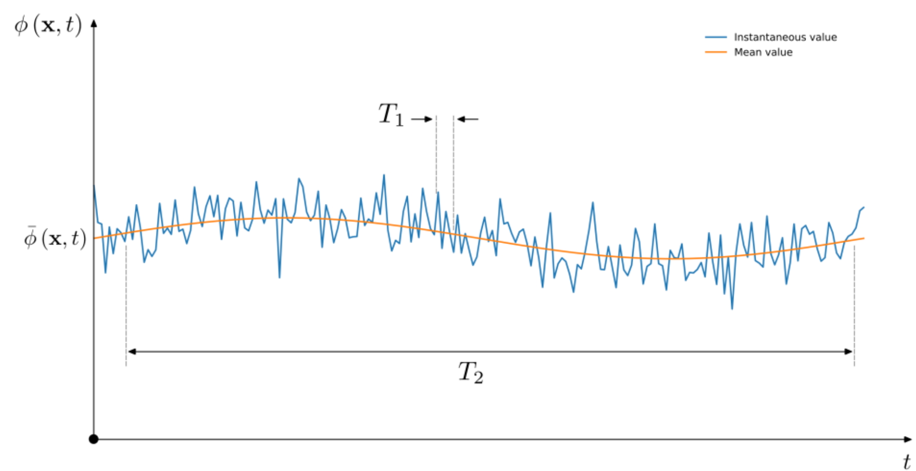
Instantaneous fluctuations – Removing small scales

Steady mean value



$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\ value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\ part}$$

Unsteady mean value



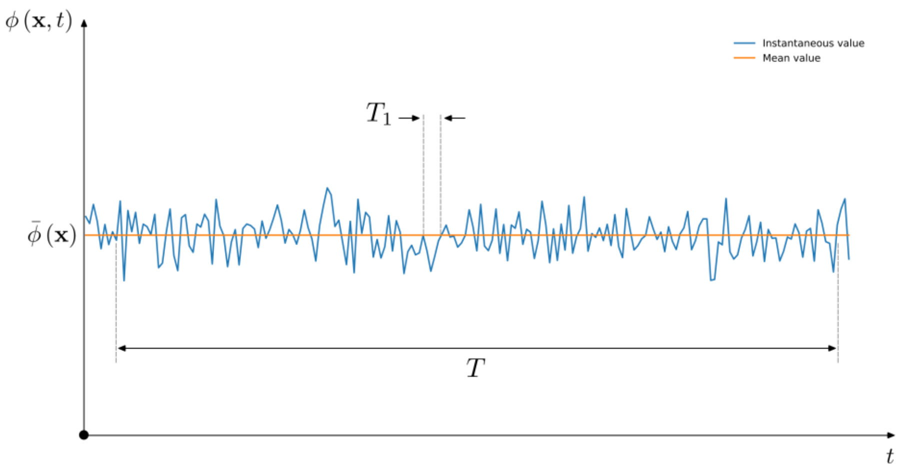
$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x},t)}_{mean\ value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\ part}$$

- We have seen that turbulent flows are characterize by instantaneous fluctuations of velocity, pressure, and all transported quantities.
- In most engineering applications is not of interest resolving these instantaneous fluctuations.
- Therefore, we need to somehow remove these fluctuations when numerically approximating the governing equations.

Governing equations – Reynolds averaging

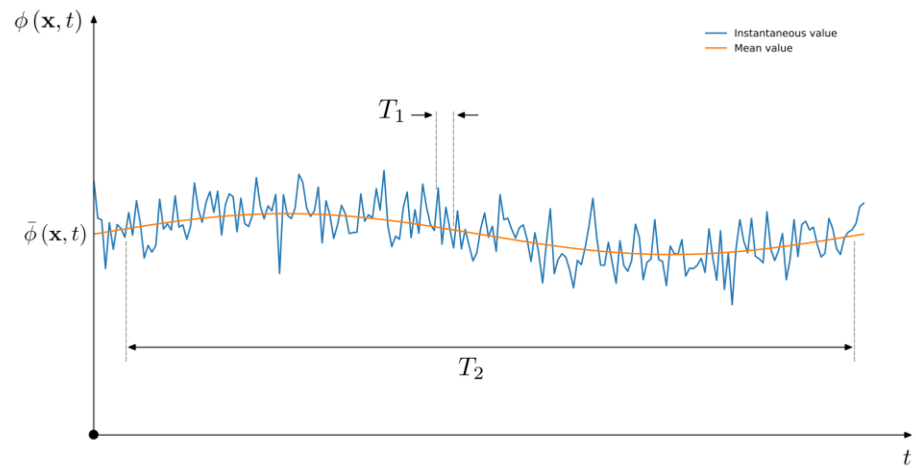
Instantaneous fluctuations – Removing small scales

Steady mean value



$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\ value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\ part}$$

Unsteady mean value



$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x},t)}_{mean\ value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\ part}$$

- To avoid the need to resolve the instantaneous fluctuations (or small scales), two methods are often used:
  - Reynolds averaging.
  - Filtering of the governing equations.
- If you want to resolve all scales, you conduct DNS simulations, which are computational expensive.

Governing equations – Reynolds averaging

Instantaneous fluctuations – Removing small scales

- Two methods can be used to eliminate the need to resolve the small scales:
  - Reynolds averaging (RANS/URANS):
    - All turbulence scales are modeled.
    - Can be 2D and 3D.
    - Can be steady or unsteady.
  - Filtering (LES/DES):
    - Resolves large eddies.
    - Models small eddies.
    - Intrinsically 3D and unsteady.
- Both methods introduce additional terms in the governing equations that must be modeled.
    - These terms are related to the instantaneous fluctuations.

- **The final goal of turbulence modeling is to find the closure equations to model these additional terms (usually a stress tensor correlating fluctuating variables).**

Overview of the main turbulence modeling approaches

MODELING APPROACH

RANS
Reynolds-**A**veraged **N**avier-**S**tokes equations

URANS
Unsteady **R**eynolds-**A**veraged **N**avier-**S**tokes equations

- Many more acronyms that fit between **RANS/URANS** and **SRS**.
- Some of the acronyms are used only to differentiate approaches used in commercial solvers.

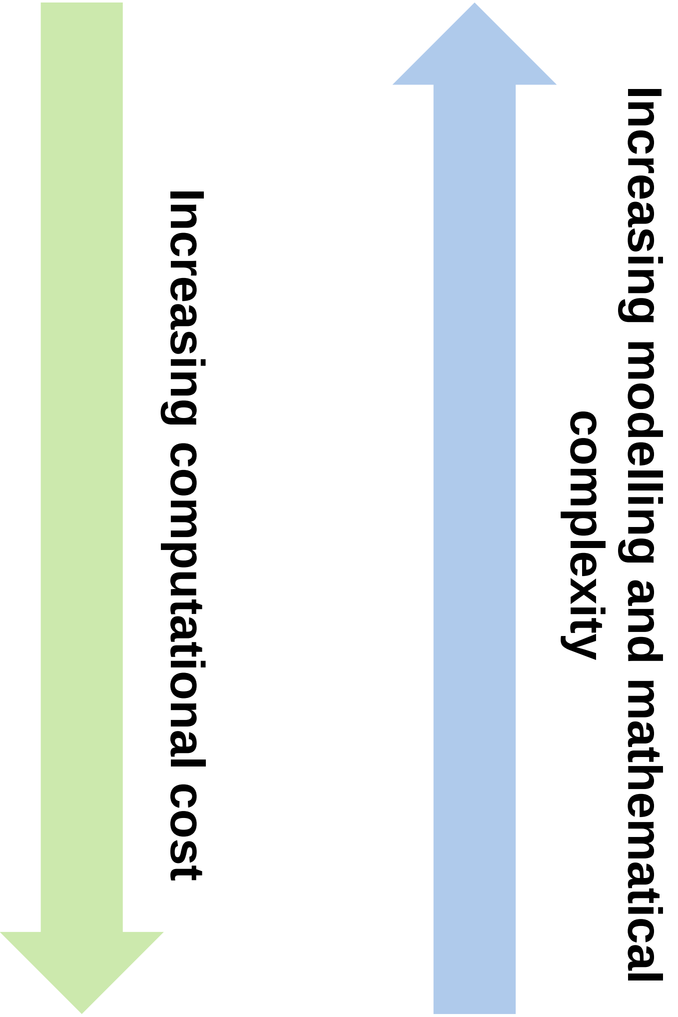
PANS, **SAS**, **RSM**, **EARSM**, **PITM**, **SBES, ELES**

DES
Detached **E**ddy **S**imulations

LES
Large **E**ddy **S**imulations

DNS
Direct **N**umerical **S**imulations

SRS
Scale-**R**esolving **S**imulations



Governing equations – Reynolds averaging

Turbulence modeling – Starting equations

Exact NSE

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\
\dfrac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \tau + \mathbf{S_u} \\
\dfrac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) = -\nabla \cdot q - \nabla \cdot (p\mathbf{u}) + \boldsymbol{\tau}\mathbf{:}\nabla \mathbf{u} + \mathbf{S}_{e_t}
\end{cases}
$$

$$+$$

Additional equations to close the system (thermodynamic variables)

Additional relationships to relate the transport properties

Additional closure equations for the turbulence models

- Turbulence models equations cannot be derived from fundamental principles.
- All turbulence models contain some sort of empiricism.
- Some calibration to observed physical solutions is contained in the turbulence models.
- Also, some intelligent guessing is used.
- A lot of uncertainty is involved!

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- Let us write down the governing equations for an incompressible, isothermal, Newtonian flow.
- When conducting DNS simulations (no turbulence models involved), this is our starting point,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- These are the **exact** governing equations, where we have not introduced approximations.
- Sometimes these equations are referred as the laminar Navier-Stokes equations.
- This does not mean that this set of equations are only valid to laminar regime.
- In fact, they are valid for laminar and turbulent regimes.
- But if you use this set of equations in turbulent regime, you need to resolve all turbulent scales (in space and time), and this requires very fine meshes and very small time-steps.
- A lot of computational power is involved!

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- When using RANS/URANS turbulence models, we use the following set of governing equations,

If we retain this term, we talk about **URANS** equations and if we drop it, we talk about **RANS** equations

Reynolds stress tensor
This term requires modeling

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \bar{p}) + \nu \nabla^2 \bar{\mathbf{u}} - \nabla \cdot \left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

- In these equations, the overbar represents mean quantities and the prime symbol represent fluctuating quantities (small scales).
- So far, we have not introduced approximations to model the fluctuations.
  - These are the **exact** Navier-Stokes RANS/URANS equations.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- The previous set of equations can be rewritten as,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \overline{p}) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}^R$$

- Where τ^R is the Reynolds stress tensor, and it can be written as,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- The Reynolds stress tensor τ^R represents the transfer of momentum due to turbulent fluctuations.
- It correlates the velocity fluctuations.

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

- The Reynolds stress tensor is symmetric; therefore, it has six components.
- The diagonal represents normal stresses and the off-diagonal shear stresses.
- Notice that the Reynolds stress tensor is not actually a stress, it must be multiplied by density in order to have dimensions corresponding to stresses.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- To derive the incompressible Navier-Stokes RANS equations, we need to apply Reynolds averaging to the governing equations.
- Reynolds averaging simple consists in:
    1. Splitting the instantaneous values of the primitive variables into a mean component and a fluctuating component (Reynolds decomposition).
    2. Averaging the quantities (time average, spatial average, or ensemble average).
    3. Applying a few averaging rules to simplify the equations.
    4. Doing some algebra.

- When we use Reynolds averaging, we are taking a statistical approach to turbulence modeling.
- When we do DNS, we take a deterministic approach to turbulence modeling.
- Usually, we are interested in the mean behavior of the flow.
- Therefore, by applying Reynolds averaging, we are only solving for the averaged variables and the fluctuations are modeled.

Incompressible Navier-Stokes RANS equations

- The Reynolds decomposition consists in splitting the instantaneous value of a variable into a mean component and a fluctuating component, as follows,

Instantaneous value

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}) + \phi'(\mathbf{x}, t)$$

Mean component

Fluctuating component

- In our notation, the overbar represents the average (or mean) value and the prime (or apostrophe) represents the fluctuating part.
- We will use this notation consistently during the lectures.
- But have in mind that you will find different notations in literature.

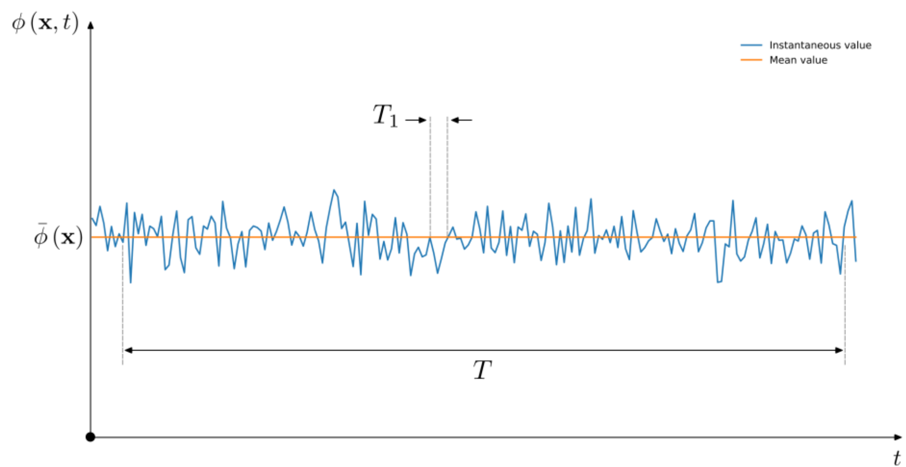
Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- To compute the average (or mean) quantities, we can use time averaging,

$$\bar{\phi}(\mathbf{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} \phi(\mathbf{x}, t) dt$$

- Here, T represents the averaging interval.
- This interval must be large compared to the typical time scales of the fluctuations so it will yield to a stationary state.
- Time averaging is appropriate for stationary turbulence or slowly varying turbulent flows, *i.e.*, a turbulent flow that, on average, does not vary much with time.
- Notice that we are not making the distinction between steady or unsteady flow.
- We are only saying that if we take the average between different ranges or values of *t*, we will get approximately the same mean value.
- The time average can be in time (unsteady simulations) or iterative (steady simulations).



$$\phi(\mathbf{x}, t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\ value} + \underbrace{\phi'(\mathbf{x}, t)}_{fluctuating\ part}$$

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- We can also use spatial averaging and ensemble averaging.
- Spatial averaging is appropriate for homogenous turbulence and is defined as follows,

$$\bar{\phi}(t) = \lim_{V \to \infty} \frac{1}{V} \int_V \phi(\mathbf{x}, t) dV$$

Volume of the domain

- Ensemble averaging is appropriate for unsteady turbulence.

$$\bar{\phi}(\mathbf{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \phi(\mathbf{x}, t)$$

Number of realizations

- In ensemble averaging, the number of realizations (or experiments) or cycles must be large enough to eliminate the effects of fluctuations.
  - This type of averaging can be used with steady or unsteady flows.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

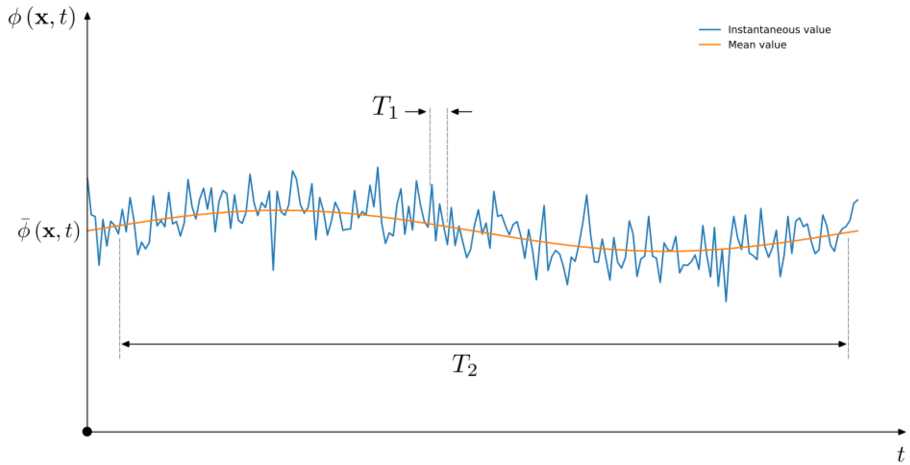
- If the mean quantities varies in time, such as,

$$\phi(\mathbf{x},t) = \bar{\phi}(\mathbf{x},t) + \phi'(\mathbf{x},t)$$

- We simple modify time averaging, as follows,

$$\bar{\phi}(\mathbf{x},t) = \lim_{T\to\infty} \frac{1}{T}\int_{t}^{t+T} \phi(\mathbf{x},t)dt \qquad T_1 << T << T_2$$

- Where T_2 is the time scale characteristic of the slow variations in the flow that we do not wish to regard as belonging to the turbulence.
- In this kind of situations, it might be better to use ensemble averaging.
- However, ensemble averaging requires running many experiments. This approach is better fit for experiments as CFD is more deterministic.
- Ensemble average can also be used when having periodic signal behavior. However, you will need to run for long times (many cycles) in order to take good averages.
- Another approach is the use of phase averaging.



Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- Any of the previous time averaging rules can be used without loss of generality.
- But from this point on, we will consider only time averaging.
- Before continuing, let us recall a few averaging rules that we will use when deriving the RANS equations.

$$\overline{\phi'} = 0,$$

$$\overline{\bar{\phi}} = \bar{\phi},$$

$$\overline{\bar{\phi}} = \overline{\bar{\phi} + \phi'} = \bar{\phi},$$

$$\overline{\phi + \varphi} = \bar{\phi} + \bar{\varphi},$$

$$\overline{\bar{\phi}\varphi} = \overline{\bar{\phi}\bar{\varphi}} = \bar{\phi}\bar{\varphi},$$

$$\overline{\bar{\phi}\varphi'} = \bar{\phi}\overline{\varphi'} = 0,$$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \bar{\phi}}{\partial x}, \quad \overline{\frac{\partial \phi}{\partial t}} = \frac{\partial \bar{\phi}}{\partial t},$$

The differentiation operator commutes in space and time

$$\begin{aligned}
\overline{\phi\varphi} &= \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')} \\
&= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'} \\
&= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'} \\
&= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'},
\end{aligned}$$

$$\overline{\phi'^2} \neq 0,$$

$$\overline{\phi'\varphi'} \neq 0,$$

$$\overline{\int \phi ds} = \int \bar{\phi} ds$$

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- Let us write down the Reynolds decomposition for the primitive variables of the incompressible Navier-Stokes equations (NSE),

$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t),$$
$$p(\mathbf{x},t) = \bar{p}(\mathbf{x}) + p'(\mathbf{x},t)$$

- By substituting the previous decompositions into the exact incompressible NSE, using the previous averaging rules, and doing some algebra, we arrive to the exact incompressible RANS/URANS equations,

If we retain this term, we talk about **URANS** equations and if we drop it, we talk about **RANS** equations

Reynolds stress tensor
This term requires modeling
$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \bar{p}) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- The **exact Navier-Stokes** equations are very similar to the **exact Navier-Stokes RANS** equations.

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Exact Navier-Stokes equations.
No turbulence models are being used.
These equations are valid for DNS.

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla \overline{p}) + \nu \nabla^2 \overline{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

Exact RANS/URANS equations

- The differences are that all quantities have been averaged in the exact Naiver-Stokes RANS equations (the overbar over the primitive variables).
- And the appearance of the Reynolds stress tensor τ^R .

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- Notice that the Reynolds stress tensor τ^R is not actually a stress, it must be multiplied by density in order to have dimensions corresponding to stresses,

Vector notation	Index notation
$\boldsymbol{\tau}^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right)$	$\tau_{ij}^R = -\rho \left(\overline{u_i' u_j'} \right)$

- In the derivation of the RANS equation, we multiplied and divided by density the velocity correlation resulting from the Reynolds averaging procedure.
- The complete derivation of the incompressible exact RANS equations is covered in Appendix 4 of the lecture notes.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- Also notice that in the literature, different authors will define the Reynolds stress tensor τ^R differently.
- Sometimes is defined as shown here, sometimes is defined with the opposite sign, and sometimes without the density included in the definition.
- This different terminology does not matter, as long as consistency is maintained throughout the derivation.
- From now on, and unless otherwise specified, we will consistently use this definition,

$$\boldsymbol{\tau}^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right)$$

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- As previously mentioned, the Reynolds stress tensor τ^R arises from the Reynolds averaging and it can be written as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

$$\mathbf{uu} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{bmatrix}$$

- In CFD we do not want to resolve the velocity fluctuations as it requires very fine meshes and small time-steps.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled.
- The rest of the terms appearing in the governing equations can be computed from the mean flow.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations

- In the additional notes, we go thru all the steps to derive the incompressible Navier-Stokes RANS/URANS equations.

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations – Additional comments

- The Reynolds stress tensor τ^R represents the transfer of momentum due to turbulent fluctuations.
- The Reynolds stress tensor is responsible for the increased mixing and larger wall shear stresses.
- Remember, increased mixing and larger wall shear stresses are properties of turbulent flows.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled.
- The question now is, how do we model the Reynolds stress tensor τ^R ?

Governing equations – Reynolds averaging

Incompressible Navier-Stokes RANS equations – Additional comments

- It is possible to derive a set of governing equations for the Reynolds stress tensor τ^R.
    - Six new equations as the tensor is symmetric.
- This approach is known as Reynolds stress models (RSM), which we will address in Lecture 6.
- Probably, this is the most physically sound RANS model (RSM) as it avoids the use of hypothesis/assumptions to model the Reynolds stress tensor.
- However, it is much simpler to model the Reynolds stress tensor.
- The most widely hypothesis/assumption used to model the Reynolds stress tensor is the Boussinesq hypothesis, that we will study in next section.

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\underbrace{\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}}$$

- Using the RSM approach we derive a transport equation for each of these terms.
- Six equations in total because the tensor is symmetric.

Incompressible Navier-Stokes RANS equations – Additional comments

- We just outlined the incompressible Navier-Stokes RANS/URANS equations.
- The road to derive the compressible Navier-Stokes RANS/URANS equations is similar.
- But to derive them, we use Favre average (which can be seen as a mass-weighted averaging operation) and a few additional averaging rules.
- Also, we need to retain many terms that we dropped in the incompressible governing equations.
- Remember, if we drop the time derivative in the governing equations, we are dealing with steady turbulence or RANS simulations.
- On the other hand, if we keep the time derivative, we are dealing with unsteady turbulence or URANS simulations.

Incompressible Navier-Stokes RANS equations – Additional comments

- If you can afford it, ensemble averaging is recommended.
  - If you have a time series that is periodic or oscillatory (e.g., time signal of the forces), you can use ensemble averaging. In this case, ensemble averaging consists in computing the statistics in the span of many cycles or periods.
  - You can also do ensemble averaging by conducting several simulations (or realizations) and averaging the bulk of the results, but this is expensive.
    - And as CFD is deterministic, we should start each realization using different initial conditions and boundary conditions fluctuations to obtain different outcomes.
- In CFD, time average (steady or unsteady) is preferred.
- The derivation of the LES equations is very similar, but instead of using averaging, we filter the equations in space, and we solve the time scales (no need to average in time).
- LES/DES models are intrinsically unsteady and three-dimensional.
- We will address LES/DES methods in Lecture 10.

Governing equations – Reynolds averaging

Scalar transport RANS equation

- Evolution and transport of scalar fields, such as temperature, internal energy, species concentration, and so on, can also be modeled in RANS/URANS.
- The exact scalar transport equation of a scalar quantity ϕ (no models used), can be written as follows,

$$\nabla_t \rho\phi + \nabla \cdot \rho\mathbf{u}\phi - \nabla \cdot \Gamma_\phi \nabla\phi = S_\phi$$

- Where Γ_ϕ is the molecular (or laminar) diffusion coefficient.
- To derive the exact scalar transport RANS/URANS equation, we proceed in the same way as for the incompressible exact RANS/URANS Navier-Stokes equations.
- Note that the transport equations of the turbulent quantities take the same form of the general scalar transport equation.

Governing equations – Reynolds averaging

Scalar transport RANS equation

- Let us define the Reynolds decomposition for the velocity vector **u** and the scalar ϕ, such as,

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$$

$$\phi = \overline{\phi} + \phi'$$

- Introducing the Reynolds decomposition and by time averaging the whole equation, we obtain,

$$\nabla_t \rho \overline{\left(\overline{\phi} + \phi'\right)} + \nabla \cdot \rho \overline{(\overline{\mathbf{u}} + \mathbf{u}')\left(\overline{\phi} + \phi'\right)} - \nabla \cdot \Gamma_\phi \nabla \overline{\left(\overline{\phi} + \phi'\right)} = S_\phi$$

Governing equations – Reynolds averaging

Scalar transport RANS equation

- Let us define the following averaging rules,

$$
\begin{aligned}
\overline{\phi'} &= 0, \\
\bar{\bar{\phi}} &= \bar{\phi}, \\
\bar{\bar{\phi}} = \overline{\bar{\phi} + \phi'} &= \bar{\phi}, \\
\overline{\phi + \varphi} &= \bar{\phi} + \bar{\varphi}\,, \\
\overline{\bar{\phi}\varphi} = \bar{\bar{\phi}}\bar{\varphi} &= \bar{\phi}\bar{\varphi}, \\
\overline{\bar{\phi}\varphi'} = \bar{\phi}\overline{\varphi'} &= 0, \\
\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \bar{\phi}}{\partial x}, &\quad \overline{\frac{\partial \phi}{\partial t}} = \frac{\partial \bar{\phi}}{\partial t},
\end{aligned}
$$

$$
\begin{aligned}
\overline{\phi\varphi} &= \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')} \\
&= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'} \\
&= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'} \\
&= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'}, \\
\overline{\phi'^2} &\neq 0, \\
\overline{\phi'\varphi'} &\neq 0, \\
\overline{\int \phi ds} &= \int \bar{\phi} ds
\end{aligned}
$$

The differentiation operator commutes in space and time

Governing equations – Reynolds averaging

Scalar transport RANS equation

- After doing some algebra and using the previously defined averaging rules, we get,

$$\nabla_t \rho \left(\overline{\overline{\phi}} + \cancel{\overline{\phi'}} \right) + \nabla \cdot \rho \left(\overline{\overline{\mathbf{u}}\overline{\phi}} + \cancel{\overline{\overline{\mathbf{u}}\phi'}} + \cancel{\overline{\mathbf{u}'\overline{\phi}}} + \overline{\mathbf{u}'\phi'} \right) - \nabla \cdot \Gamma_\phi \nabla \left(\overline{\overline{\phi}} + \cancel{\overline{\phi'}} \right) = S_\phi$$

- After simplifying, we obtain the following equation,

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \left(\overline{\mathbf{u}}\overline{\phi} + \overline{\mathbf{u}'\phi'} \right) - \nabla \cdot \Gamma_\phi \nabla \overline{\phi} = S_\phi$$

- Rearranging and regrouping, we obtain the exact scalar transport RANS/URANS equation,

$$\nabla_t \rho \overline{\phi} + \nabla \cdot \rho \overline{\mathbf{u}}\overline{\phi} = \nabla \cdot \left(\Gamma_\phi \nabla \overline{\phi} - \rho \overline{\mathbf{u}'\phi'} \right) + S_\phi$$

If we retain this term, we talk about **URANS** equations and if we drop it, we talk about **RANS** equations

Scalar flux turbulent diffusion
This term requires modeling

Governing equations – Reynolds averaging

Scalar transport RANS equation

- The **exact scalar transport** equation is very similar to the **exact scalar transport RANS/URANS** equation.

$$\nabla_t \rho\phi + \nabla\cdot\rho\mathbf{u}\phi - \nabla\cdot\Gamma_\phi\nabla\phi = S_\phi$$

Exact scalar transport equation

Scalar flux

$$\nabla_t \rho\overline{\phi} + \nabla\cdot\rho\overline{\mathbf{u}}\,\overline{\phi} = \nabla\cdot\left(\Gamma_\phi\nabla\overline{\phi} - \rho\overline{\mathbf{u}'\phi'}\right) + S_\phi$$

Exact scalar transport RANS/URANS equation

- The main differences are:
  - All quantities have been averaged in the exact scalar transport RANS/URANS equation (the overbar over the primitive variables).
  - And the appearance of the extra velocity-scalar covariance turbulent diffusion term $\overline{\mathbf{u}'\phi'}$.
  - This extra term is a vector and is called scalar flux.
  - It represents the flux (or flow rate per unit area) of the scalar due to the fluctuating velocity field.

Governing equations – Reynolds averaging

Scalar transport RANS equation

- The scalar flux $\overline{\mathbf{u}'\phi'}$ plays an analogous role to that of the Reynolds stresses in the RANS equations.
- As for the Reynolds stresses, the scalar flux results in a closure problem.
- Even if the mean velocity $\overline{\mathbf{u}}$ is known, the exact scalar transport equation cannot be solved for the transported quantity ϕ, without a prescription for $\overline{\mathbf{u}'\phi'}$.
- We need to model the scalar flux term somehow.
- The most widely used hypothesis/assumption to model the scalar flux vector is the gradient diffusion hypothesis, that we will study in next section.

$$\left(\overline{\mathbf{u}'\phi'}\right) = \begin{pmatrix} \overline{u'\phi'} \\ \overline{v'\phi'} \\ \overline{w'\phi'} \end{pmatrix}$$

Governing equations – Reynolds averaging

Scalar transport RANS equation

- Let us use the following definitions of the transported scalar variable and the diffusion coefficient,

$$\phi = \overline{h} = c_p \overline{T}$$

$$\Gamma_\phi = \alpha = k$$

- Thermal diffusivity.
- In the literature, sometimes is defined as α, and sometimes as *k (do* not confuse TKE).

- After substituting in the exact scalar transport RANS/URANS equation we obtain,

- Turbulent thermal heat flux
- This term requires modeling

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}} \overline{T} = \nabla \cdot \left(k_L \nabla \overline{T} - \rho c_p \overline{\mathbf{u}' T'} \right) + S_T$$

- Molecular thermal diffusivity.
- Also called laminar thermal diffusivity (henceforth the subindex $_L$).

Scalar transport RANS equation

- This is the turbulent heat transfer equation (or energy equation) with constant density and negligible viscous dissipation.

$$\nabla_t \rho c_p \overline{T} + \nabla \cdot \rho c_p \overline{\mathbf{u}}\overline{T} = \nabla \cdot \left(k_L \nabla \overline{T} - \rho c_p \overline{\mathbf{u}'T'} \right) + S_T$$

- Note that the energy equation for high-speed compressible flows (the most general form), is more complex than this one, as it involves complex interactions between different terms.
- Nevertheless, the steps to follow to arrive to the turbulent formulation are similar.
- As for the exact Navier-Stokes RANS/URANS equations, we need to introduce models to close the system.
- In this case, we need to model the turbulent thermal flux (the circled term).

Governing equations – Reynolds averaging

Final comments

- To model the Reynolds stress tensor in the exact Navier-Stokes RANS/URANS equations we can use the Boussinesq assumption.

$$-\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t \bar{\mathbf{S}}^R - \frac{2}{3}\rho k \mathbf{I}$$

- Similarly, to model the turbulent scalar flux in the exact scalar transport equation we use the gradient diffusion hypothesis.

$$-\rho\overline{\mathbf{u}'\phi'} = \Gamma_T \nabla\overline{\phi}$$

- Both hypotheses (or assumptions) relate a single covariance to a single gradient.
- We will study both hypotheses in the next section.