Turbulence and CFD models: Theory and applications

Roadmap to Lecture 3

1. **Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales to integral scales**
2. **Energy spectrum and energy cascade. Integral length scale and grid length scale**
3. **Turbulence near the wall - Law of the wall**
4. **A glimpse to a turbulence model**

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Turbulence modeling – Scales of turbulence. From Kolmogorov scales to Taylor microscales to integral scales

- We are going to derive a few relations.
- Many of the derivations are based on dimensional analysis. So, I invite you to dust your notes.
- At this point, let me remind you a few base and derived quantities that we will use.

Base quantity	Symbol	Dimensional units	SI units
Length	-	L	m
Mass	-	M	kg
Time	-	T	s

Turbulence modeling – Scales of turbulence. From Kolmogorov scales to Taylor microscales to integral scales

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Derived quantity	Symbol	Dimensional units	SI units
Velocity	-	LT^{-1}	m/s
Density	ρ	ML^{-3}	kg/m^3
Kinematic viscosity	ν	L^2T^{-1}	m^2/s
Dynamic viscosity	μ	$ML^{-1}T^{-1}$	kg/m-s
Energy dissipation rate per unit mass	ϵ	L^2T^{-3}	m^2/s^3
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Wavenumber	κ	L^{-1}	1/m
Energy spectral density per wavenumber	$E(\kappa)$	L^3/T^2	m^3/s^2
Stress	τ	$ML^{-1}T^{-2}$	$kg/(m\text{-}s^2)$ or N/m^2 or Pa

Turbulence modeling – Scales of turbulence

- Let us recall our definition of turbulence.
  - Unsteady, aperiodic motion in which all transported quantities fluctuate in space and time.
  - Every transported quantity shows similar fluctuations (pressure, temperature, species, concentration, and so on)
  - Turbulent flows contains a wide range of eddy sizes (scales):
    - Large eddies derives their energy from the mean flow and are anisotropic. These eddies are unstable and they break-up into smaller eddies.
    - At the scales of the smallest eddies, the turbulent energy is dissipated. The behavior of the small eddies is more universal in nature.
    - In between the large and small eddies, there are some intermediate scales (that will not address for the moment).
    - The turbulent kinetic energy is transferred from the largest eddies to the smallest ones.

Turbulence modeling – Scales of turbulence

- The instantaneous fluctuations are random both in space and time.
- Therefore, they are difficult and expensive to resolve as they require fine meshes and small time-steps.
- So, the question is, how can we determine/characterize the scales in turbulence, in particular, the smallest scales?
- Also, how the turbulent energy is transferred from the large eddies to the smallest ones? What is the mechanism?
- These questions and more, are answered (to some extension) using Kolmogorov's universal equilibrium theory or K41 (the K stand for Kolmogorov and 41 for the year when this theory was proposed – 1941 –).

Turbulence modeling – Scales of turbulence

- It is important to stress that the energy transfer mechanism still is not well understood.
- Kolmogorov's theory does not answer all questions.
- In fact, we can argue that this theory is very speculative.
- There are many references that aim at rebuking this theory (experimentally and numerically).
- Nevertheless, the K41 theory gives a good picture that have been confirmed with experimental and numerical measurements.
- This theory is not definitive, and as such, it cannot be written in stone.
- Turbulence is too difficult to be addressed by one single (and questionable) theory.
- By the way, to introduce many concepts we are going to make many assumptions that sometimes are difficult to digest.

Turbulence modeling – Scales of turbulence

- It is important to emphasize that the energy cascade concept plays an important role in the Kolmogorov theory and the study of turbulence.
- The energy cascade process, states that the turbulent energy is transferred to successively smaller and smaller eddies.
- This process continues until the Reynolds number of the eddies is sufficiently small so that the eddy motion is stable, and molecular viscosity is effective in dissipating the kinetic energy.
- As previously stated, this mechanism is not well understood.
- In fact, it has been found that energy transfer can go from the smallest scales to the largest scales. This is known as backscatter.
- The energy cascade was memorably described in this poem by Lewis F. Richardson in 1922 [1].

"Big whorls have little whorls,
which feed on their velocity,
and little whorls have lesser whorls,
and so on to viscosity"

[1] L. F. Richardson. "Weather Prediction by Numerical Process". Cambridge University Press, 1922.

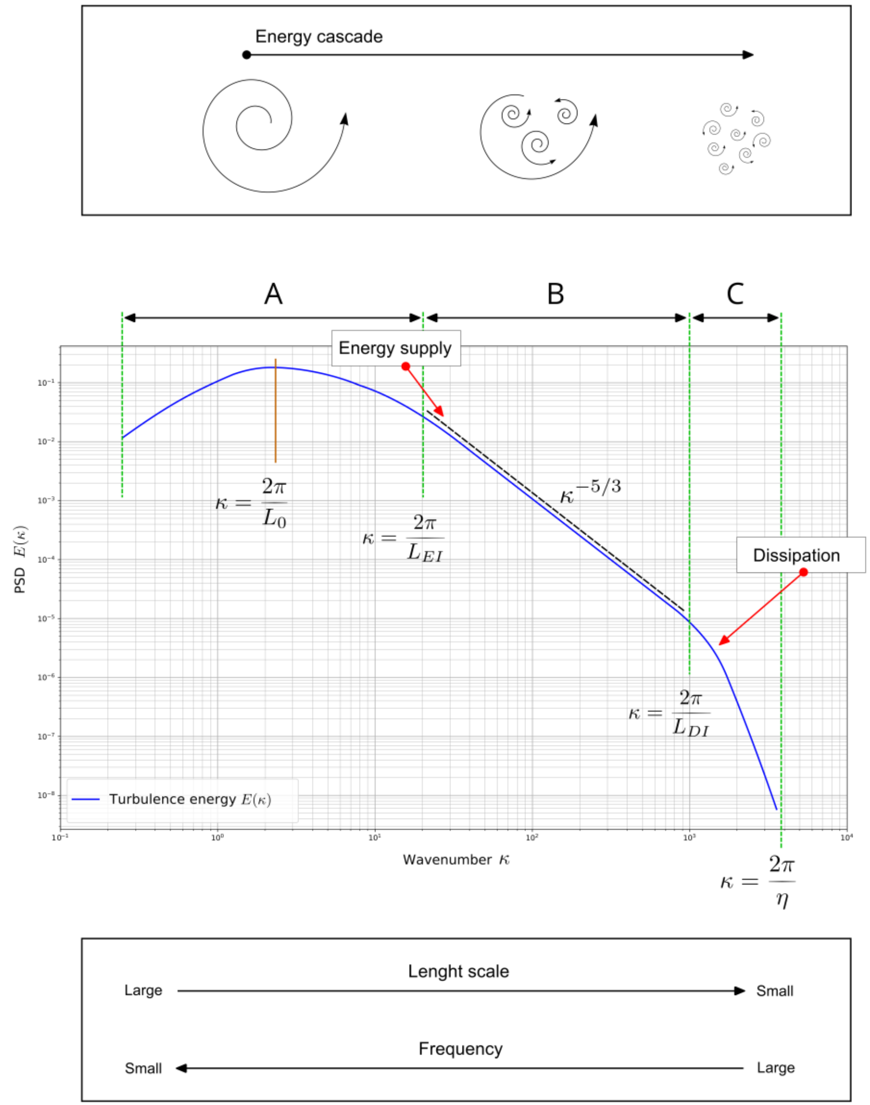
Turbulence modeling – Scales of turbulence

- Kolmogorov's universal equilibrium theory (K41) was originally stated in the form of three hypotheses, which are summarized hereafter:
  - **Kolmogorov's hypothesis of local isotropy.** The small-scale turbulent motions are statistically isotropic.
  - **Kolmogorov's first similarity hypothesis.** The small-scale motions have a universal form that is uniquely determined by ν (viscosity) and ϵ (dissipation).
  - **Kolmogorov's second similarity hypothesis.** The statistics of the motions between the large and small-scales have a universal form that is uniquely determined by ϵ (dissipation), independent of ν (viscosity).
- As a consequence of these hypotheses, the velocity and time scales decrease as the eddies decrease.
- Starting from here, let us derive the Kolmogorov scales and the energy spectrum relationship.
- For a complete description of these hypotheses, the interested reader should refer to references [1, 2].

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[2] H. Tennekes and J. L. Lumley. "A first course in turbulence". MIT Press, 1972.

Turbulence modeling – Scales of turbulence

- Kolmogorov's hypotheses and the energy cascade can be explained better by using a plot of the turbulent energy spectrum.
- Since turbulence contains a large spectrum of scales, it is convenient to plot the scales in terms of the spectral distribution of energy.
- The turbulent power spectrum represents the distribution of the turbulent kinetic energy across the various length scales.
- In general, a spectral representation is a Fourier decomposition into wavenumbers κ , or, equivalently, wavelengths.
- In this plot, large wavenumbers corresponds to small eddies with large frequencies.
- In a similar way, small wavenumbers corresponds to large eddies with small frequencies.
- We will study later how to reproduce this plot from numerical simulations or experimental measurements.

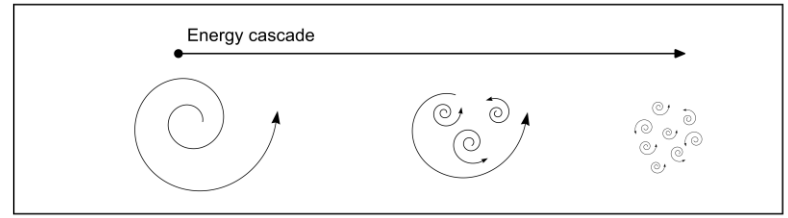


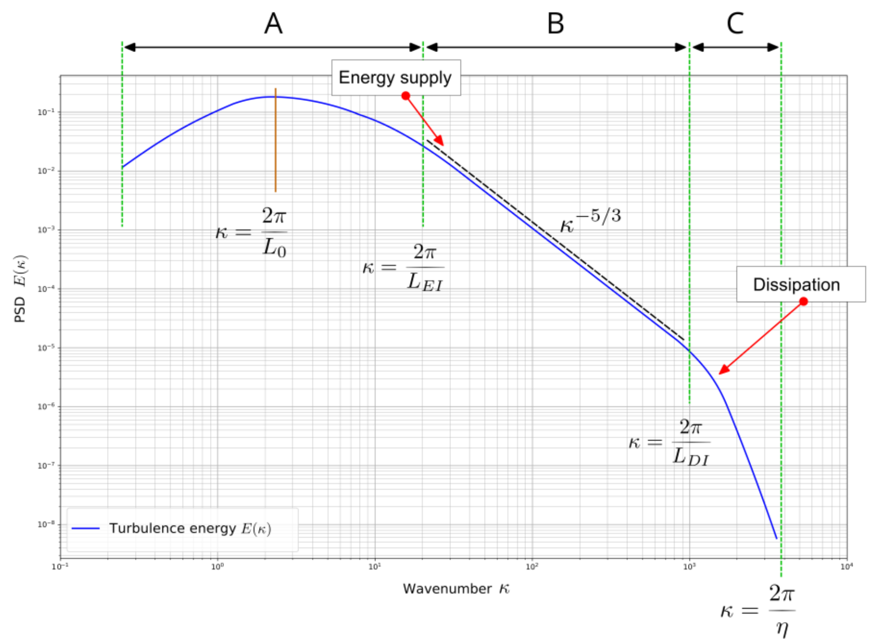
Turbulence modeling – Scales of turbulence

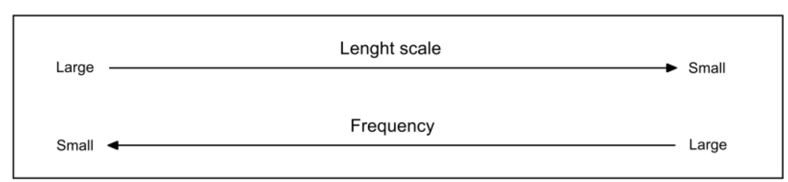
- According to this plot, the energy containing eddies or L_0 (region A in the figure), supply energy to smaller and smaller eddies.
- This energy supply happens at a constant rate in the inertial sub-range (region B in the figure).
- On dimensional grounds, the energy supply in the inertial sub-range can be expressed as follows (more on this later),

$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- The existence of this inertial sub-range has been verified by many physical experiments and numerical simulations.
- When the <u>eddies' Reynolds number</u> (Re_T) is sufficiently small, molecular viscosity is effective in dissipating the kinetic energy (region C in the figure).
- Dissipation of the turbulent kinetic energy happens at the smallest scales or η (Kolmogorov scales).







Turbulence modeling – Scales of turbulence

- Summarizing the previous hypotheses, and according to Kolmogorov's universal equilibrium theory (K41), the motion at the smallest scales should depend only upon:
  - The rate at which the larger eddies supply energy,

Turbulent dissipation rate → $$\epsilon = -\frac{dk}{dt}$$ ← Turbulent kinetic energy

  - The kinematic viscosity ν.

- Having established that ϵ have the following dimensional units L^2T^{-3}, and ν have the following dimensional units L^2T^{-1}, we can derive the Kolmogorov's scales,

$$\eta \rightarrow \text{Length scale}$$
$$\tau \rightarrow \text{Time scale}$$
$$\upsilon \rightarrow \text{Velocity scale}$$

Turbulence modeling – Scales of turbulence

- By using dimensional analysis and the similarity hypotheses (and a lot of intuition and maybe good luck), Kolmogorov derived the following relations that determine the smallest scales in turbulence (Kolmogorov scales),

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

Length scale

$$\tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}$$

Time scale

$$\upsilon_\eta = (\nu\epsilon)^{1/4}$$

Velocity scale

- These scales are indicative of the smallest eddies, that is, the scales at which the energy is dissipated in turbulent flows.
- Remember, turbulence is a continuum phenomenon; therefore, the Kolmogorov length scale is much larger than any molecular length scale.
- By the way, by simple inspection you can verify that the dimensional groups in the Kolmogorov scales all match.

Turbulence modeling – Scales of turbulence

- From the Kolmogorov scales, we can derive two important expressions.
- The first expression, is given as follows,

$$Re_\eta = \frac{\eta u_\eta}{\nu} = 1$$

- Where Re_η is the Kolmogorov Reynolds number.
- The fact that the Kolmogorov Reynolds number is equal to 1, is consistent with the notion that the energy cascade proceeds to smaller and smaller scales until the Reynolds number is small enough for dissipation due to viscosity to be effective.
- At the Kolmogorov Reynolds number, viscous effects dominate over convective effects.
- Therefore, these small eddies are dissipated at a rate ϵ (turbulent dissipation rate).

Turbulence modeling – Scales of turbulence

- The second expression, is given as follows,

$$\epsilon = \nu \left(\frac{v_\eta}{\eta} \right)^2 = \frac{\nu}{\tau_\eta^2}$$

> - To arrive to this relation, recall that turbulent dissipation rate is equivalent to turbulent kinetic energy per unit time with base units m^2/s^2.
> - Then, look back at Kolmogorov's first similarity hypothesis. According to this hypothesis, the dissipation rate should be a function of $\epsilon = f(\nu, u_\eta, \eta)$
> - Match the dimensional groups and do some algebra.

- Where ϵ is the dissipation rate.
- From this equation, we can get the following relationship,

$$\frac{u_\eta}{\eta} = \frac{1}{\tau_\eta}$$

- This relationship provides a consistent characterization of the velocity gradients of the dissipative eddies.

Turbulence modeling – Scales of turbulence

- The turbulent dissipation rate ϵ represents the dissipated turbulent kinetic energy per unit time.
- Or in other words, the amount of turbulent kinetic energy transformed into heat per unit time.
- The turbulent dissipation rate has the following base units,

$$\frac{m^2}{s^2}\frac{1}{s} = \frac{m^2}{s^3}$$

Base units of turbulent kinetic energy

- Or in terms of velocity, time, and length,

$$velocity^2\left(\frac{1}{time}\right) = velocity^2\left(\frac{velocity}{length}\right) = \left(\frac{velocity^3}{length}\right)$$

Turbulence modeling – Scales of turbulence

- The turbulent kinetic energy k or TKE, is the kinetic energy (per unit mass) associated with the eddies, and it has the following base units,

$$\frac{m^2}{s^2}$$

- Or in terms of velocity, time, and length,

$$velocity^2$$

- We will give later a formal definition of TKE.

Turbulence modeling – Scales of turbulence

- From dimensional analysis, the large scales, or turbulence producing eddies l_0 have length scales characterized by [1,2],

$$l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- The large-scale eddies have time scales of the order of,

$$\tau_0 \sim \frac{l_0}{k^{1/2}} \sim \frac{k}{\epsilon}$$

- Combining these large scales (or integral scales) with the Kolmogorov scales, we can compute the ratio of large scales to small scales.

[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

Turbulence modeling – Scales of turbulence

- Using the subscript $_0$ to denote the largest scales, and combining them with the Kolmogorov scales, we can derive the following relations,

Largest eddies →
Smallest eddies →

$$\frac{l_0}{\eta} \sim Re_T^{3/4} \qquad \frac{\tau_0}{\tau_\eta} \sim Re_T^{1/2} \qquad \frac{u_0}{v_\eta} \sim Re_T^{1/4}$$

- Where Re_T is known as the turbulence Reynolds number and it corresponds to the Reynolds number of the integral length scales or the largest eddies.

$$Re_T = \frac{k^{1/2} l_0}{\nu}$$

- The size of the largest eddies or l_0, is comparable to the flow scale, *i.e.*, it depends on the geometry and boundary conditions.
- The characteristic velocity of the largest eddies or u_0, is on the order of the mean flow velocity.
- From these relations, it can be seen that for large Re_T (turbulent flows), the length, time, and velocity scales of the smallest eddies are small compared to those of the largest eddies.

Turbulence modeling – Scales of turbulence

- The turbulence Reynolds number Re_T corresponds to the Reynolds number of the integral length scales or the largest eddies.
- Re_T is defined as follows,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu}$$

- This is not any more the Re of the system.
- It is related to the Re of the integral length scales l_0 or the large eddies.
- And where l_0 can approximated as follows,

$$l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- In turbulent flows, the Reynolds number of the largest eddies is high (convection is dominant) and the effects of viscosity are small.
- But as the eddies become smaller and smaller, the effects of viscosity becomes dominant.
- The system Reynolds number Re usually is 10 to 1000 times larger than the Re_T.
- Or from another point of view, Re_T is few orders of magnitude lower than the characteristic Reynolds number.

Turbulence modeling – Scales of turbulence

- Re_T is defined as follows,

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- And where l_0 can approximated as follows,

$$l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- In the previous definition, k is the turbulent kinetic energy (or TKE) and is defined as,

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

- The overbar means time-average.
- The prime indicates that we are dealing with fluctuations.

- And l_0 are the integral scales that can be estimated using turbulence models, correlations, or physical experiments (for example, via two points correlations measurements).

Turbulence modeling – Scales of turbulence

- For turbulent flows at high Reynolds number, dimensional analysis suggests, and measurements confirm that k (or TKE) can be expressed in terms of ϵ and l_0 (turbulence length scale) as follows [1],

$$k \sim (\epsilon l_0)^{2/3} \quad \Longrightarrow \quad \epsilon \sim \frac{k^{3/2}}{l_0} \quad \Longrightarrow \quad l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- Where the TKE can be computed as follows,

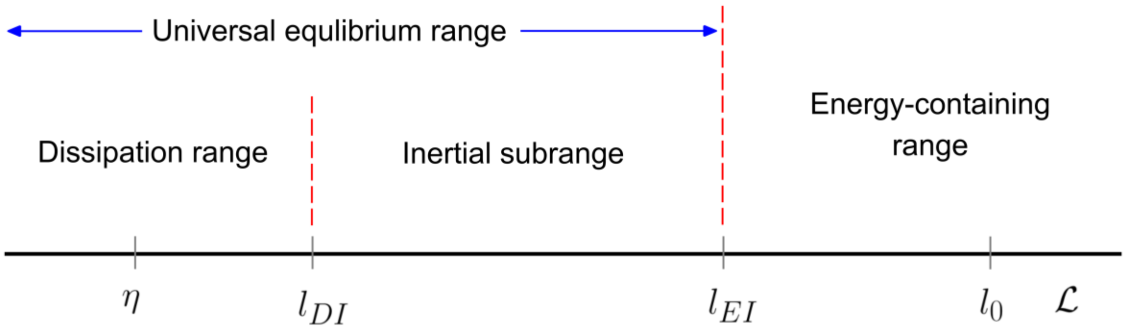
$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

- The TKE is related to the velocity fluctuations, and it is anisotropic (all components are different).
- We will talk a lot more about TKE later.

[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.

Turbulence modeling – Scales of turbulence

- So far, we addressed the scales in the energy-containing range (large eddies) and the scales in the dissipation range (Kolmogorov).
- Between these two ranges there are many eddies that are too small to behave as integral length scales and too large to behave as Kolmogorov eddies.
- This range is known as the inertial range where the motion of the eddies are determined by inertial effects (viscous effects are negligible).
- In this range, the second Kolmogorov 's similarity hypothesis is valid.
- The eddies found in this range are characterized by the Taylor microscales.



- l_{EI} represents the end of the energy-containing range and the beginning of the inertial sub-range.
- l_{DI} represents the beginning of the dissipation range and the end of the inertial sub-range.
- The energy is transferred at a constant rate from l_{EI} to l_{DI} .
- 4/5 of the energy is contained in the energy-containing range.
- 1/5 of the energy is contained in the inertial subrange.
- The extreme l_{EI} is characterized by TKE.
- The extreme l_{DI} is characterized by dissipation.
- Taylor scales can be seen as hybrid eddies.

For a more detailed explanation refer to:
S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Turbulence modeling – Scales of turbulence

- Let us define the Taylor microscales.

$$\lambda = \left(\frac{10\nu k}{\epsilon}\right)^{1/2}$$

Length scale

$$\tau_\lambda = \left(\frac{15\nu}{\epsilon}\right)^{1/2}$$

Time scale

$$u_\lambda = \frac{\lambda}{\tau_\lambda}$$

Velocity scale

- Remember, these scales are contained between the integral scales and the Kolmogorov scales (inertial subrange).
- The Taylor microscales ratio between the largest scales and smallest scales can be computed as follows,

$$\frac{\lambda}{l_0} = \sqrt{10} Re_T^{-1/2}$$

$$\frac{\lambda}{\eta} = \sqrt{10} Re_T^{1/4}$$

Recall that,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu}$$

Note: the constants' values might change from author to author, nevertheless, they are close.

Turbulence modeling – Scales of turbulence

- Because the Taylor microscale is generally too small to characterize large eddies and too large to characterize small eddies, it has generally been ignored in most turbulence modeling research [1].
- The Taylor scale Reynolds number Re_λ is defined as follows,

$$Re_\lambda = \frac{u'\lambda}{\nu} \quad \text{where} \quad k = \frac{2}{3}u'^2$$

This implies some kind of isotropic behavior

- The Taylor scale Reynolds number is related to the turbulent Reynolds number as follows,

$$Re_\lambda = \left(\frac{20}{3}Re_T\right)^{1/2}$$

- We can also relate the timescale of Taylor microscales to the Kolmogorov scales,

$$\frac{\lambda}{u'} = \left(\frac{15\nu}{\epsilon}\right)^{1/2} = \sqrt{15}\tau_\eta \quad \text{where} \quad k = \frac{2}{3}u'^2$$

[1] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

Turbulence modeling – Scales of turbulence

- Some additional relationships,

$$\epsilon = \frac{15 \nu u'^2}{\lambda^2}$$

$$\lambda = \left(\frac{10 \nu k}{\epsilon} \right)^{1/2} = \sqrt{10} \eta^{2/3} l_0^{1/3}$$

$$u_\lambda = \frac{\lambda}{\tau_\lambda} = \left[\left(\frac{10 k \nu}{\epsilon} \right) \left(\frac{\epsilon}{15 \nu} \right) \right]^{1/2} = \left(\frac{2k}{3} \right)^{1/2}$$

Where,

$$k = \frac{2}{3} u'^2$$

- It might appear that the previous Taylor relations were pulled out of thin air, we will not go into details on the derivation, because they are not used very often.
- In any case, the interested reader should refer to references [1, 2, 3, 4].
- From time to time, you will find the Taylor scales are used to characterize grid turbulence.
- Remember to always check the dimensions.

References:
[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.
[3] P. Davidson. Turbulence. An Introduction for Scientists and Engineers. Oxford University Press, 2015
[4] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.

Turbulence modeling – Implications of scales

- The previous relationships indicate that the range of turbulent scales may span orders of magnitude for high Reynolds number flows.
- Since the Kolmogorov length scale η is much smaller that the large or integral scales (*i.e.,* wing chord, channel height, blockage ratio) associated with the flow of interest, it is easy to see that in numerical simulations a large amount of grid points/cells are required to fully simulate turbulent flows.
- For example, in a direct numerical simulation (DNS), where all scales of turbulence are resolved, the relationship,

$$\frac{l_0}{\eta} \sim Re_T^{3/4}$$

- Implies that the number of grid points/cells in one direction is directly proportional to,

$$Re_T^{3/4}$$

Remember, this is the turbulent Reynolds number which is related to the integral length scales

- Thus, in a DNS simulation the meshing requirements scales proportional to $Re_T^{9/4}$ (to resolve all dimensions), or approximately proportional to Re_T^3 for a single time step.

Turbulence modeling – Implications of scales

- And the number of time steps N_t needed to satisfy a Courant condition of CFL < 1, is on the order of,

Total duration of the simulation → T

$$N_t \sim \frac{T}{\Delta t} \sim \frac{T}{\eta/u} \sim \frac{T}{l_0/u} Re^{3/4}$$

$$\Delta t \sim \Delta x/u \sim \eta/u$$

Recall that,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu}$$

- That is, the number of time steps is in the order of $\mathcal{O}(Re_T^{3/4})$
- And the number of operations required for a DNS simulation is approximately proportional to $N_{xyz}^3 N_t$ (where N_{xyz} is the number of grid points/cells in every dimension).
- So, the computing time scales proportional to Re_T^3.
- Again, these numbers are huge. Even with todays (as of 2022) most advanced supercomputers.
- And this is without considering the energy consumption costs.
- The previous are just estimates that are not written in cement, but they give a good idea of the computational cost of resolving all turbulent scales.

Turbulence modeling – Turbulent kinetic energy

- So far, we used here and there the turbulent kinetic energy k or TKE, without given a formal definition.
- TKE is the kinetic energy per unit mass of the fluctuating turbulent velocity (u', v', w').
- Recall that the base dimensional units of TKE are L^2T^{-2}.
- TKE is associated with the eddies in turbulent flows and can be computed as follows:

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

- The overbar means time-average.
- The prime indicates that we are dealing with fluctuations.

Turbulence modeling – Turbulent kinetic energy

- For high Reynolds number (turbulent flows), dimensional analysis suggests, and measurements confirm, that k can be expressed in terms of ϵ and l_0 (turbulence length scale) as follows [1,2],

$$k \sim (\epsilon l_0)^{2/3} \quad \Longrightarrow \quad \epsilon \sim \frac{k^{3/2}}{l_0}$$

- Where l_0 is a length scale associated with the energy containing eddies ($l_0 >> \eta$), or integral length scales, which can be quantified with the turbulence model or using correlations.

[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

Turbulence modeling – Turbulent kinetic energy

- A naïve explanation of the origin of the TKE equation, can be obtained from the equation of the kinetic energy of an object or moving material volume,

$$K = \frac{1}{2} m v^2$$

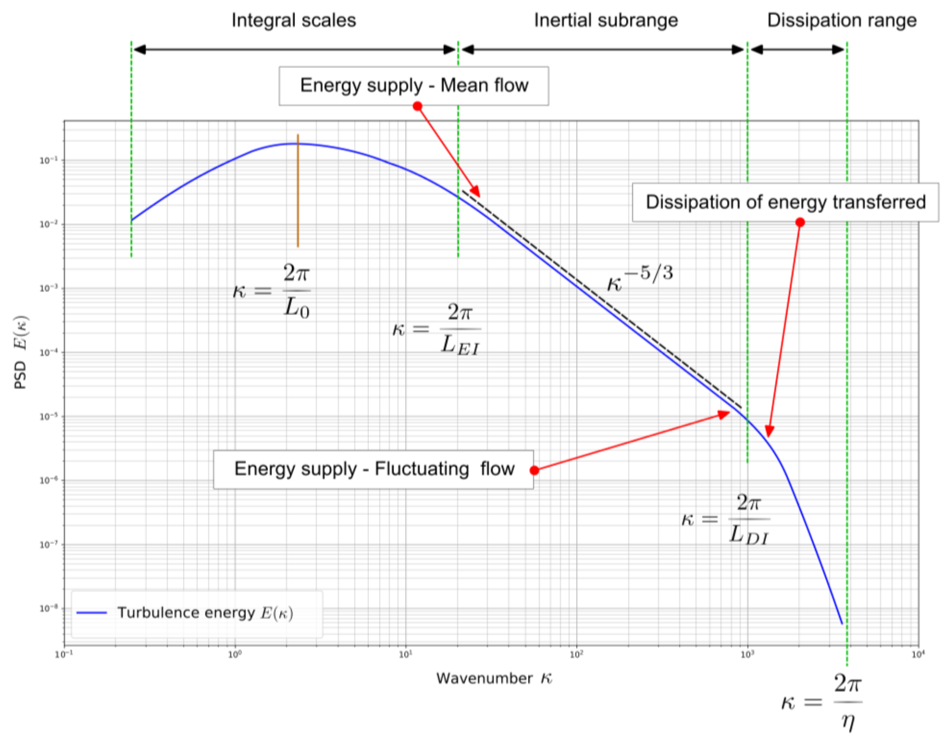
- If you compute *K* for the fluctuating velocity and divide by the mass (per unit mass), you obtain the previous TKE equation, that is,

$$k = \frac{1}{2} \left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right)$$

- As we will see later, TKE can also be calculated from the turbulence model or from the Reynolds stress tensor.
- We will also derive later a transport equation for the TKE.

Turbulence modeling – Turbulent kinetic energy

- Let us talk about the mean kinetic energy (related to the mean flow) and the turbulent kinetic energy (related to the velocity fluctuations or small scales).
- Large scales (mean flow), are very energetic. They supply energy to the flow. Therefore, large scales loss energy.
- This energy is transferred at a constant rate in the inertial subrange.
- The energy loss of the mean flow is an energy gain of the turbulent kinetic energy.
- At the end of the inertial subrange, the turbulent kinetic energy is dissipated.
- It is worth noting that we have assumed the energy transfer follows one direction.
- However, it has been observed flow of energy from small scales to large scales.
- This is known as backscatter.
- Let us take a look at the transport equations of the kinetic energy.



Turbulence modeling – Turbulent kinetic energy

- The transport equation of the mean kinetic energy (large scales), is defined as follows,

$$\frac{\partial \bar{K}}{\partial t} + \bar{U}_j \frac{\partial \bar{K}}{\partial x_j} = \underbrace{-\tau_{ij} \frac{\partial \bar{U}_i}{\partial x_j}}_{P^{\bar{K}}} - \nu \frac{\partial \bar{U}_i}{\partial x_j} \frac{\partial \bar{U}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \left[\left(\nu \frac{\partial \bar{K}}{\partial x_j} \right) - \left(\overline{u_i' u_j'} \bar{U}_i \right) - \left(\frac{1}{\rho} \bar{p} \bar{U}_j \right) \right]$$

$$\bar{K} = \frac{1}{2} \bar{U}_i \bar{U}_i$$

- The transport equation of the turbulent kinetic energy (velocity fluctuations), is defined as follows,

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \underbrace{\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{P^{k}} - \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j} \left[\left(\nu \frac{\partial k}{\partial x_j} \right) - \left(\frac{1}{2} \overline{u_i' u_i' u_j'} \right) - \left(\frac{1}{\rho} \overline{p' u_j'} \right) \right]$$

$$k = \frac{1}{2} \overline{u_i' u_i'}$$

- Notice that both equations are very similar, the main difference is the circled term.
- This term is known as the production term or P.
- In the mean kinetic energy equation, the circle term is negative (loss of energy).
- Whereas, in the turbulent kinetic energy equation is positive (gain of energy).
- We will derive these equations in Lecture 6.

Turbulence modeling – Turbulent kinetic energy

- The transport equation of the mean kinetic energy (large scales), is defined as follows,

$$\frac{\partial \bar{K}}{\partial t} + \bar{U}_j \frac{\partial \bar{K}}{\partial x_j} = -\tau_{ij} \frac{\partial \bar{U}_i}{\partial x_j} - \underbrace{\nu \frac{\partial \bar{U}_i}{\partial x_j} \frac{\partial \bar{U}_i}{\partial x_j}}_{\epsilon} + \frac{\partial}{\partial x_j} \left[\left(\nu \frac{\partial \bar{K}}{\partial x_j} \right) - \left(\overline{u_i' u_j'} \bar{U}_i \right) - \left(\frac{1}{\rho} \bar{p} \bar{U}_j \right) \right]$$

$$\bar{K} = \frac{1}{2} \bar{U}_i \bar{U}_i \qquad \epsilon_{ii} = \epsilon = \nu \frac{\partial \bar{U}_i}{\partial x_j} \frac{\partial \bar{U}_i}{\partial x_j}$$

- The transport equation of the turbulent kinetic energy (velocity fluctuations), is defined as follows,

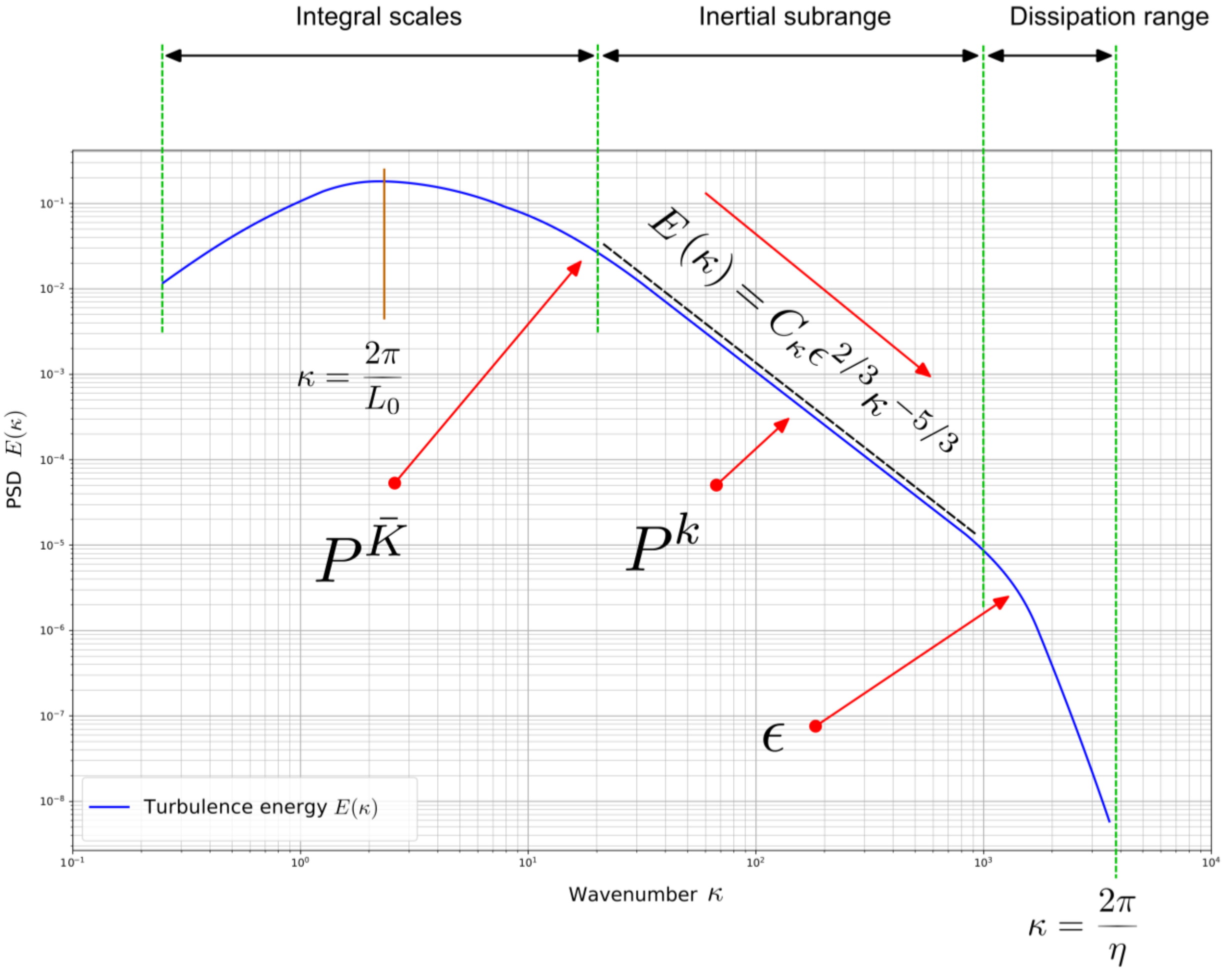
$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}}_{\epsilon} + \frac{\partial}{\partial x_j} \left[\left(\nu \frac{\partial k}{\partial x_j} \right) - \left(\frac{1}{2} \overline{u_i' u_i' u_j'} \right) - \left(\frac{1}{\rho} \overline{p' u_j'} \right) \right]$$

$$k = \frac{1}{2} \overline{u_i' u_i'} \qquad \epsilon_{ii} = \epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$$

- The circled term in both equations represents the dissipation rate ϵ.
- As it is dissipation, it is negative in both equations.
- However, dissipation at the large scales is much smaller than that at small scales, therefore, it is often not considered.
- We will derive these equations in Lecture 6.

Turbulence modeling – Turbulent kinetic energy

- Transfer of kinetic energy in the energy spectrum.



Turbulence modeling – Grid requirements

- We will talk about RANS, DES, LES and DNS simulations later.
- The grid requirements of these turbulence modeling techniques in CFD can be summarize as follows.

  - DNS simulations requires no modeling, but it demands resolution from the large scales all the way through at least the beginning of the dissipation scales.
  - This results in a grid scaling proportional to Re_T^3, or worse.

Turbulence modeling – Grid requirements

- We will talk about RANS, DES, LES and DNS simulations later.
- The grid requirements of these turbulence modeling techniques in CFD can be summarize as follows.

  - LES simulations requires modeling of part of the inertial sub-range and into the beginning of the dissipation scales.
  - The amount of required modeling is set by the grid resolution but is unlikely that the grid will scale worse than Re_T^2 .
  - Even if this requirement appears to be high, they are less than DNS but much more than RANS.
  - LES simulation are starting to become affordable thanks to the advances in supercomputing and algorithms.
  - DES simulations have similar requirements to LES but with some peculiarities. We will talk about DES later.

Turbulence modeling – Grid requirements

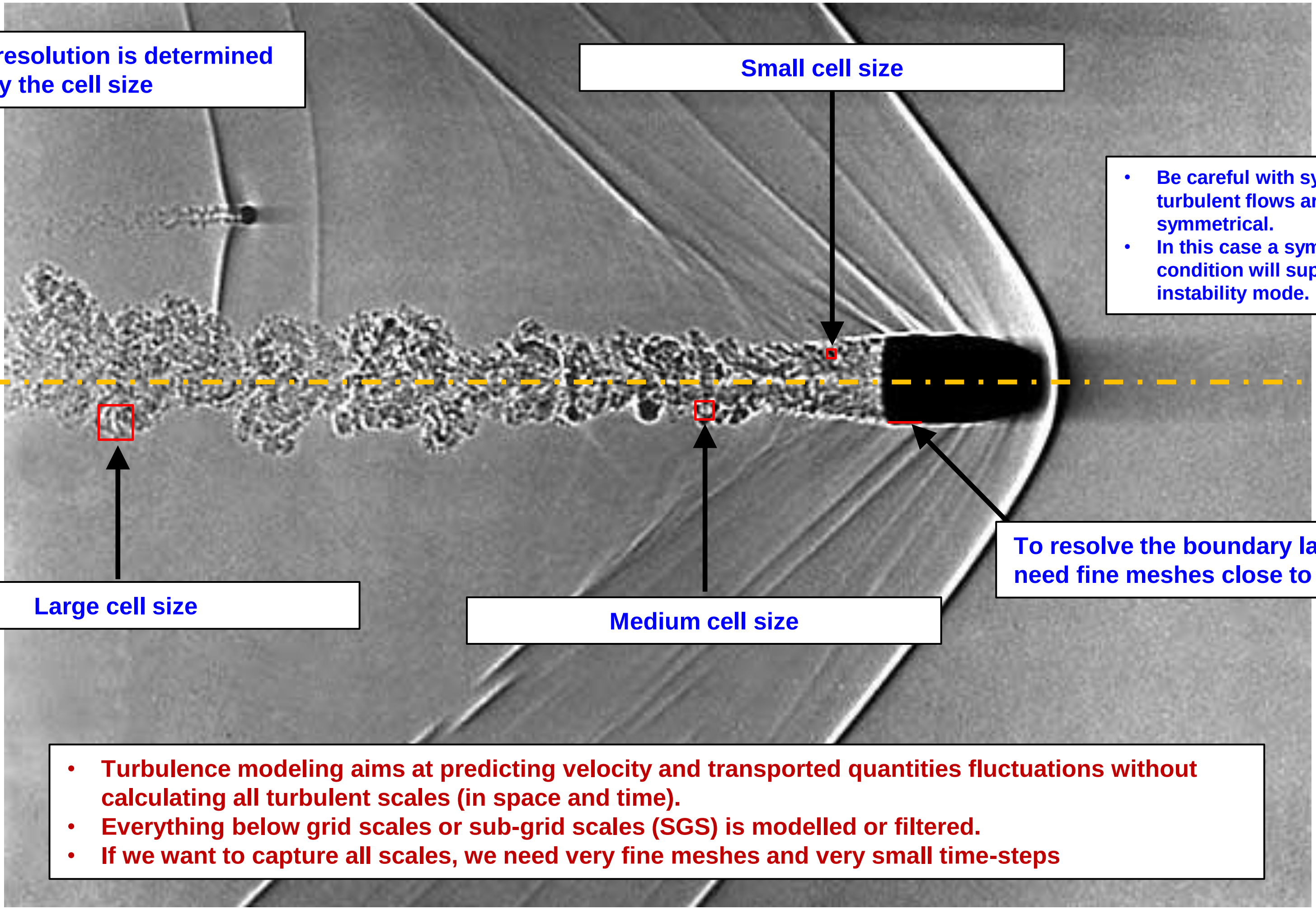
- We will talk about RANS, DES, LES and DNS simulations later.
- The grid requirements of these turbulence modeling techniques in CFD can be summarize as follows.

  - RANS simulations requires modeling of everything from the integral scales into the dissipation range.
  - As a consequence, the grid scaling is a weak function of the Reynolds number.

Turbulence modeling – Grid requirements

- As you can see from the previous requirements, RANS simulations are very affordable (steady and unsteady).
- RANS simulations are the workhorse of turbulence modeling in industrial applications.
  - Steady state RANS simulations will remain the dominant simulation method for turbulent flows for many years.
- RANS models are accurate, robust, fast, mesh insensitive, and valid for a wide range of physics.
- Nevertheless, the use of scale resolving simulations (SRS) for industrial applications is foreseen in the future.
  - It is expected that LES and DES will become more affordable in the years to come.
- DNS is only used in research and for low Reynolds number.
  - DNS simulations are used to calibrate RANS models.

Turbulence modeling – Grid requirements

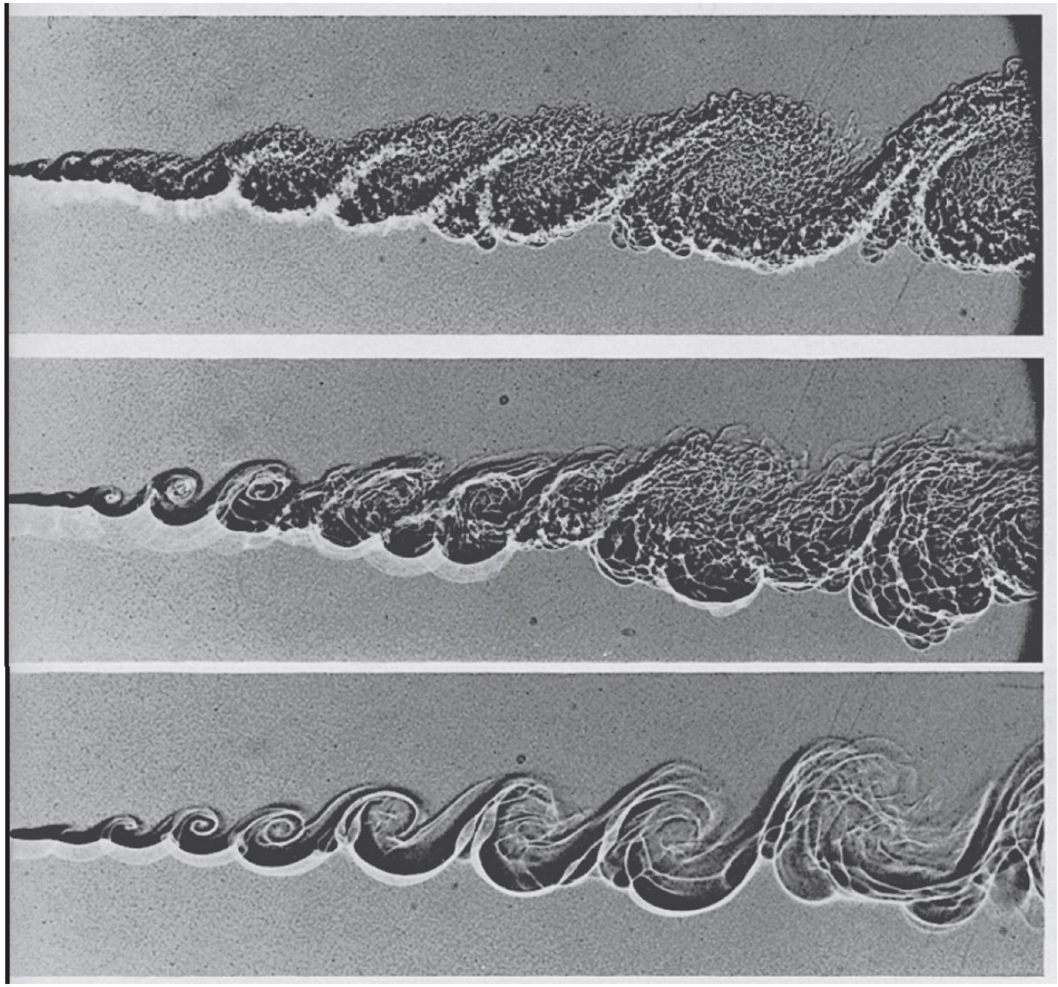


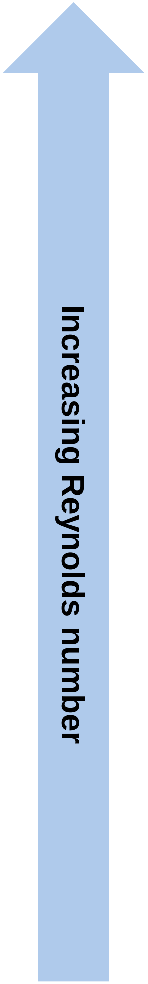
- Turbulence modeling aims at predicting velocity and transported quantities fluctuations without calculating all turbulent scales (in space and time).
- Everything below grid scales or sub-grid scales (SGS) is modelled or filtered.
- If we want to capture all scales, we need very fine meshes and very small time-steps

Bullet at Mach 1.5

Photo credit: Andrew Davidhazy. Rochester Institute of Technology.

Turbulence modeling – Grid requirements





- The large integrals scales are roughly the same as the Reynolds number increases.
- However, at larger Reynolds number there are more microscales.
- To resolve the large integral scales in all images, we can use the same mesh.
- However, to resolve the smallest microscales at large Reynolds number, we need very fine meshes.
- As can be seen, at large Reynolds number, the ratio between the largest and smallest scales is very large.

Photo credit: G. Brown, A. Roshko. 1974. On density effects and large structure in turbulent mixing layers. J. Fluid Mech. 64:775–816.

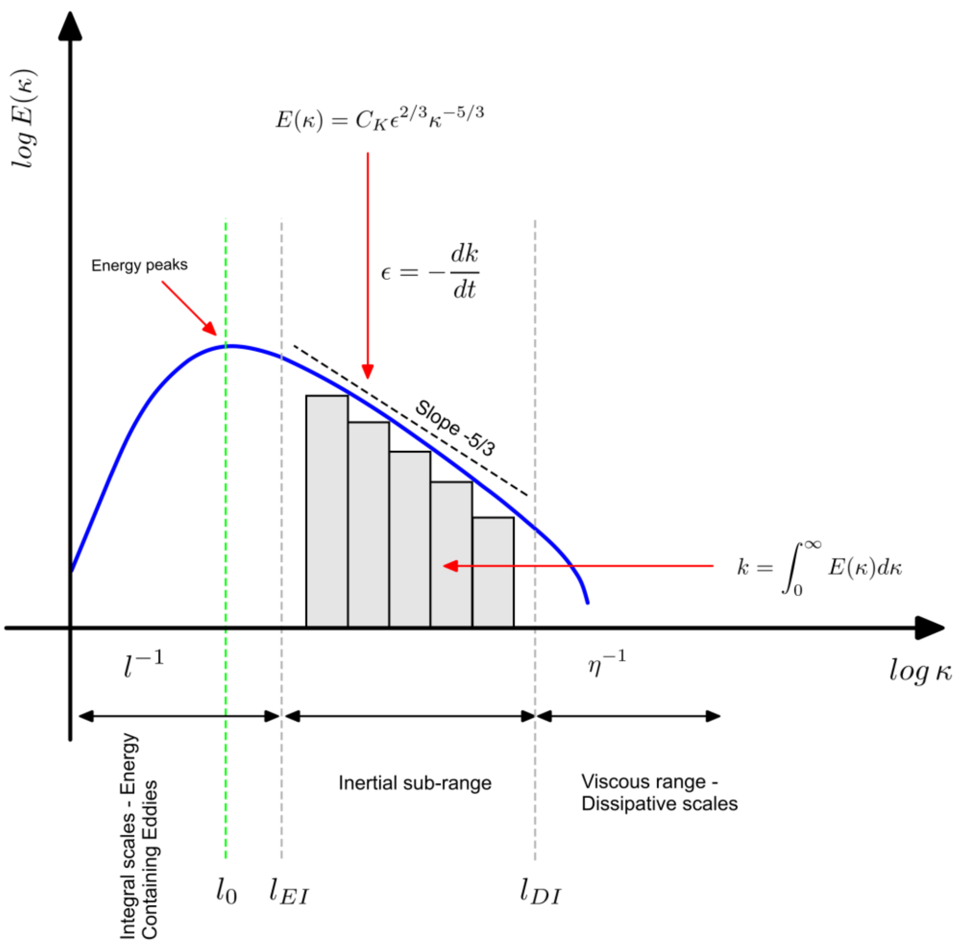
Turbulence modeling – Grid requirements

- As you can see, resolving all turbulence scales in CFD (in space and time) requires a formidable amount of computational power.
- To this, you need to add the IO overhead, storage requirements, and the qualitative and quantitative post-processing, which can be as expensive as the simulations.
- Therefore, the importance of using turbulence models to alleviate the incredible requirements of resolving all turbulence scales.

Roadmap to Lecture 3

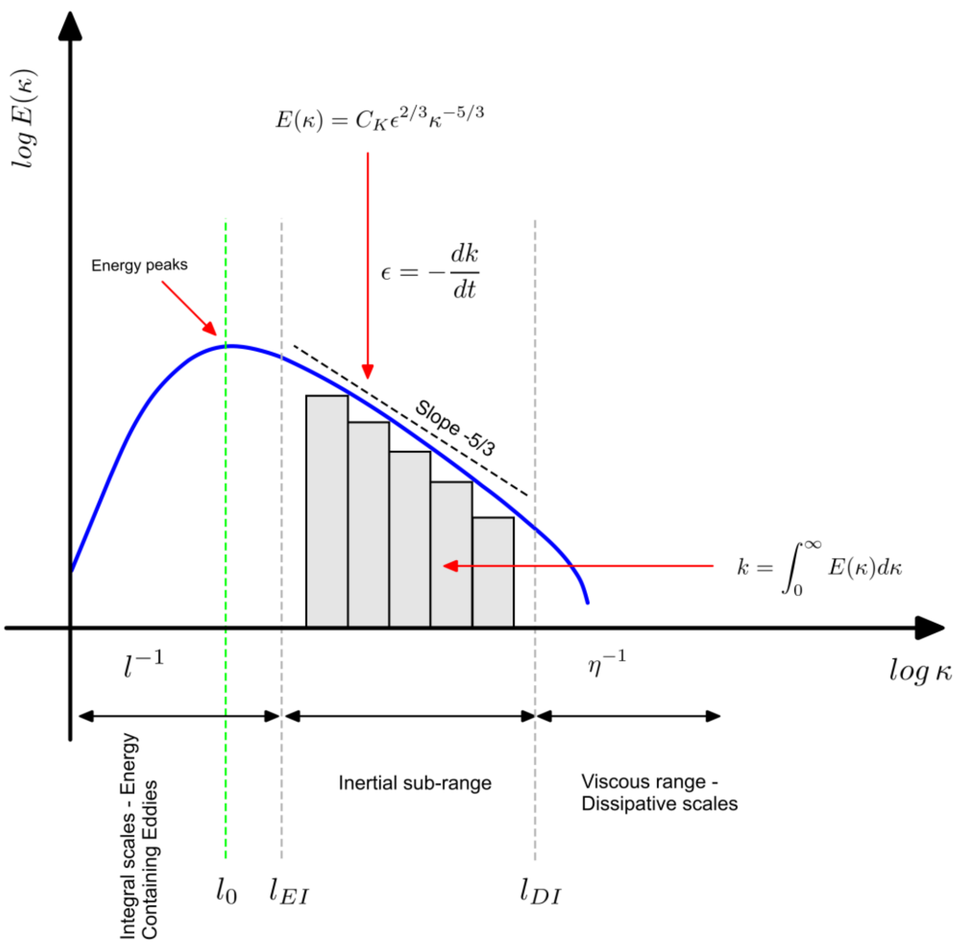
1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales to integral scales~~
2. **Energy spectrum and energy cascade. Integral length scale and grid length scale**
3. Turbulence near the wall - Law of the wall
4. A glimpse to a turbulence model

Energy spectrum and energy cascade



- The existence of a wide separation of scales is a central assumption Kolmogorov made as part of his universal equilibrium theory.
- Kolmogorov hypothesized that there is a range of eddy sizes between the largest and smallest for which the cascade process is independent of the statistics of the energy containing eddies and of the effect of viscosity.
- As a consequence, a range of wavenumbers exists in which the energy transferred by inertial effects, wherefore $E(\kappa)$ depends only upon ϵ and κ.

Energy spectrum and energy cascade

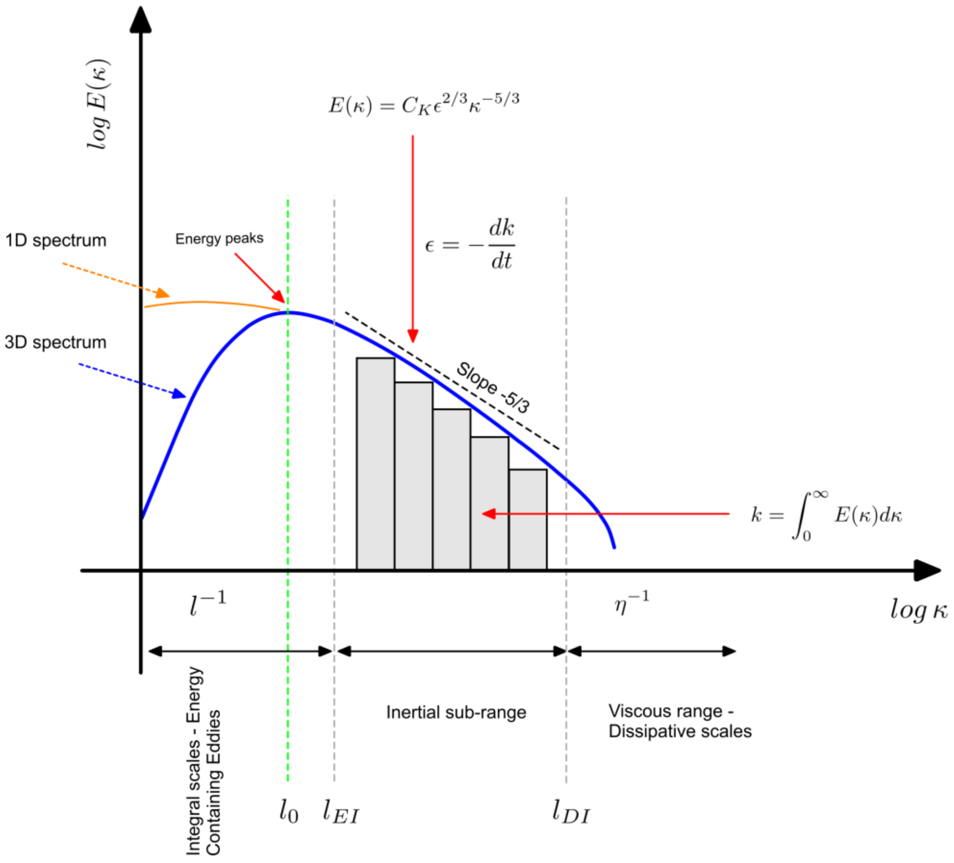


- On dimensional grounds, Kolmogorov concluded that,

$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- This relationship is sometimes known as the Kolmogorov -5/3 law.
- The energy spectrum equation $E(\kappa)$ is fundamental in turbulence modeling.
- In this equation, C_K is the Kolmogorov constant.
- After Kolmogorov formulated this theory, it took a while to find the value of C_K.

Energy spectrum and energy cascade

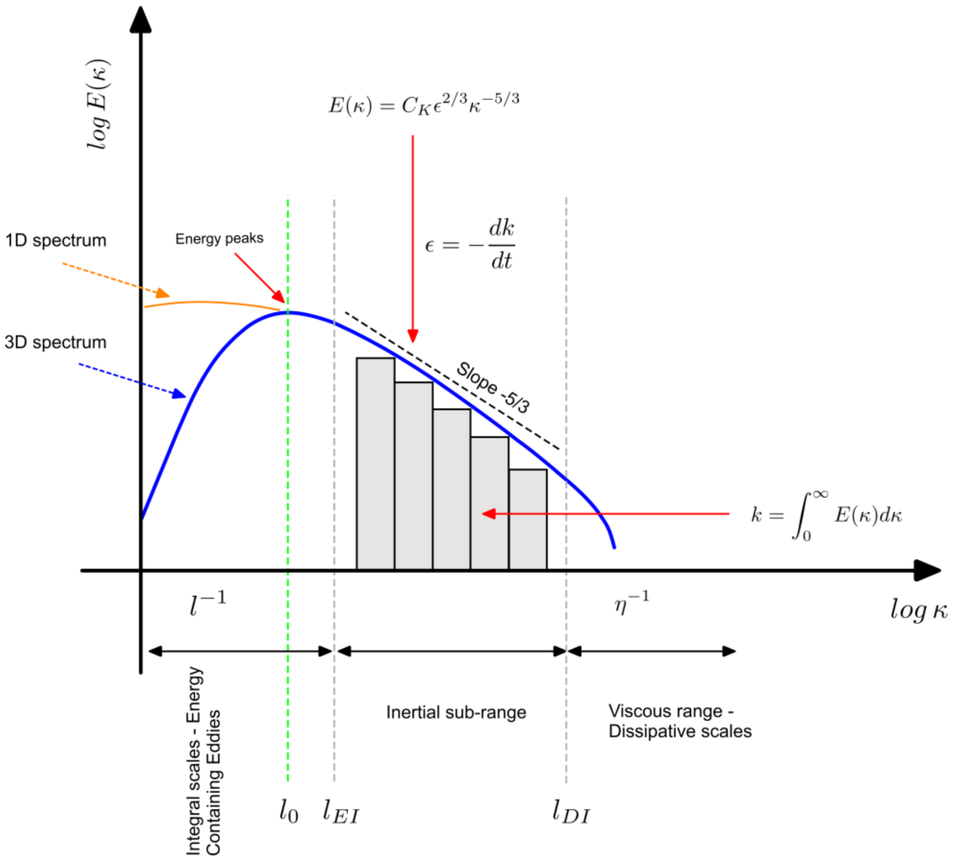


$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- For 3D spectra (using all velocity fluctuation components), the Kolmogorov constant has been found to be anywhere between 1.2 and 1.8 [1, 2].
- The most common value that you will find in literature is 1.5.
- The range of values of the Kolmogorov constant have been confirmed with experimental and numerical (DNS) measurements.

[1] K. R. Sreenivasan. On the universality of the Kolmogorov constant. Physics of Fluids 7, 2778 (1995).
[2] P. K. Yeung, Y. Zhou. On the universality of the Kolmogorov constant in numerical simulations of turbulence. ICASE Report No. 97-64, 1997.

Energy spectrum and energy cascade

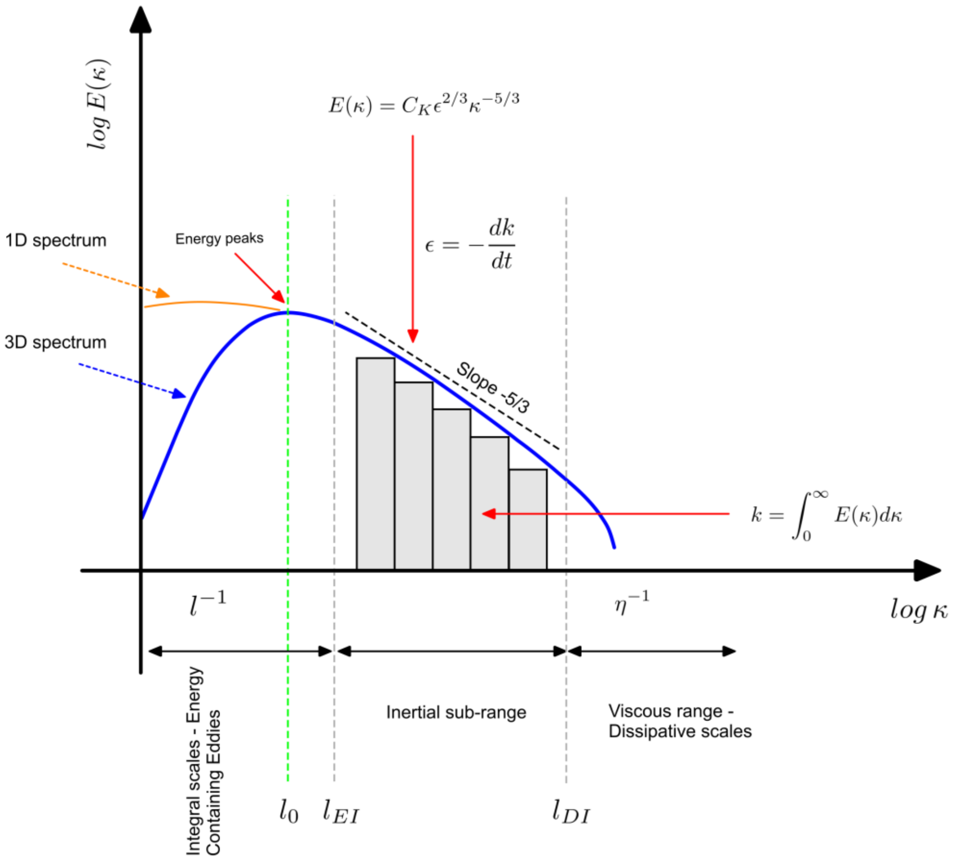


$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- For a 1D spectra, for example, K_{11} or the streamwise component, the Kolmogorov constant has been found to be anywhere between 0.4 and 0.6 [1, 2].
- The most common value that you will find in literature is 0.5.
- For the spanwise component or K_{22}, the values are different.
- But independently of the different values of the Kolmogorov constant, all the spectra show a similar behavior in the inertial range.

[1] K. R. Sreenivasan. On the universality of the Kolmogorov constant. Physics of Fluids 7, 2778 (1995).
[2] P. K. Yeung, Y. Zhou. On the universality of the Kolmogorov constant in numerical simulations of turbulence. ICASE Report No. 97-64, 1997.

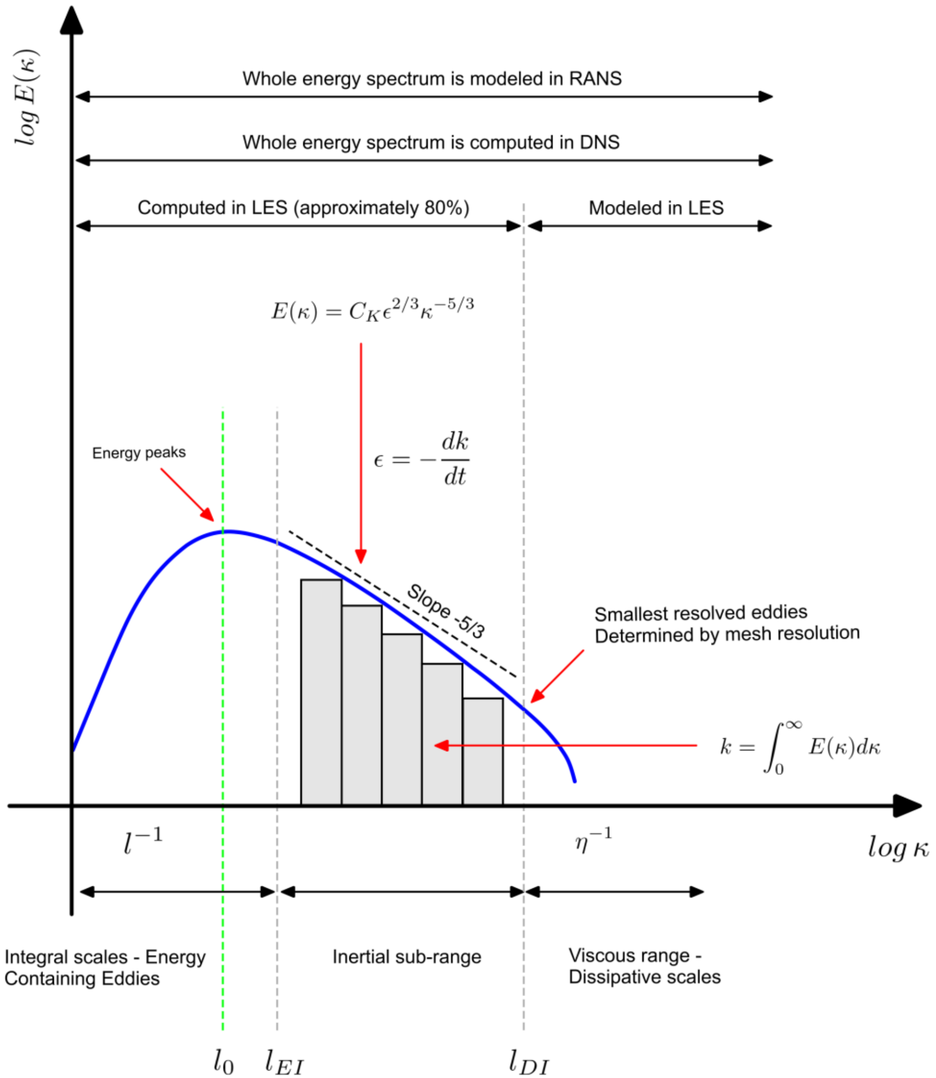
Energy spectrum and energy cascade



- The main difference between the 1D and 3D spectra curves is seen in the range of the integral scales.
- The 1D spectrum does not break down like the 3D spectrum, as shown in the figure.
- Usually, the 1D spectrum is used for physical experiments.
- Why? Because measuring locally all velocity fluctuations (u', v', w') at the same time is not easy in physical experiments.
- Instead, in numerical simulations we can use the 1D or 3D spectrum.

[1] K. R. Sreenivasan. On the universality of the Kolmogorov constant. Physics of Fluids 7, 2778 (1995).
[2] P. K. Yeung, Y. Zhou. On the universality of the Kolmogorov constant in numerical simulations of turbulence. ICASE Report No. 97-64, 1997.

Energy spectrum and energy cascade

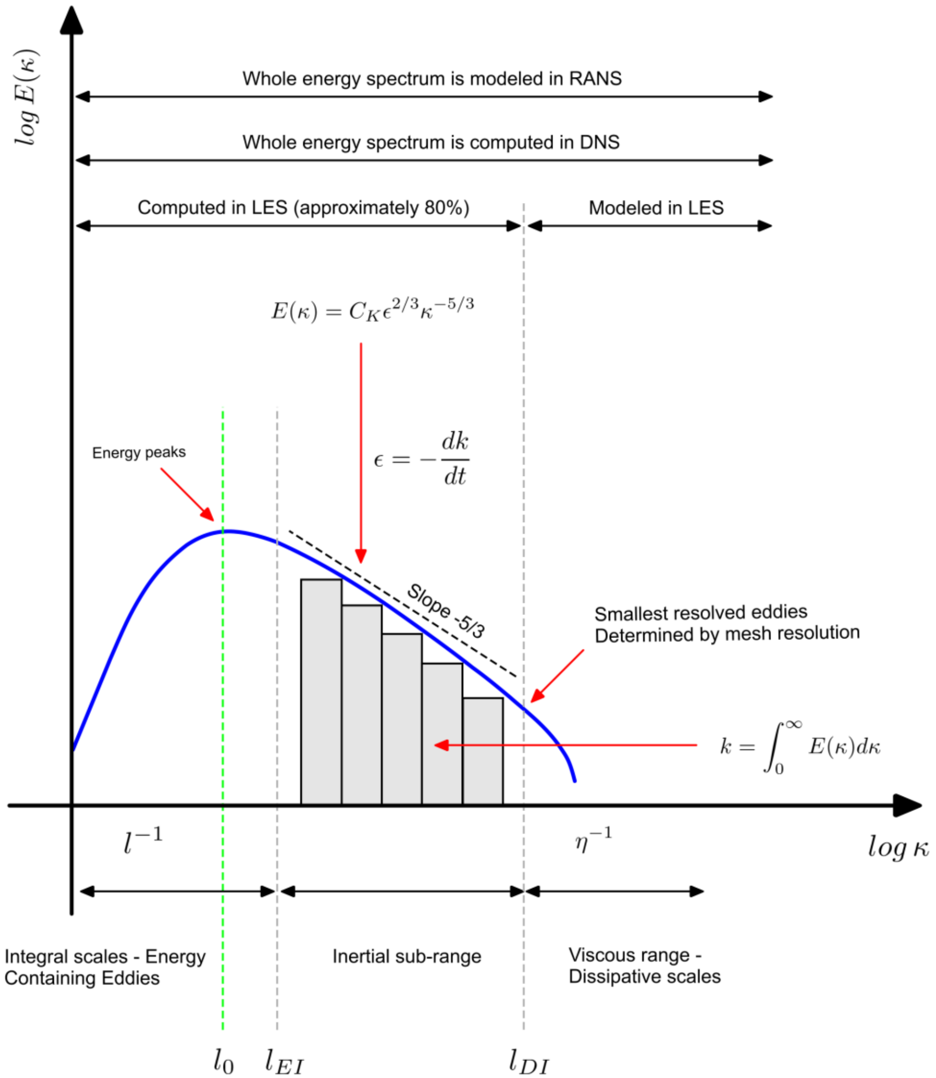


- Note that this type of graph is local.
- It will be different for each and every point in the domain.
- In the x axis the wavenumber is plotted,

$$\kappa = \frac{2\pi}{l} = \frac{2\pi f}{U}$$

- The energy spectrum density or energy spectrum, $E(\kappa)$ is related to the Fourier transform of k.
- The turbulent power spectrum represents the distribution of the turbulent kinetic energy k across the various length scales.
- It is a direct indication of how energy is dissipated with eddies size.

Energy spectrum and energy cascade



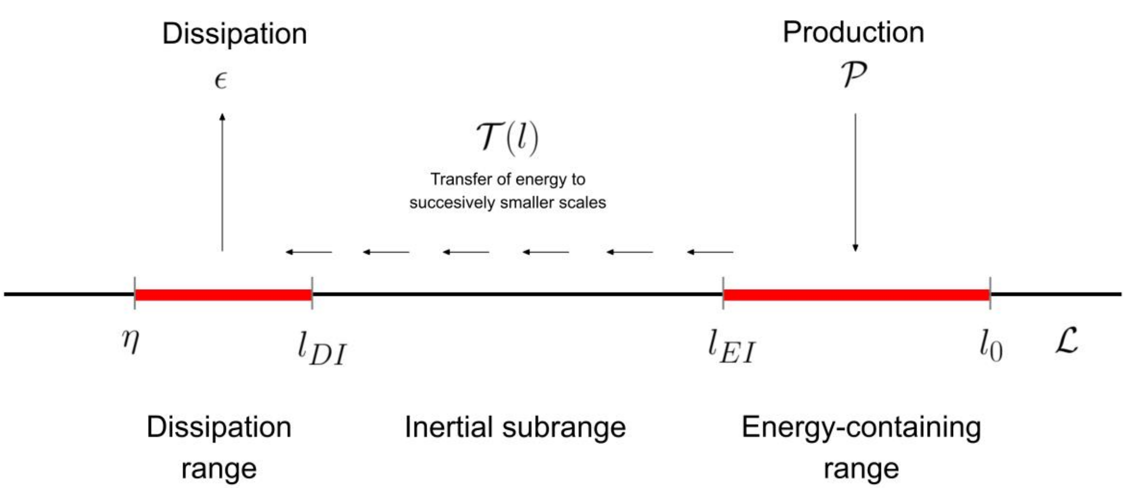
- Note that this type of graph is local.
- It will be different for each and every point in the domain.
- In the x axis the wavenumber is plotted,

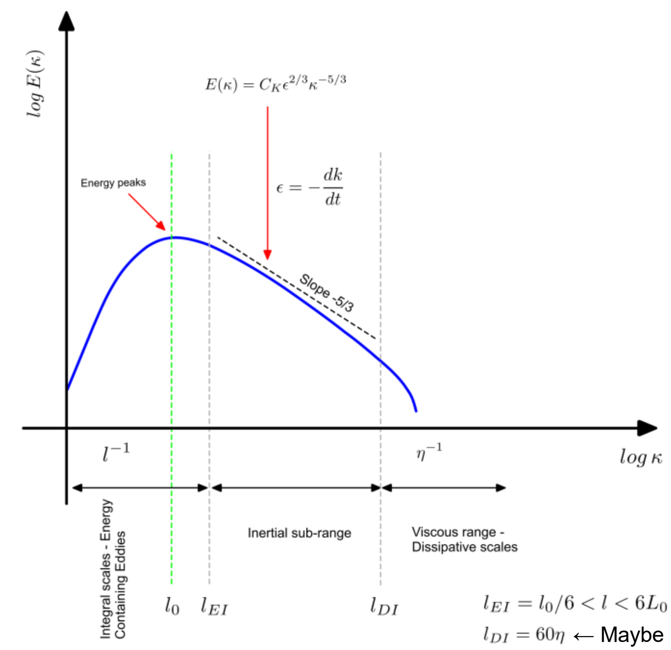
$$\kappa = \frac{2\pi}{l} = \frac{2\pi f}{U}$$

- The mesh resolution determines the fraction of the energy spectrum directly resolved.
- In a DNS simulation, the whole spectrum is resolved.
- In a good LES simulation, approximately 80% of the spectrum is resolved.
- In a URANS simulation, the whole spectrum is modeled.
- In CFD eddies cannot be resolved down to the molecular dissipation limit.

Energy spectrum and energy cascade

$$\mathcal{T}_{EI} \equiv \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \epsilon$$



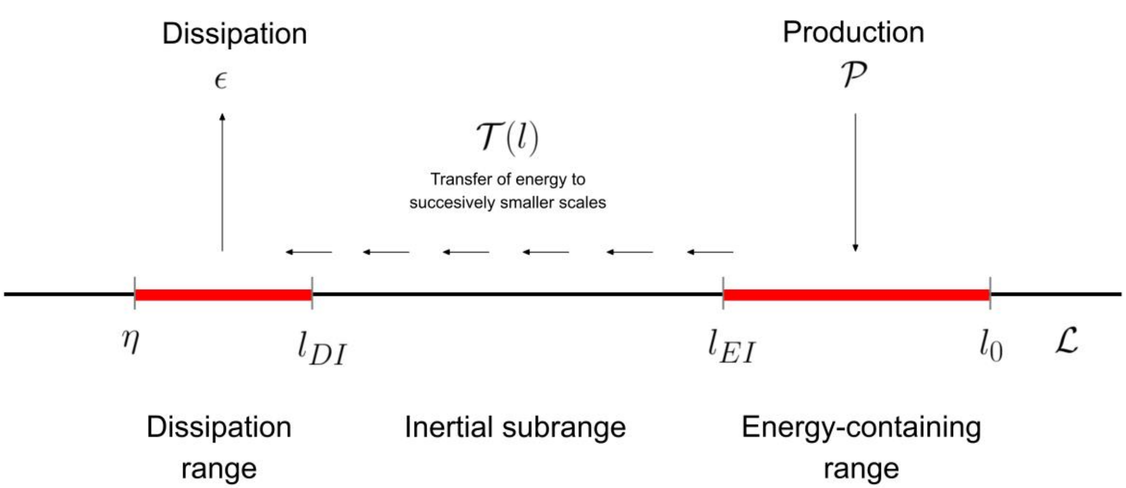


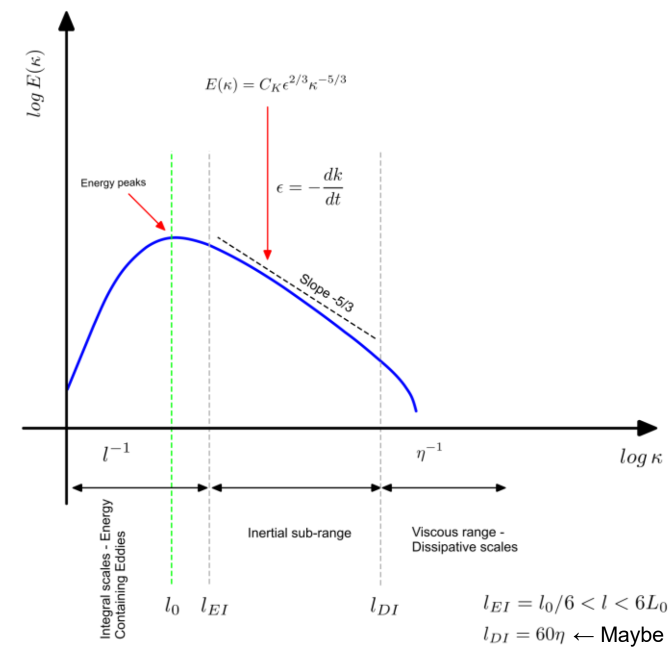
- l_{EI} represents the end of the energy-containing range and the beginning of the inertial sub-range.
- l_{DI} represents the beginning of the dissipation range and the end of the inertial sub-range.
- Under equilibrium conditions, turbulence production is equal to turbulence dissipation.
- The energy is transferred at a constant rate from l_{EI} to l_{DI} .
- The extension of the -5/3 law (the inertial sub-range region) is wider for larger Reynolds number.
- At low Reynolds numbers, it is difficult to distinct between l_{EI} and l_{DI}.

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and energy cascade

$$\mathcal{T}_{EI} \equiv \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \epsilon$$



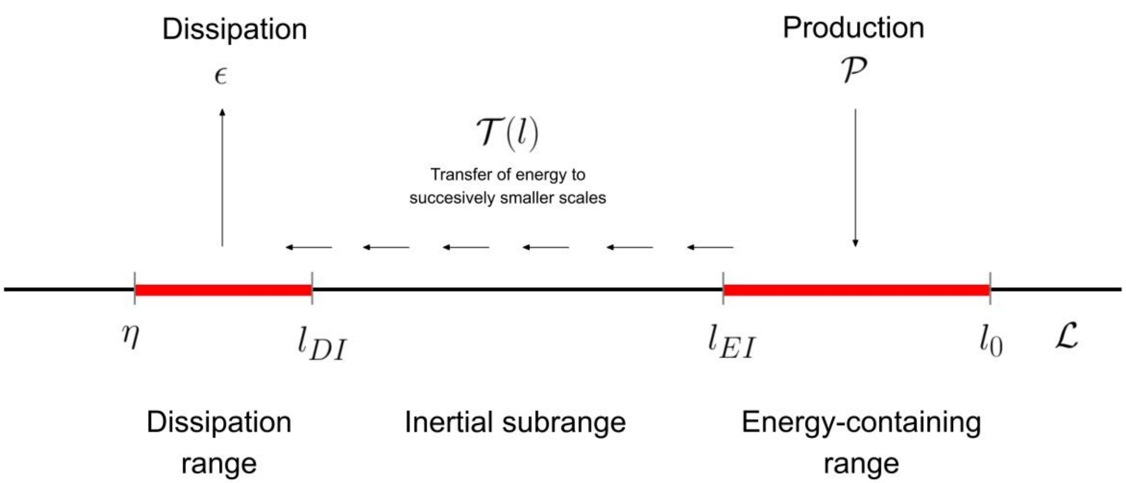


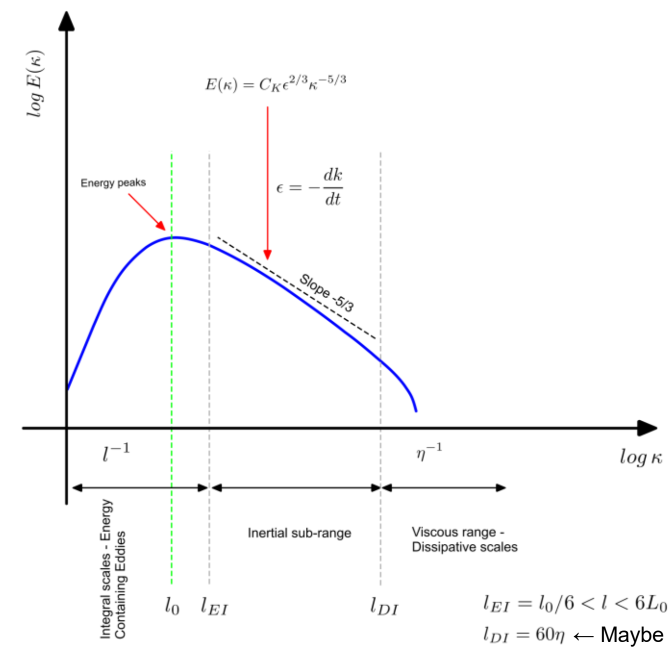
- Note that dissipation takes place at the end of the sequence of this process (l_{DI}).
- Under equilibrium conditions (production = dissipation), the rate of dissipation ϵ is determined by the first process in the sequence (l_{EI}), which is the transfer of energy from the largest eddies.

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and energy cascade

$$\mathcal{T}_{EI} \equiv \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \epsilon$$



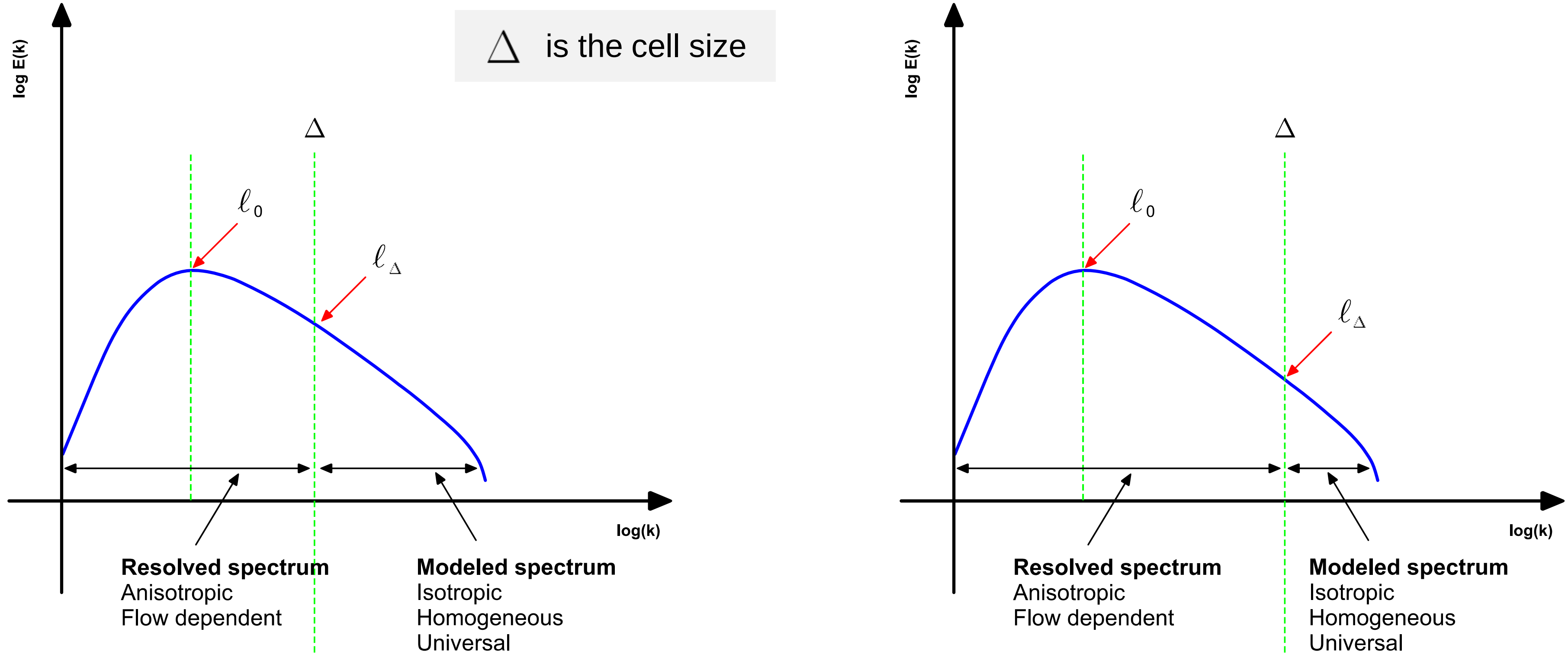


- The energy dissipation rate per unit mass ϵ is given by the following equation, which comes from the transport equation of the turbulent kinetic energy (which we will derive later),

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}$$

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and energy cascade



Coarse mesh – More energy is modeled

Fine mesh – Less energy is modeled

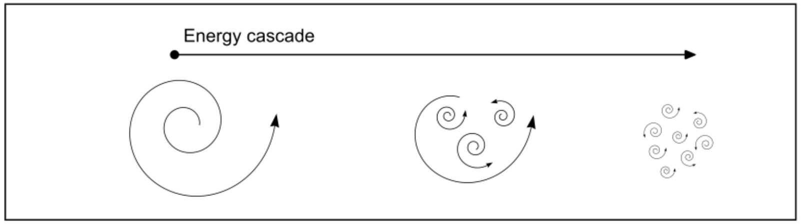
- The finer the mesh the less energy that is being modeled.
- Turbulent kinetic energy peaks at integral length scale l_0.
- In SRS simulations, this scale must be sufficiently resolved.

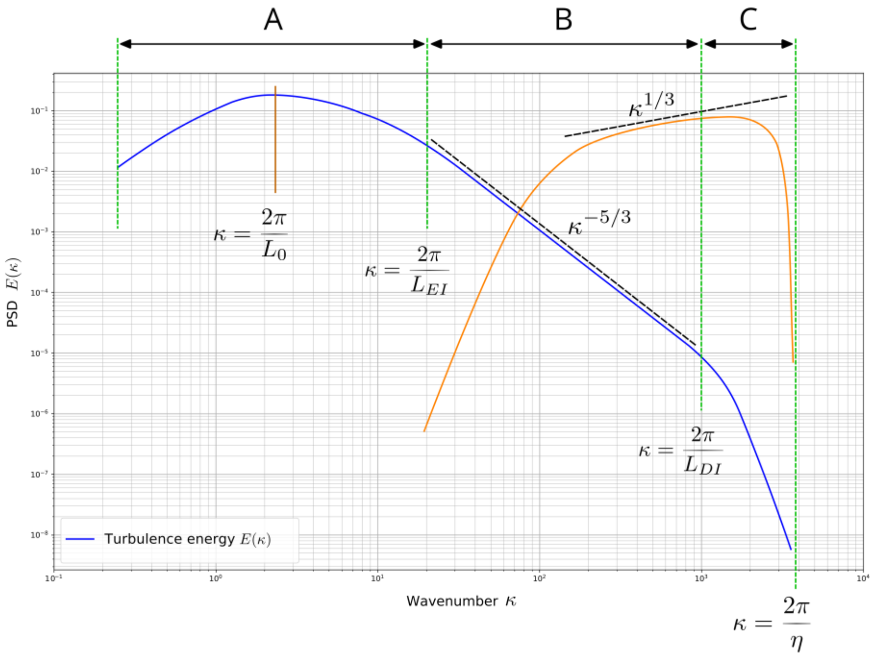
Energy spectrum and energy cascade

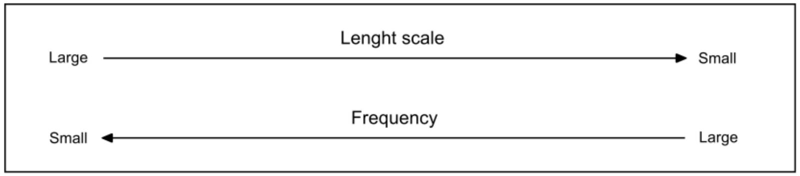
- In the same manner as $E(\kappa)$ describe the wavenumber dependence of the turbulent kinetic energy, another distribution, $D(\kappa)$, describes the dependence of the dissipation rate on the wavenumber in the subrange C (refer to the figure).
- In this range, viscosity plays a central role.
- Again, by using dimensional analysis, we can derive an expression for the dissipation spectrum $D(\kappa)$,

$$D(\kappa) = C_{\epsilon} \nu \epsilon^{2/3} \kappa^{1/3} \qquad \text{with} \qquad C_{\epsilon} \approx 2C_k$$

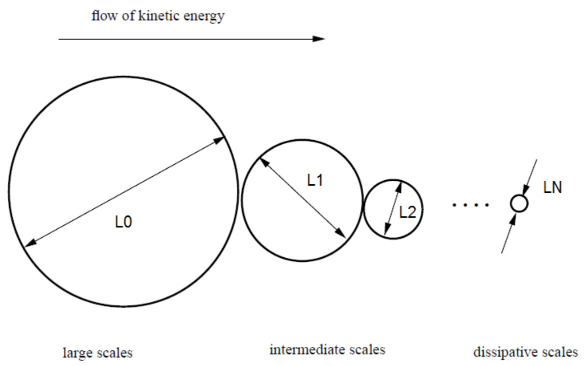
- In the figure, the variation of $D(\kappa)$ in function of the wavenumber is illustrated (orange line).
- As it can be seen, the dissipation starts to rise in the inertial subrange (B in the figure), and then declines rapidly (in the subrange C in the figure).
- Similar observations to those of the $E(\kappa)$ spectrum apply to the $D(\kappa)$ spectrum, namely, slope, determination of the constant, experimental observations, and so on.

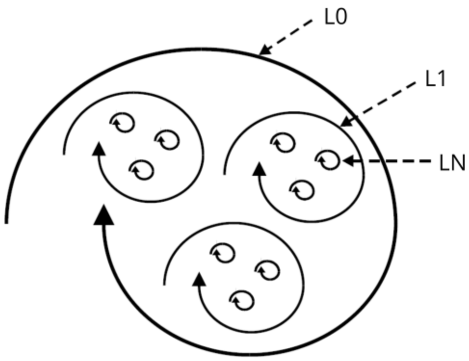






Energy spectrum and energy cascade





- The energy-containing eddies are denoted by L0.
- L1 and L2 denotes the size of the eddies in the inertial sub-range such that L2 < L1 < L0.
- LN is the size of the dissipative eddies.
- The large, energy containing eddies transfer energy to smaller eddies via vortex stretching.
- Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.
- Large eddies derive their energy from the mean flow.
- The size and velocity of large eddies are on the order of the mean flow.
- Large eddies (L0) are anisotropic; whereas small eddies (LN) are isotropic.
- The eddies in the inertial sub-range become more isotropic as energy is transferred from large eddies to small (dissipative) eddies.

References:
S. Pope. Turbulent Flows. Cambridge University Press. 2014.

Energy spectrum and energy cascade

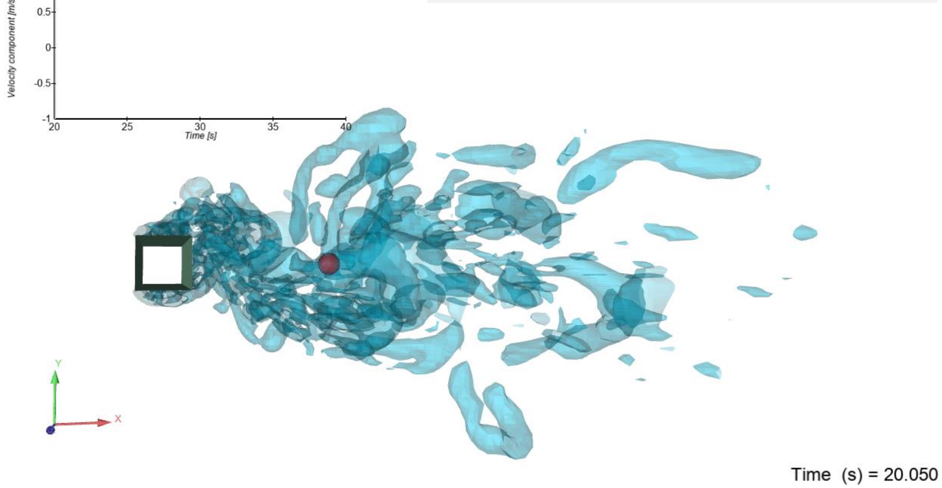
- Energy spectrum and Richardson's ode to the energy cascade [1],

> *"Big whorls have little whorls,*
> *which feed on their velocity,*
> *and little whorls have lesser whorls,*
> *and so on to viscosity"*

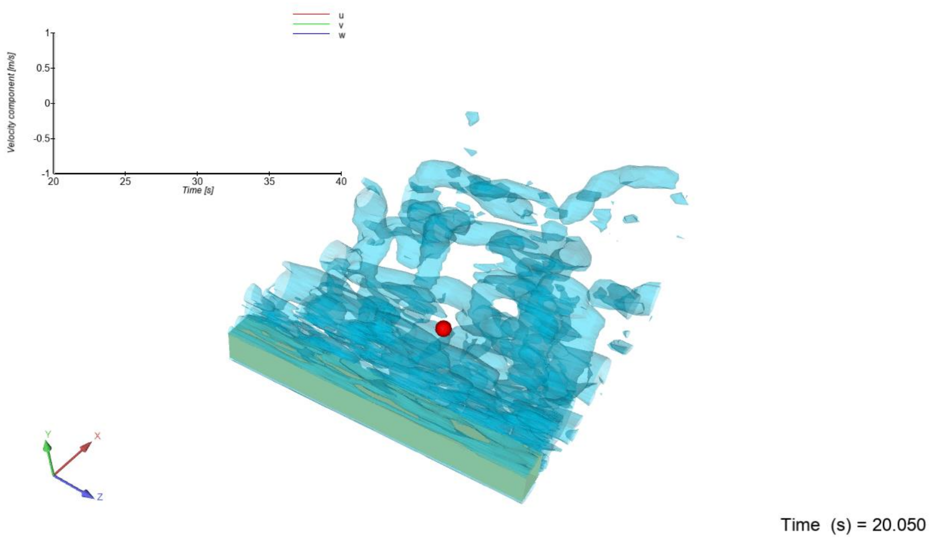
[1] L. F. Richardson. "Weather Prediction by Numerical Process". Cambridge University Press, 1922.

Energy cascade in action

Eddies = Vortices = Coherent structures



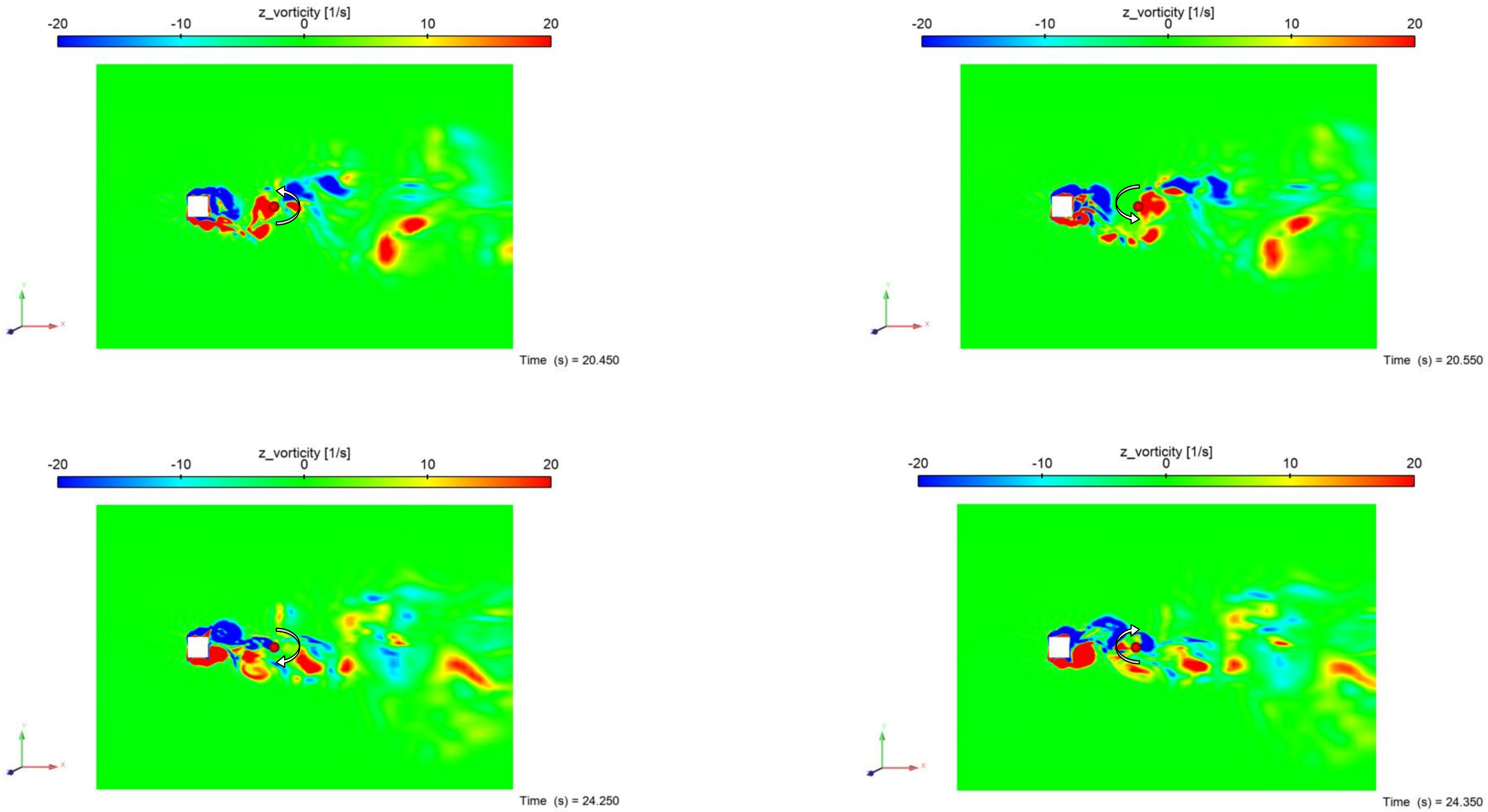
http://www.wolfdynamics.com/training/turbulence/image19.gif



http://www.wolfdynamics.com/training/turbulence/image20.gif

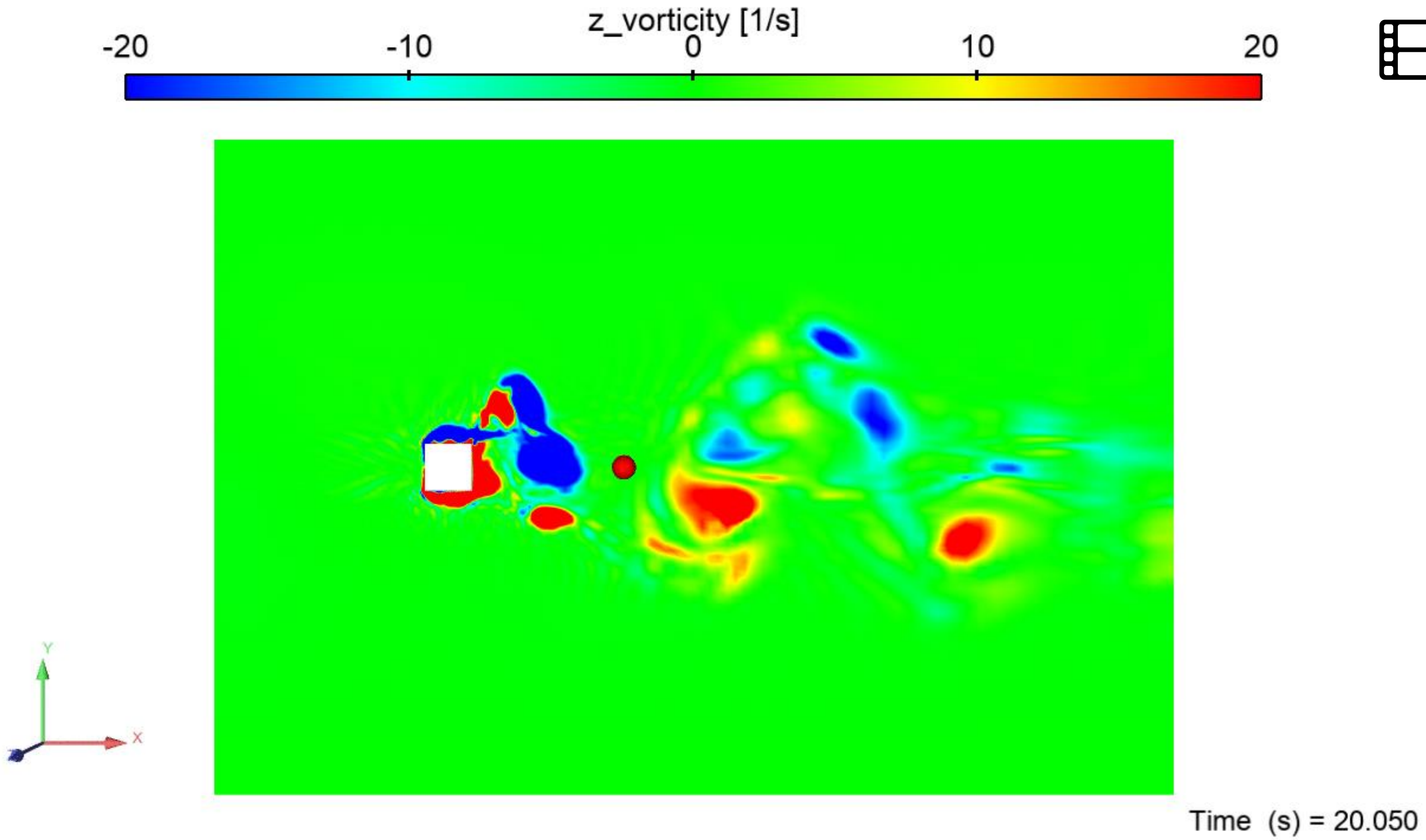
- To plot the energy spectrum, we need to sample the velocity field in a location behind the wake, the red sphere in this case.
- Then, by using signal processing (FFT or wavelets), we can convert the time domain into the frequency domain.
- The probe serve as a bucket where we accumulate all the information passing by the probe.
- Then, all the information accumulated in the time domain, can be converted into the frequency domain.
- Notice that this type of graph is local.
- It will be different for each and every point in the domain.
- In this case, we are showing the sampling in only one location (the red sphere).
- In practice, we sample the fluctuating velocity in many locations in the wake behind the body, or in the regions where fluctuations are present
- We will address this type of post-processing later.

Energy cascade in action



- All the information passing over the probe (sampling location) can be used to compute the energy spectrum and much more.

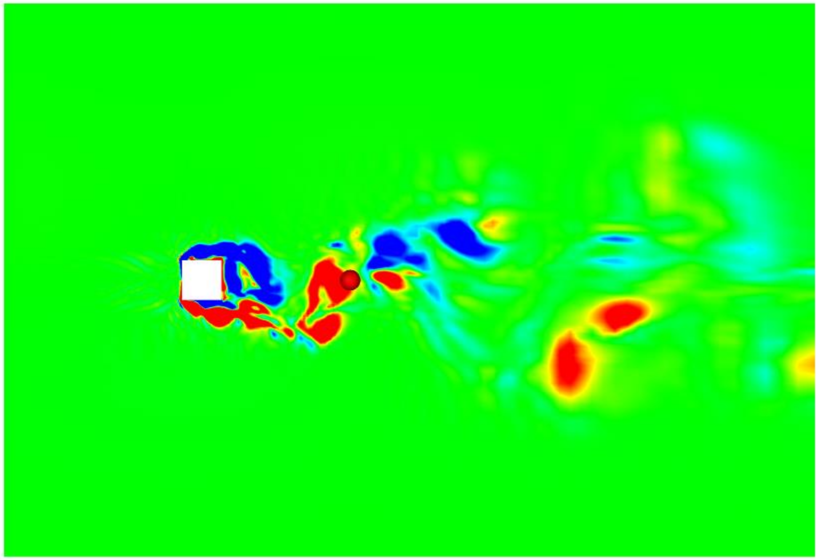
Energy cascade in action

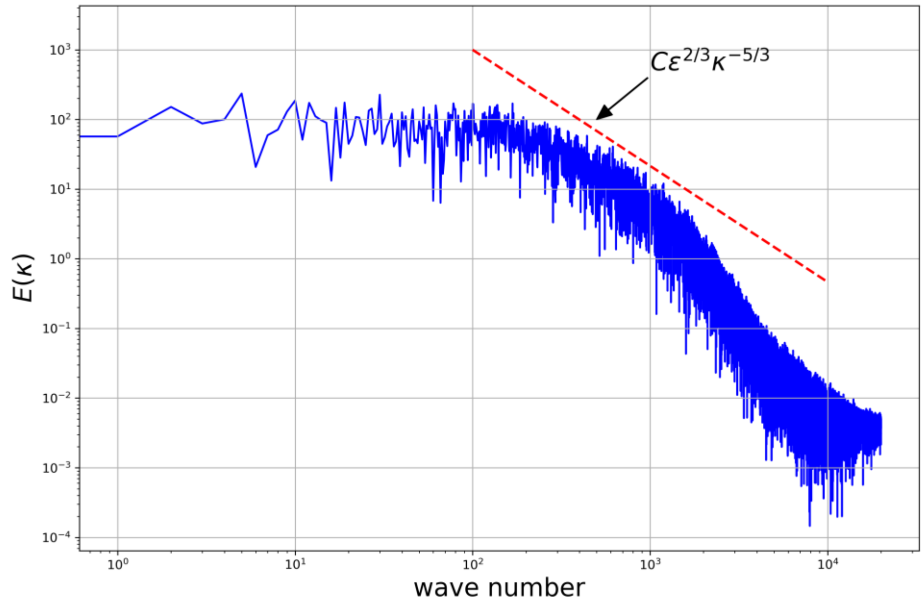


Vorticity contours – The velocity field is being sampled at the red sphere.

http://www.wolfdynamics.com/training/turbulence/image21.gif

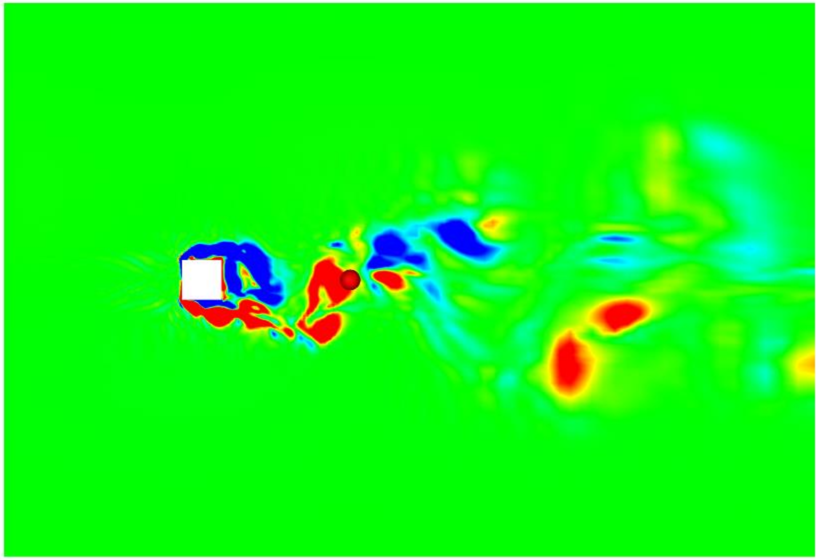
Energy cascade in action

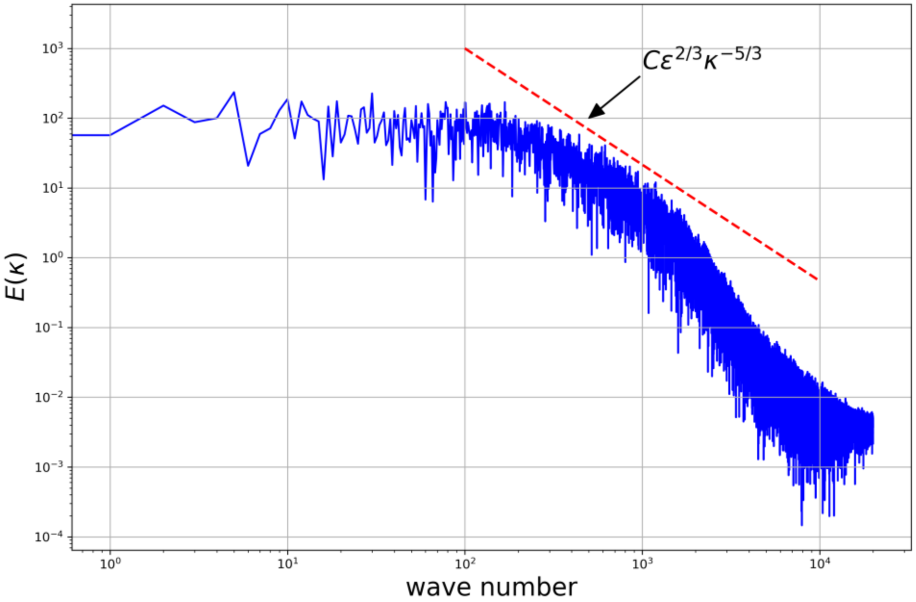




- Plot of the energy spectrum (right figure).
- Remember, this plot is local.
- To obtain this plot, you need to measure the velocity fluctuations at a given location, in this case, the red sphere in the left figure.
- The energy spectrum (right figure) represents the distribution of the TKE across the various wavenumbers or different frequencies..
- The wavenumber κ is proportional to the inverse of the eddy size l.

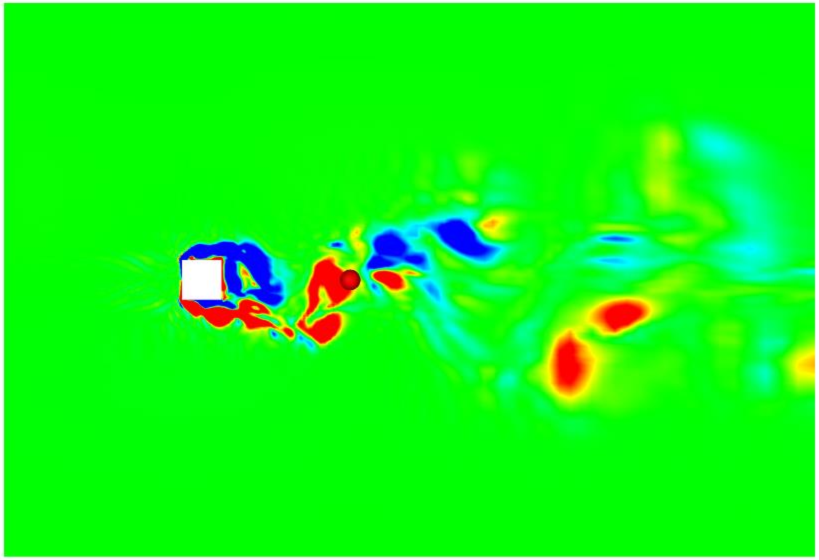
Energy cascade in action

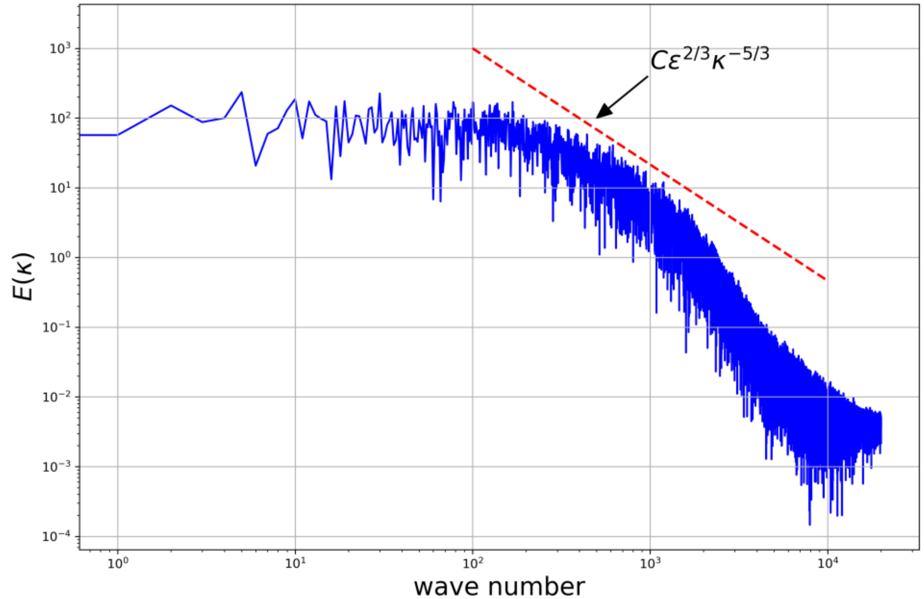




- The plot shows that the TKE peaks at largest scales (or small wavenumbers).
- Then, TKE rapidly decreases as the eddy sizes are smaller (large wavenumbers).
- And in the end, TKE is dissipated at the smallest scales.
- This is the energy cascade in action.
- In the plot, the inertial sub-range is compared against the -5/3 law (red line with a slope of -5/3).

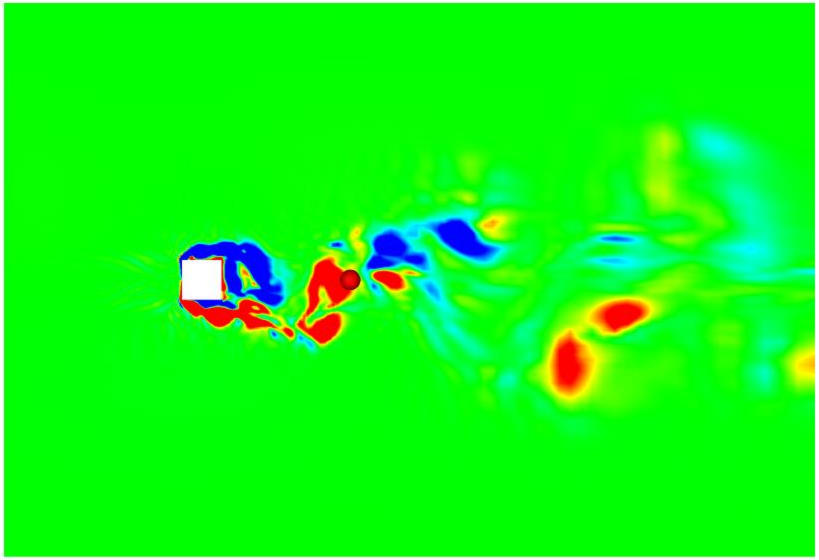
Energy cascade in action

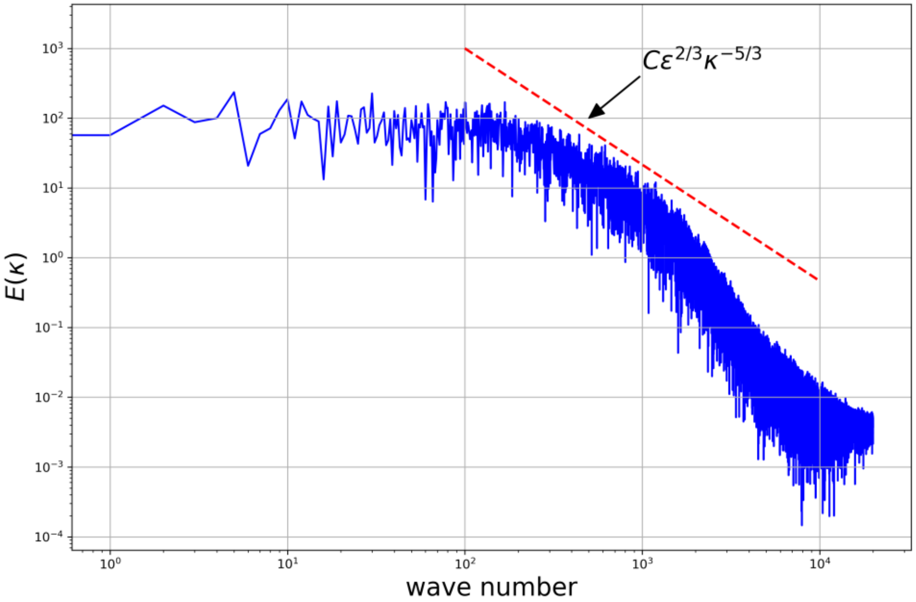




- Large wavenumbers, indicates small eddies with large frequency.
- Conversely, small wavenumbers are an indication of large eddies with low frequency.
- Have in mind that the relationship $l \propto k^{-1}$ should be treated as an order of magnitude approximation.
- In this plot, it is difficult to distinguish between k^{-1} and $2\pi k^{-1}$ (the angular wavelength of the Fourier component).
- As a general note, in typical engineering applications the smallest eddies are about 0.1 to 0.01 mm and have frequencies in the order of 10 kHz.

Energy cascade in action

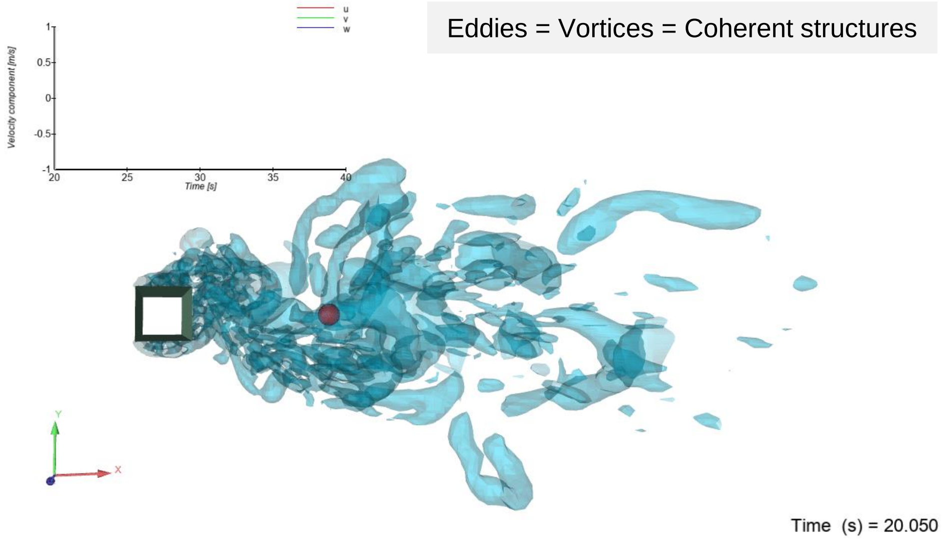




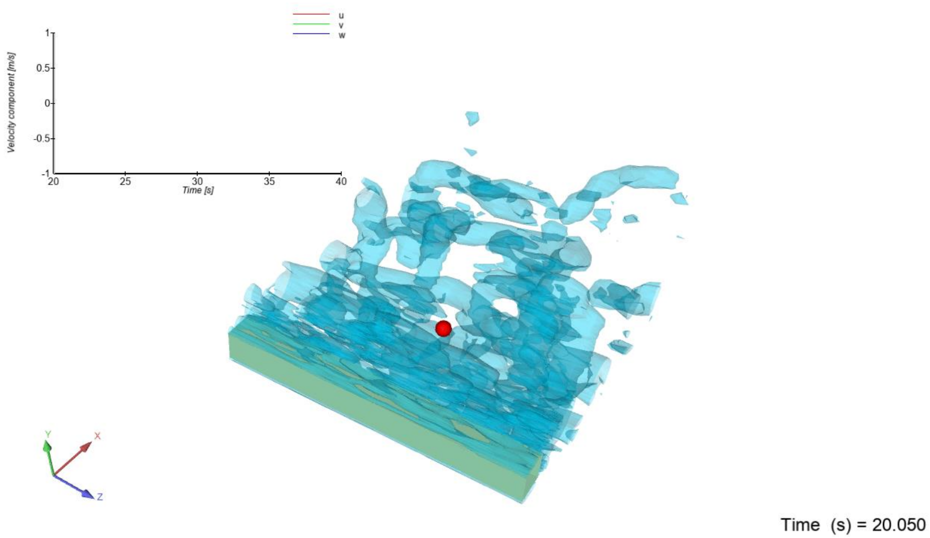
- In the energy spectrum plot, the energy should not accumulate or increase at large wavenumbers.
- If this happens, it is an indication that the mesh is too coarse, the numerical method used is too diffusive, or there are issues with the turbulence model.

Energy cascade in action

Eddies = Vortices = Coherent structures



http://www.wolfdynamics.com/training/turbulence/image19.gif



http://www.wolfdynamics.com/training/turbulence/image20.gif

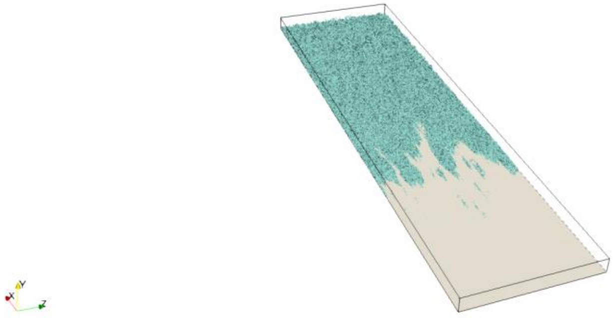
- From these animations, the following can be said regarding large and small eddies.

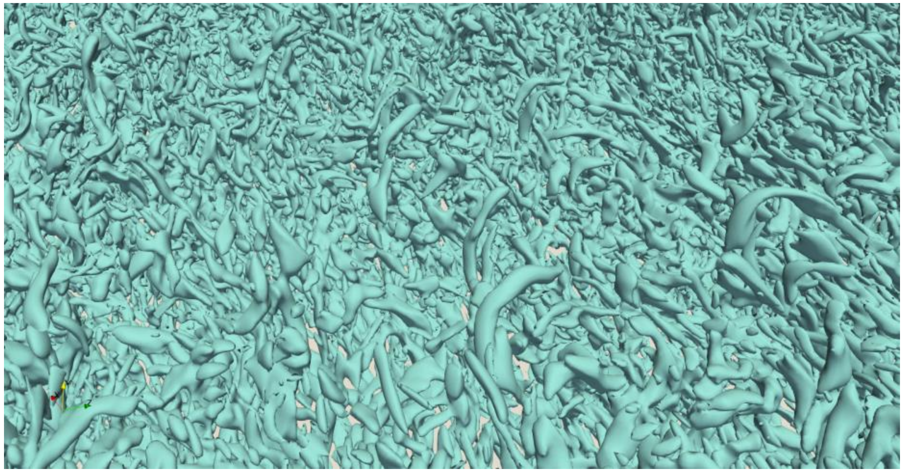
- **Large eddies:**
  - They are energy-containing and extract their energy from the mean flow.
  - Their velocity is on the order of the mean flow.
  - Their size is on the order of the mean flow or the obstacle the flow over.
  - They are anisotropic and unstable. They break-up into smaller eddies.
  - Their frequency is low compare the small eddies.

- **Small eddies (which we do not clear see in this animation):**
  - Their behavior is more universal in nature.
  - Their frequency is high.
  - Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.

Energy cascade in action

Eddies = Vortices = Coherent structures





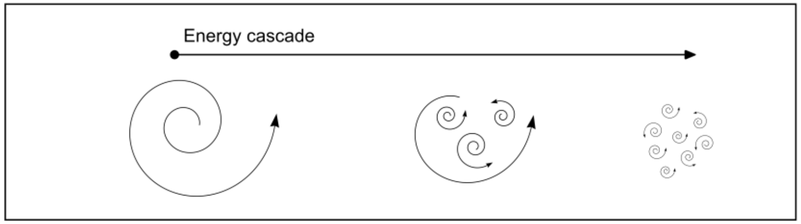
- From these figures, the following can be said regarding the small eddies.

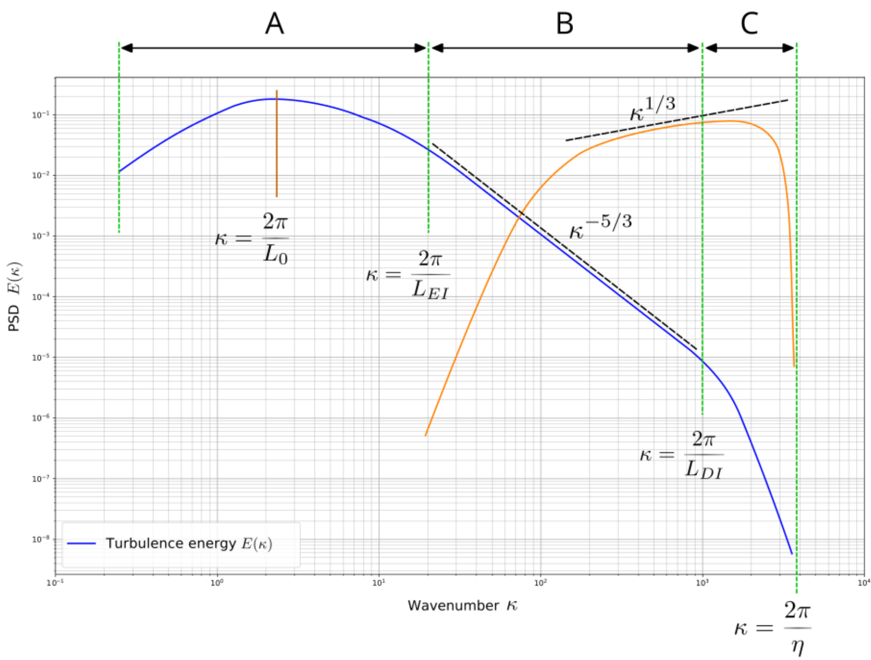
- **Small eddies (which we can see better in these images):**
  - Their behavior is more universal in nature.
  - Their frequency is high.
  - Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.

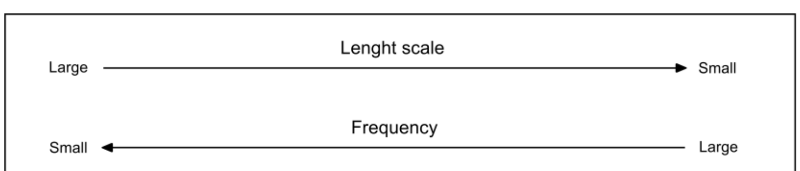
Note

- We are looking at one timestep of a DNS simulation.
- The input file is about 17 GB, and it required about 110 GB of RAM memory, a GPU of 16 GB, 16 cores, and about 5 minutes to open and manipulate the data (mesh size approximately 1.5 billion grid points).
- Dataset source:
  - http://turbulence.pha.jhu.edu/Transition_bl.aspx

Energy cascade in action







- The previous statements regarding large and small eddies also holds when we look at the plot of the turbulent energy spectrum.

- **Large eddies:**
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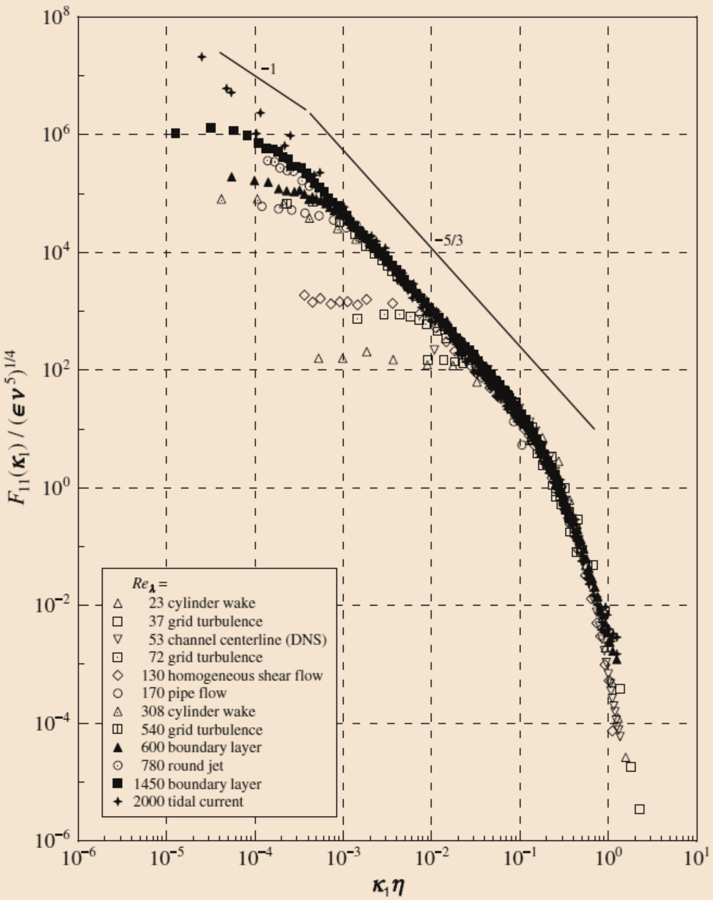
  - Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.

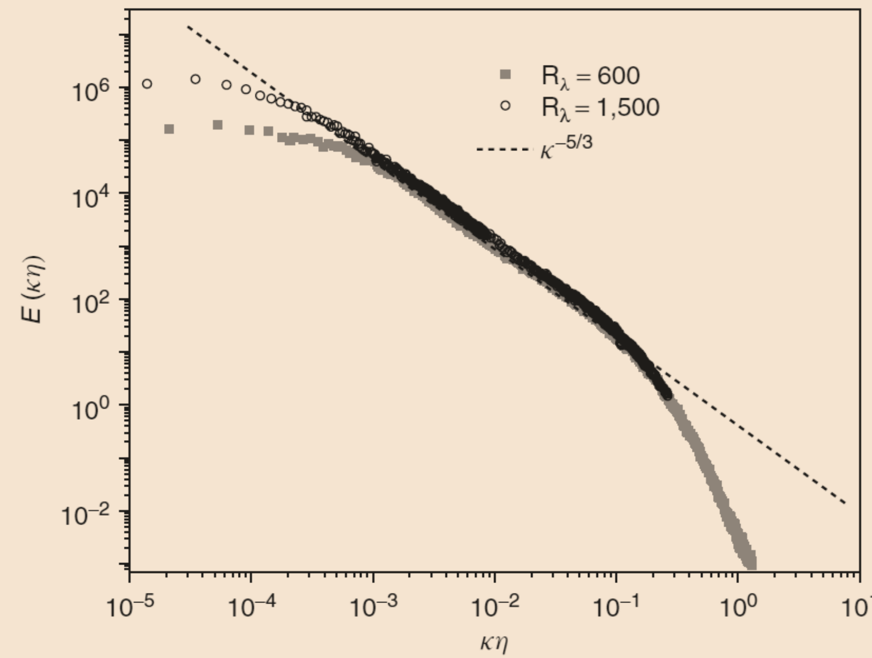
Validity of the Kolmogorov theory (K41)

- Kolmogorov’s theory is an asymptotic theory, and it has been shown to work well in the limit of very high Reynolds numbers.
- Kolmogorov’s theory assumes that the energy cascade is one way, from large eddies to small eddies.
- However, experimental studies have shown that energy is also transferred from smaller scales to larger scales (a process called backscatter), albeit at a much lower rate.
- Nevertheless, the dominant energy transfer is always for large eddies to small eddies.
- The theory assumes that turbulence at high Reynolds numbers is completely random. However, large scale coherent structures may form.
- It also assumes that the smallest scales are very isotropic. Note that in reality, they are elongated structures with a small degree of anisotropy (which means that the eddies have forgotten their initial anisotropic state).
- Kolmogorov’s theory has been confirmed using experiments and large-scale simulations (DNS or direct numerical simulations).
- It is worth mentioning that the universality of Kolmogorov theory has been questioned by many authors. Whom, to some extension, provides valid arguments to support the lack of universality.

Validity of the Kolmogorov theory (K41)



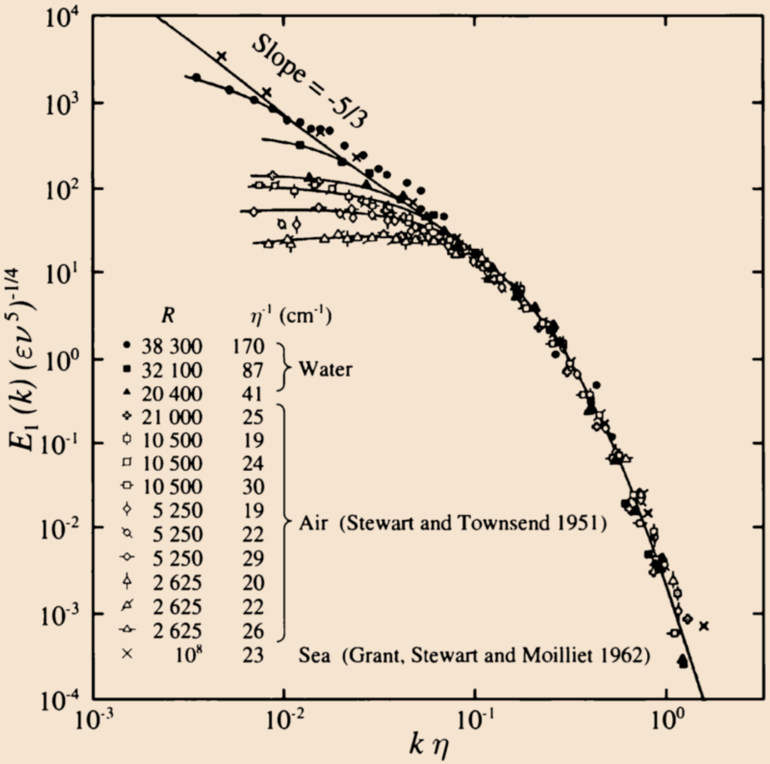
One-dimensional spectra scaled with respect to the microstructure from various turbulent flows. This demonstrates the universal character of the microstructure and illustrates the −5/3 behavior of the inertial subrange [1].



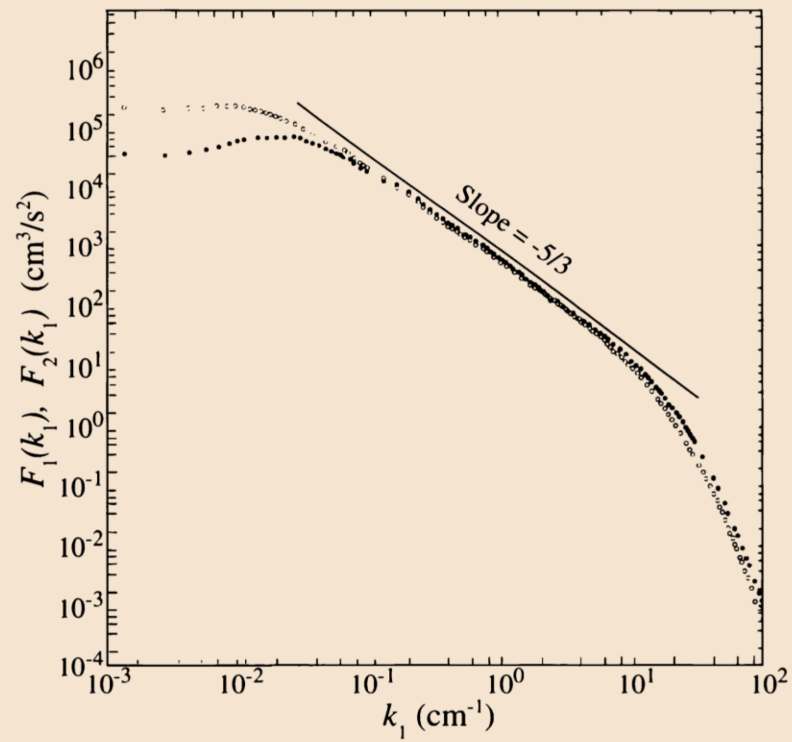
Experimental spectra measured in the boundary layer of the NASA Ames 80 x 100 foot wind tunnel [2].

[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.
[2] P. Durbin, B.A. Pettersson-Reif. Statistical Theory and Modeling for Turbulent Flow. Wiley, 2011.

Validity of the Kolmogorov theory (K41)



Normalized longitudinal velocity spectrum in the time domain according to different authors [1].



Log-log plot of the energy spectra of the streamwise components (white circles) and the lateral component (black circles) of the velocity fluctuations [2].

[1] F. Champagne. The fine-scale structure of the turbulent velocity field. J. Fluid Mech. 1978.
[2] C. Gibson, W. Schwarz. The universal equilibrium spectra of turbulent velocity and scalar fields. J. Fluid Mech. 1963.

Energy spectrum and mesh resolution

- DNS simulations are too expensive, they require a lot of grid points/cells in order to resolve all the turbulent scales (in space and time).
- As we have seen, in a DNS simulation the gridding/meshing requirements scales proportional to $Re_T^{9/4}$ or approximately proportional to Re_T^3 for a single time step.
- And every time step should be sufficiently resolved in time (CFL condition less than 1, and the ideal value should be less than 0.5).
- Those are a lot of grid points/cells and a lot of time-steps.
- To avoid the extremely high computational requirements of DNS simulations, we can use large-eddy simulations (LES).

Energy spectrum and mesh resolution

- A good LES simulation, aims at resolving 80% of the energy spectrum.
- If the mesh requirements of a LES are still too high, we can do a VLES (very large eddy simulation) where we aim at resolving 50% of the energy spectrum.
- Another alternative are detached-eddy simulations (DES).
- In DES, we use RANS close to walls and LES in the far field.
- LES, VLES, and DES are commonly called scale-resolving simulations (SRS).
- If SRS requirements are still too high, which is the case for most of the industrial applications requiring quick turn-around times, we can use RANS/URANS models.

Energy spectrum and mesh resolution

- In RANS/URANS simulations the whole energy spectrum is modeled.
- RANS/URANS also heavily relies in wall functions and insensitive y^+ near the wall treatment.
- The meshing requirements of RANS/URANS should be sufficiently to capture well integral scales l_0 and model/resolve the boundary layer (according to the near the wall treatment).
- As it can be seen from this discussion, meshing requirements are driven by the turbulent scales we would like to resolve.
- Meshing depends on the turbulence modeling approach taken.
- SRS simulations requires fine meshes to resolve the space and time scales.
- RANS/URANS can use coarse meshes, as all the scales are being modeled.
- In Lecture 4, we will address how to compute estimates of the turbulence scales and leverage this information to boundary conditions, initial conditions, and mesh spacing values.

Energy spectrum and mesh resolution

- As we have seen, the mesh resolution determines the fraction of the turbulent kinetic energy directly resolved.
- So, let us suppose that we want to resolve 80% of the turbulent kinetic energy $k(l)$ in a scale-resolving simulation (SRS).
- And recall that 80% is the minimum requirement for a good quality LES simulation.
- Then, according to some estimates that we will introduce hereafter, at least 5 to 10 cells are needed across the integral length scales l_0.
- This estimate is usually based on isotropic turbulence [1]; therefore, it tends to be very conservative.

- Summarizing,
  - For RANS/URANS/VLES simulations 5-6 cells are the absolute minimum to resolve integral scales l_0.
  - For SRS and LES simulations that resolves at least 80% of the turbulent spectrum, you will need to use at least 10 cells across the integral scales l_0.

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and mesh resolution

- To estimate the minimum mesh resolution and time step of the simulation, we need to quantify the range of length and time scales associated with a turbulent flow.
- These estimates of the maximum and minimum eddy length and time scales can be used in selecting the appropriate mesh resolution and time step for a given problem.
- The turbulent scales can be defined in terms of the turbulent kinetic energy (k), turbulent dissipation (ϵ), and the kinematic viscosity (ν).
- Provided the Reynolds number is not too small, dimensional analysis suggests, and measurements confirm that TKE can be expressed in terms of the turbulent dissipation ϵ and the turbulence length scales l_0 as follows [1,2],

$$k \sim (\epsilon l_0)^{2/3} \implies \epsilon \sim \frac{k^{3/2}}{l_0} \implies l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- Similarly, the turbulent viscosity ν_t and the characteristic time scale of the eddies τ_0 (or turnover timescale), is in the order of,

$$\nu_t \sim k^{1/2} l_0 \sim \frac{k^2}{\epsilon} \qquad \tau_0 \sim \frac{l_0}{k^{1/2}} \sim \frac{k}{\epsilon} \sim \frac{\nu_t}{k}$$

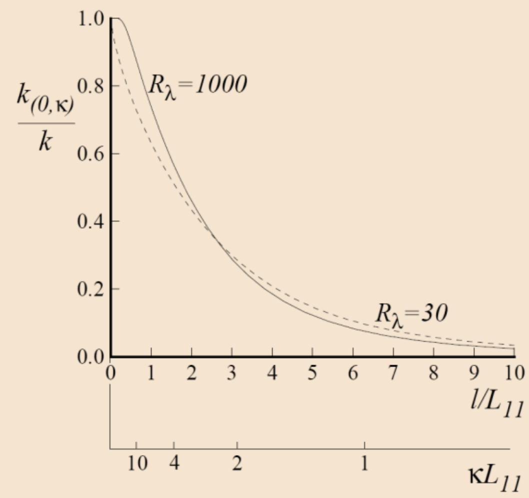
[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

Energy spectrum and mesh resolution

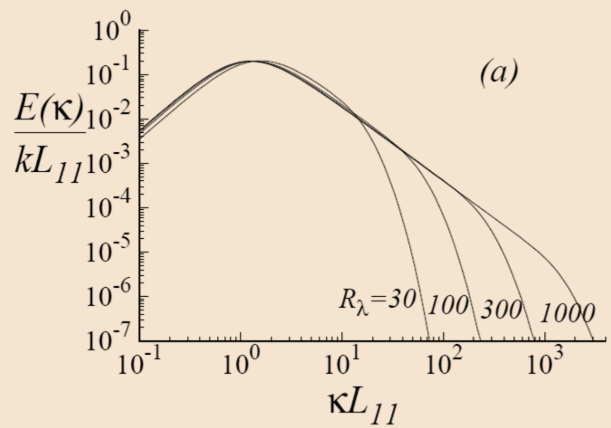
- At this point, the previous estimates of the largest scales in turbulence can be used to get a mesh resolution estimate.
- Let us introduce the ratio of integral length scales to grid length scales R_l,

$$R_l = \frac{l_0}{\Delta}$$

- This ratio represents the number of cells needed to resolve integral length scales l_0.
- With reference to the energy spectrum, the ratio R_l can be estimated such as it solves a given percentage of the spectrum.
- The ratio R_l can be roughly approximated using the cumulative turbulent kinetic energy plot against integral length scales for a model energy spectrum, numerical simulations, or physical experiments conducted at different Reynolds number.



Cumulative turbulent kinetic energy against wavenumber and wavelength for a model spectrum [1].

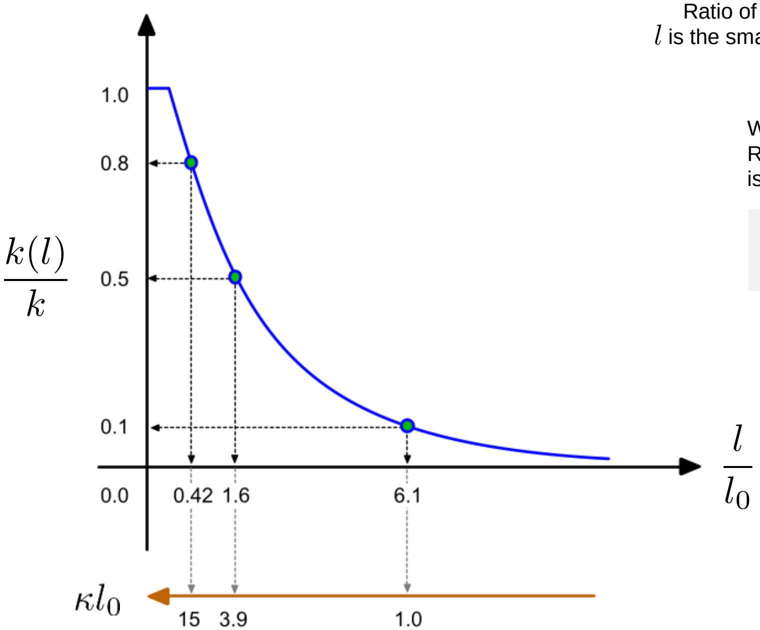


The model spectrum for various Reynolds number scaled by k and L_{11} [1].

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and mesh resolution

- The information contained in the cumulative plot can be used to estimate the ratio $R_l = l_0/\Delta$.
- In the table below, we show some of the numerical characteristics (wavenumber and length scales) of a model energy spectrum cumulative plot (illustrated in the figure below).
- It is a good practice to always aim to resolve at least 80% of the spectrum, even in RANS/URANS simulations.
- For a rigorous explanation of these results, please refer to reference [1].



Cumulative turbulent kinetic energy plot against lengths scale of eddies for a model spectrum function. The figure has been adapted from reference [1].

Wavenumber times integral length scales. Recall that the wavenumber is defined as,

$$\kappa = \frac{2\pi}{l} = \frac{2\pi f}{U}$$

Ratio of characteristic eddy size to integral length scale. l is the smallest vortex that can be resolved with the mesh.

Ratio of integral length scale to grid length scale. The grid cut-off dimension Δ is defined as,

$$\Delta = l/2$$

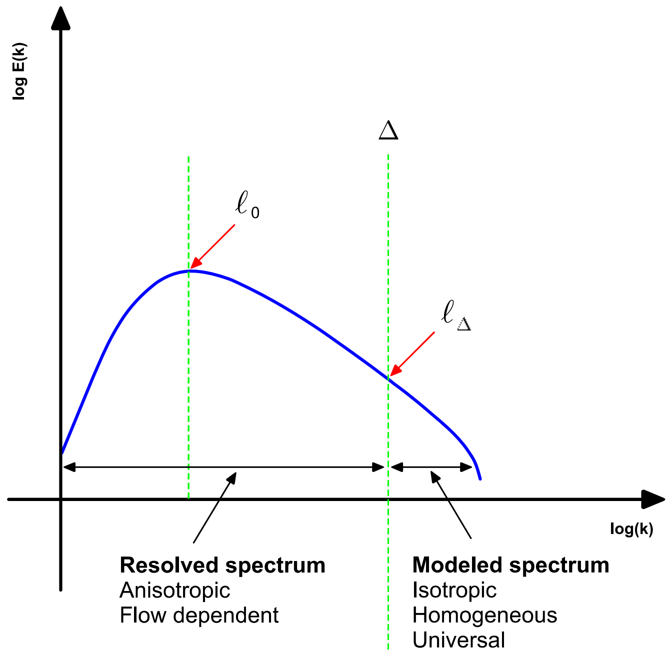
	κl_0	l/l_0	l_0/Δ
$k(l) = 0.1k \ \ (10\%)$	≈ 1.0	≈ 6.1	≈ 0.33
$k(l) = 0.5k \ \ (50\%)$	≈ 3.9	≈ 1.6	≈ 1.25
$k(l) = 0.8k \ \ (80\%)$	≈ 15	≈ 0.42	≈ 4.8
$k(l) = 0.9k \ \ (90\%)$	≈ 38	≈ 0.16	≈ 12.5

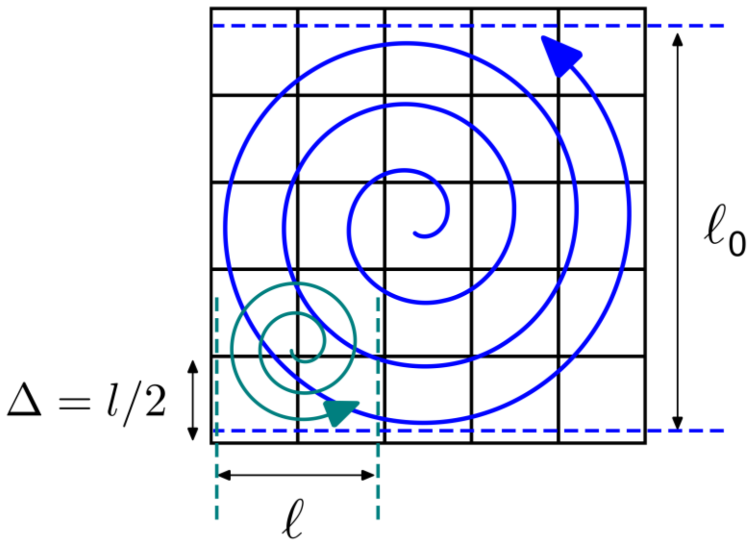
Characteristic turbulent kinetic energy and length scales of the energy spectrum.

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and mesh resolution

- From the previous estimates (very conservative by the way), we can conclude the following.
  - 5 to 6 cells are acceptable to resolve integral length scales in RANS/URANS/VLES simulations.
  - A minimum of 10 cells across the integral scales l_0 are required to solve 80-90% of the turbulent spectrum in LES simulations.
  - To resolve an eddy with a length scale l, where l is the smallest scales that can be resolved with a cell of dimension Δ in every dimension and $l << l_0$, at least a couple of cells need to be used in each direction
- Remember, eddies cannot be resolved down to the molecular dissipation limit. It is too expensive!





Integral length scale and grid length scale

- At this point the question is, how do we estimate the integral length scale l_0?
- The integral length scale l_0 can be roughly estimated as follows,
  - Based on a characteristic length, such as the size of a bluff body or pipe diameter.
  - From correlations.
  - From experimental results.
  - **From a precursor RANS simulation.**

- After identifying the integral scales, you can cluster enough cells in the domain regions where you expect to find the integral scales (or large eddies).
- In other words, put enough cells in the wake or core of the flow.
- Remember, turbulent kinetic energy peaks at integral length scale l_0.
- Therefore, these scales must be sufficiently resolved in LES/DES simulations or captured (be able to track) in RANS/URANS simulations.

Integral length scale and grid length scale

- Recall that from dimensional analysis,

$$l_0 = \frac{k^{1.5}}{\epsilon}$$

- Also, the dissipation rate ϵ can be related to the specific dissipation rate ω using the following relation [1,2],

$$l_0 = \frac{k^{0.5}}{0.09 \times \omega}$$

where

$$\epsilon = C_\mu \omega k$$

$$C_\mu = 0.09$$

[1] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.
[2] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Integral length scale and grid length scale

- Therefore, to compute the integral length scales l_0, we need to compute the turbulent kinetic energy k and the dissipation rate ϵ (or the specific dissipation rate ω).
- These quantities (including l_0), can be computed from a precursor RANS simulation.
- It is important to stress that in order to compute the turbulent kinetic energy k, the turbulent dissipation rate ϵ, and the specific dissipation rate ω, we need to use turbulence models that compute these turbulent variables.
- For example, we can use the following two equations models,
    - $k-\epsilon$ family of turbulence models.
    - $k-\omega$ family of turbulence models.
- As usual, remember to always do a dimensional check.

Derived quantity	Symbol	Dimensional units	SI units
Energy dissipation rate per unit mass	ϵ	L^2T^{-3}	m^2/s^3
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Specific dissipation rate	ω	T^{-1}	1/s
Integral length scale	l_0	L	m

Integral length scale and grid length scale

- After computing l_0, the ratio of integral length scale to grid length scale R_l can be estimated as follows,

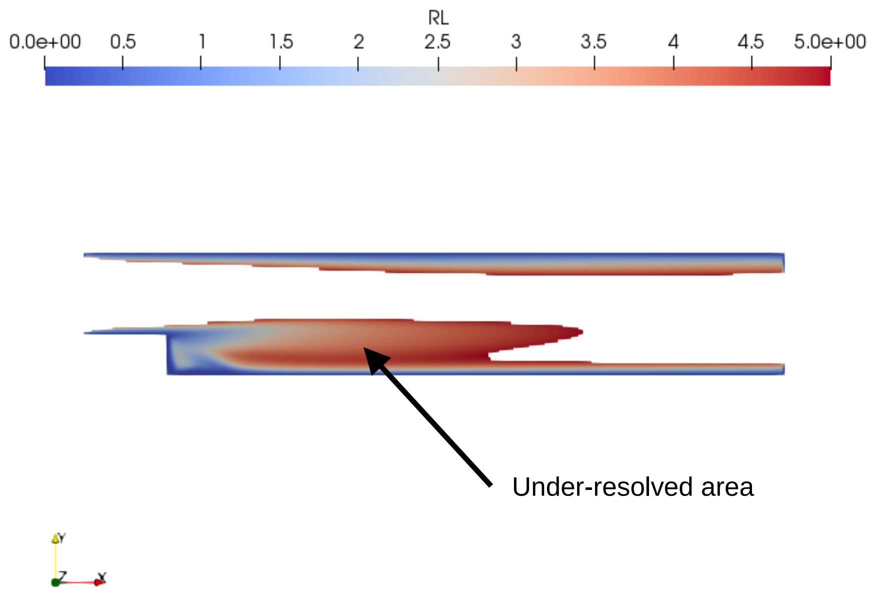
$$R_l = \frac{l_0}{\Delta}$$

where Δ can be approximated as follows $\Delta \approx \sqrt[3]{\text{cell volume}}$

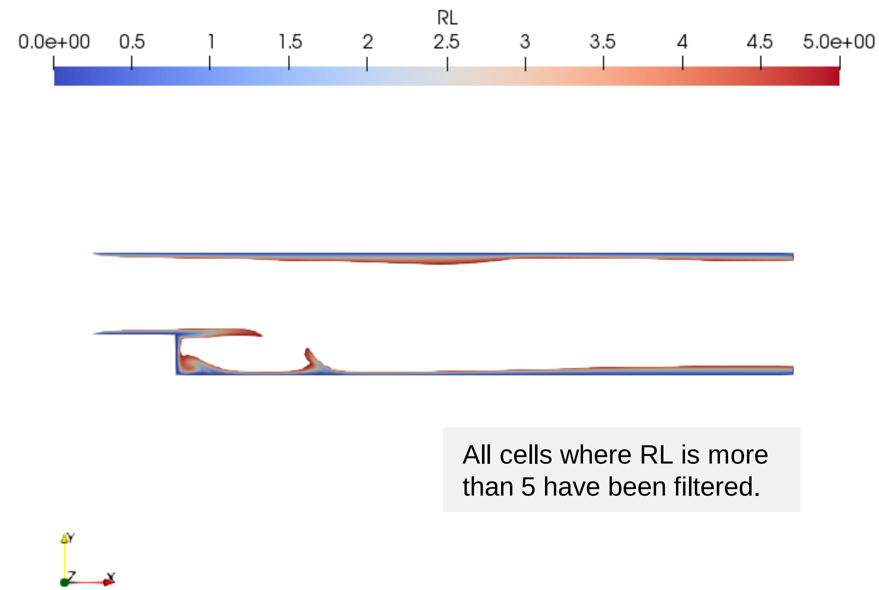
This approximation is accurate if the aspect ratio of the cells is modest (less than 1.2)

- The recommended value of R_l is about R_l > 5-10.
- Where 5 should be considered the lowest limit of resolution for RANS/URANS/VLES.
- And 10 is the desirable lower limit for LES simulations that resolve at least 80% of the turbulent spectrum.
- DES simulations have similar requirements as LES simulations (in the wake or far from the walls).
- Higher values can be used if computer power and time constrains permit.
- This is a conservative estimate, which is likely problem dependent.
- In well resolved LES simulations equal mesh resolution should be provided in all spatial directions.

Integral length scale and grid length scale



Coarse mesh

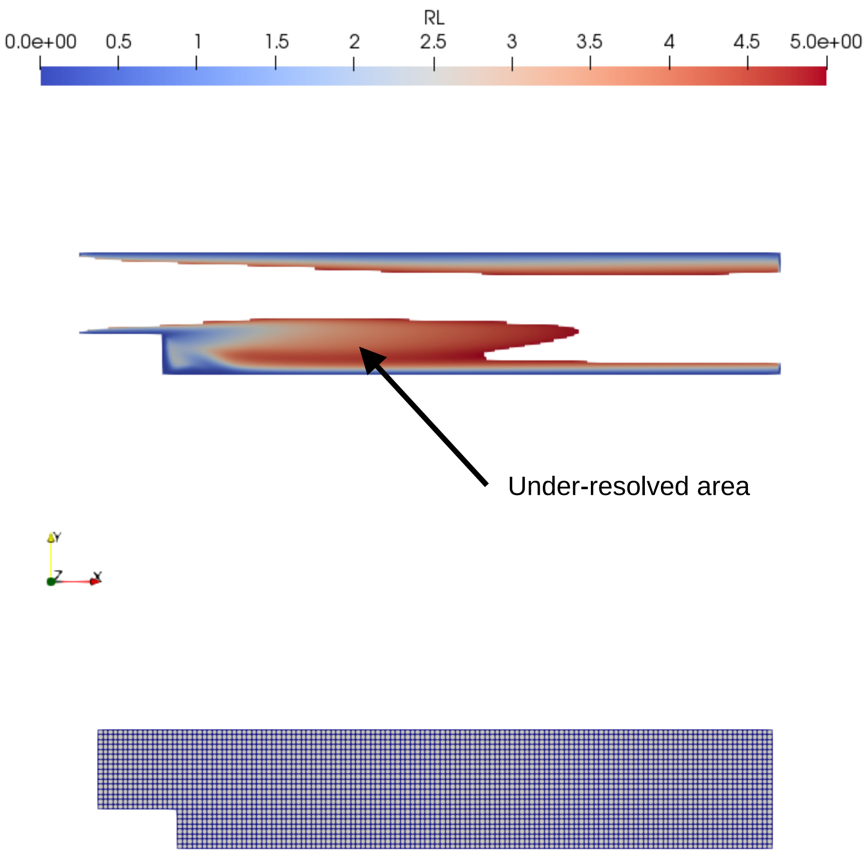


Fine mesh

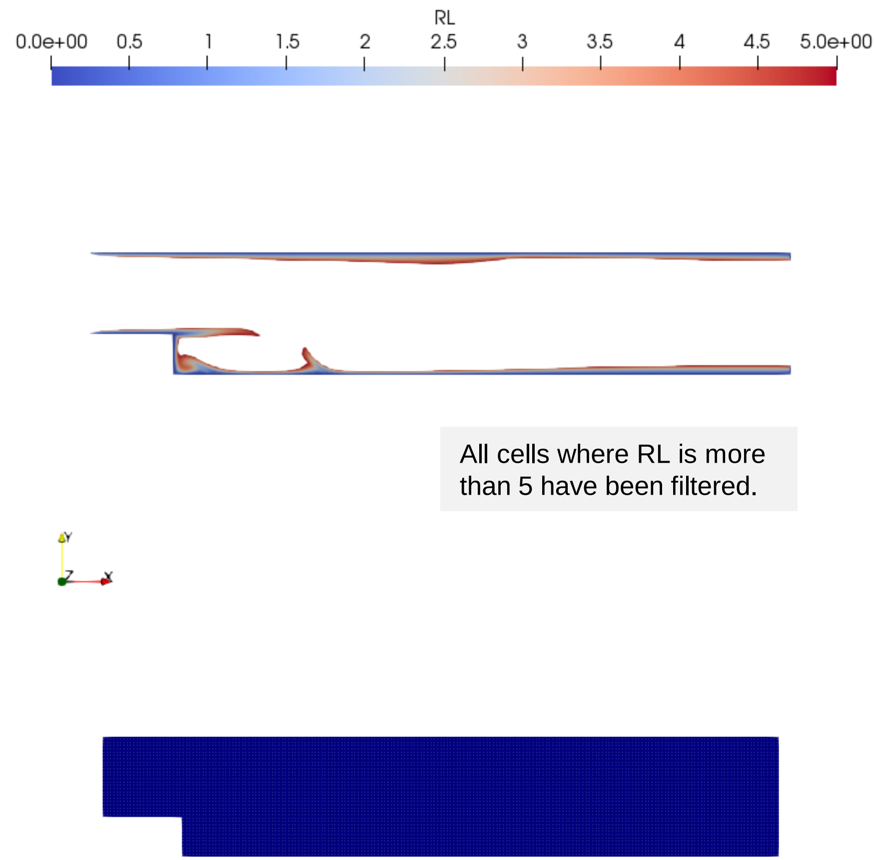
- To identify integral length scales and grid length scales you can plot contours of these quantities at different locations/planes in the domain.
- The lowest limit of R_l can be clipped so that the well resolved areas do not appear.
- In this case, we are showing (clipping) the cells where $0 < R_l < 5$.
- Under-resolved areas (the areas shown), will need finer meshes or local mesh adaption.
- Near-wall regions always pose challenges. In these areas is better to quantify the y^+ value.

Energy spectrum and energy cascade. Integral length scale and grid length scale

Integral length scale and grid length scale

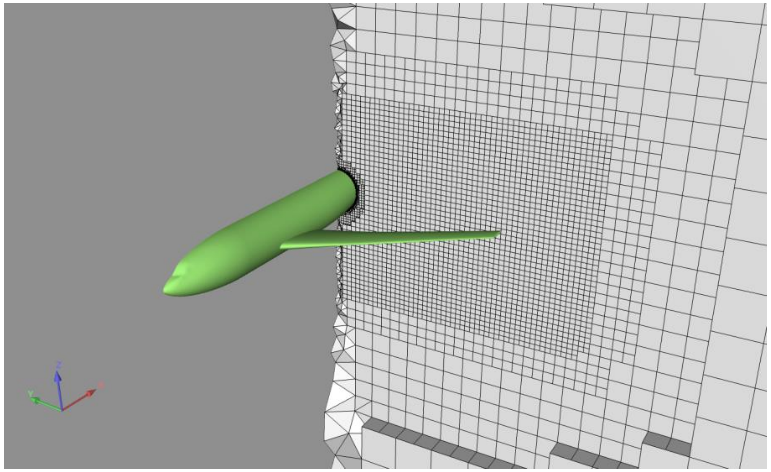


Coarse mesh

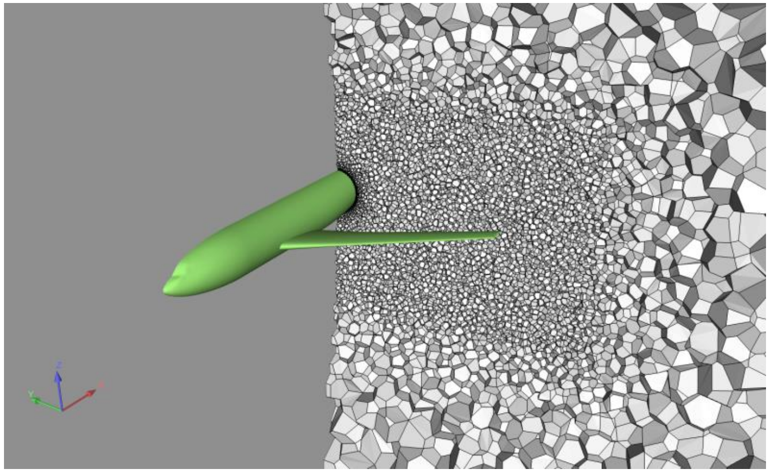


Fine mesh

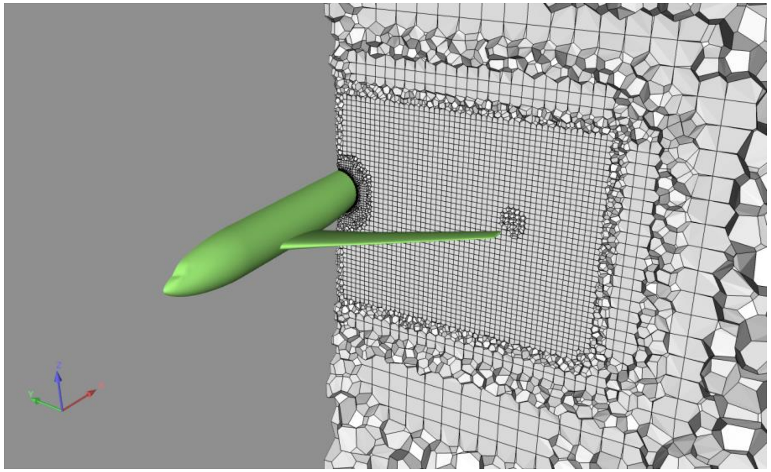
Turbulence modeling, CFD, and the mesh



Hexahedral dominant mesh with tetrahedral surface mesh



Polyhedral mesh



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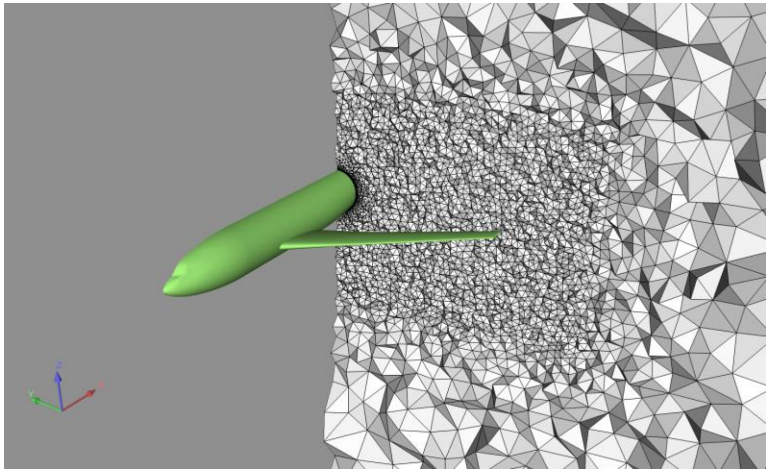
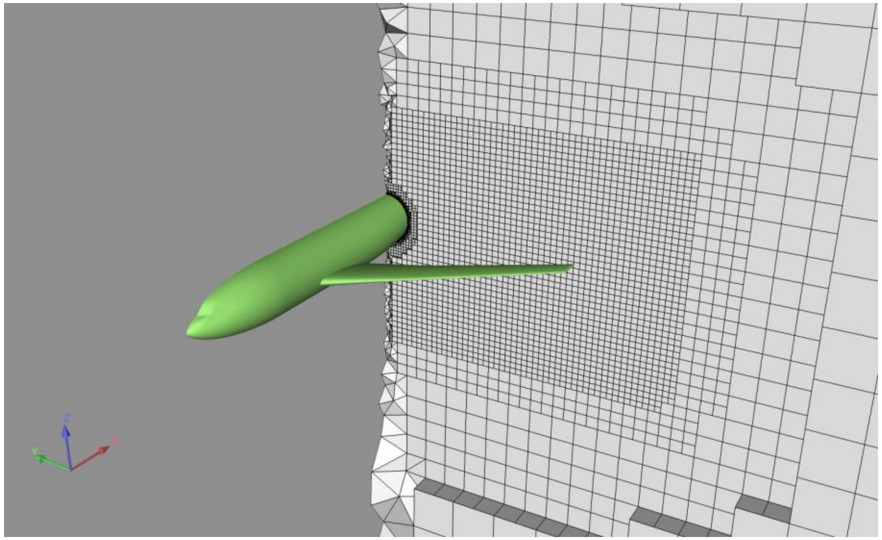


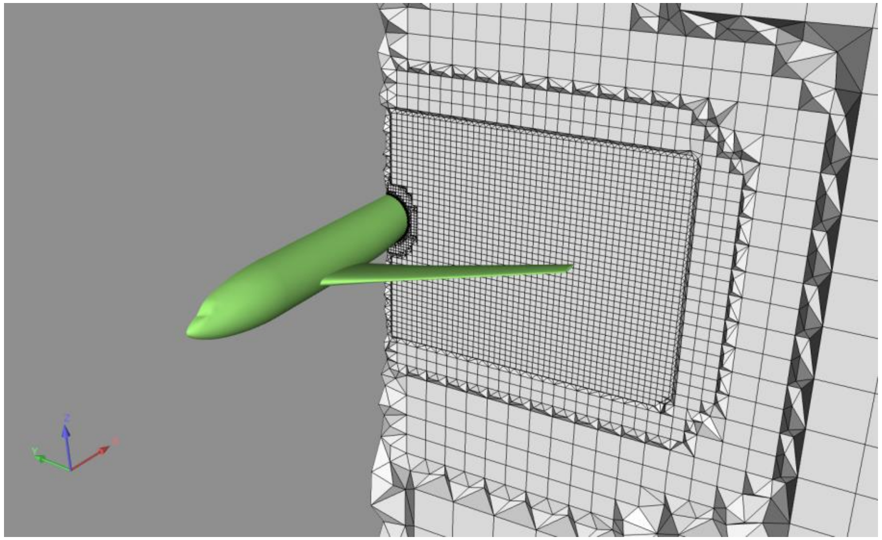
Tetrahedral mesh

- There is nothing written when it comes to best cell type, they all will give similar results if good standard practices are followed.

Turbulence modeling, CFD, and the mesh



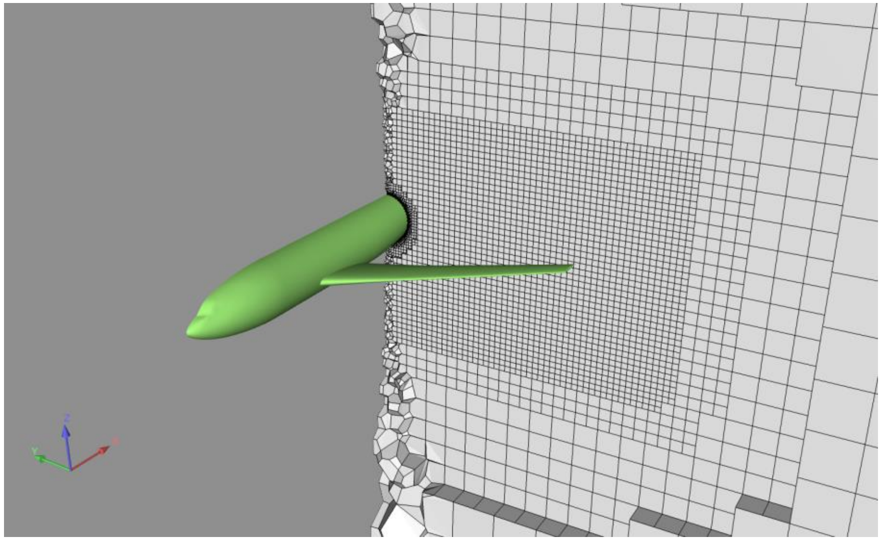
Hexahedral dominant mesh with tetrahedral surface mesh
Hanging nodes allowed – Octree subdivision



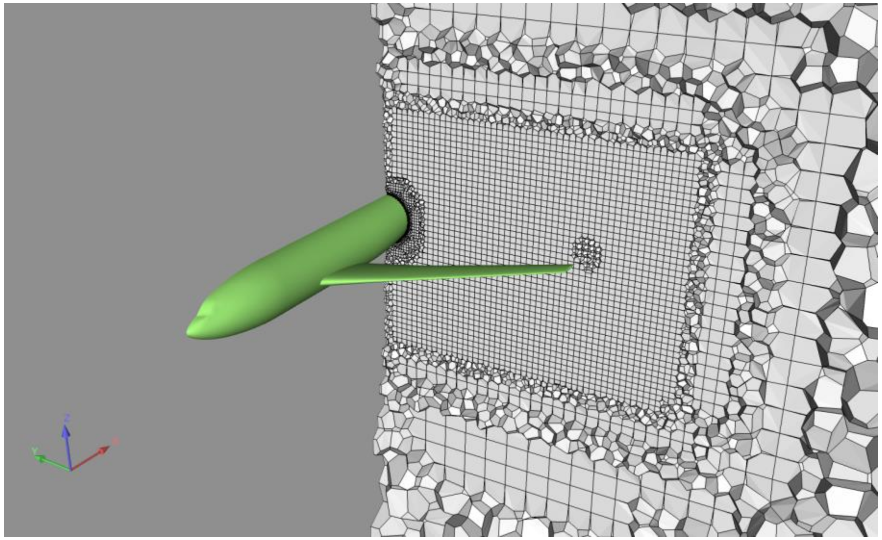
Hexahedral dominant mesh with tetrahedral surface mesh
Hanging nodes not allowed – Octree subdivision filled with pyramids

- Hanging nodes are often used by meshers and supported by solvers to ease the mesh generation process.
- However, hanging nodes tend to add a little of numerical diffusion due to the fast transition between small and large cells.
- This is especially important if you are conducting LES simulations.

Turbulence modeling, CFD, and the mesh



Hexahedral dominant mesh with polyhedral surface mesh
Hanging nodes allowed – Octree subdivision



Hexahedral dominant mesh with tetrahedral surface mesh
Hanging nodes not allowed – Octree subdivision filled with polyhedral

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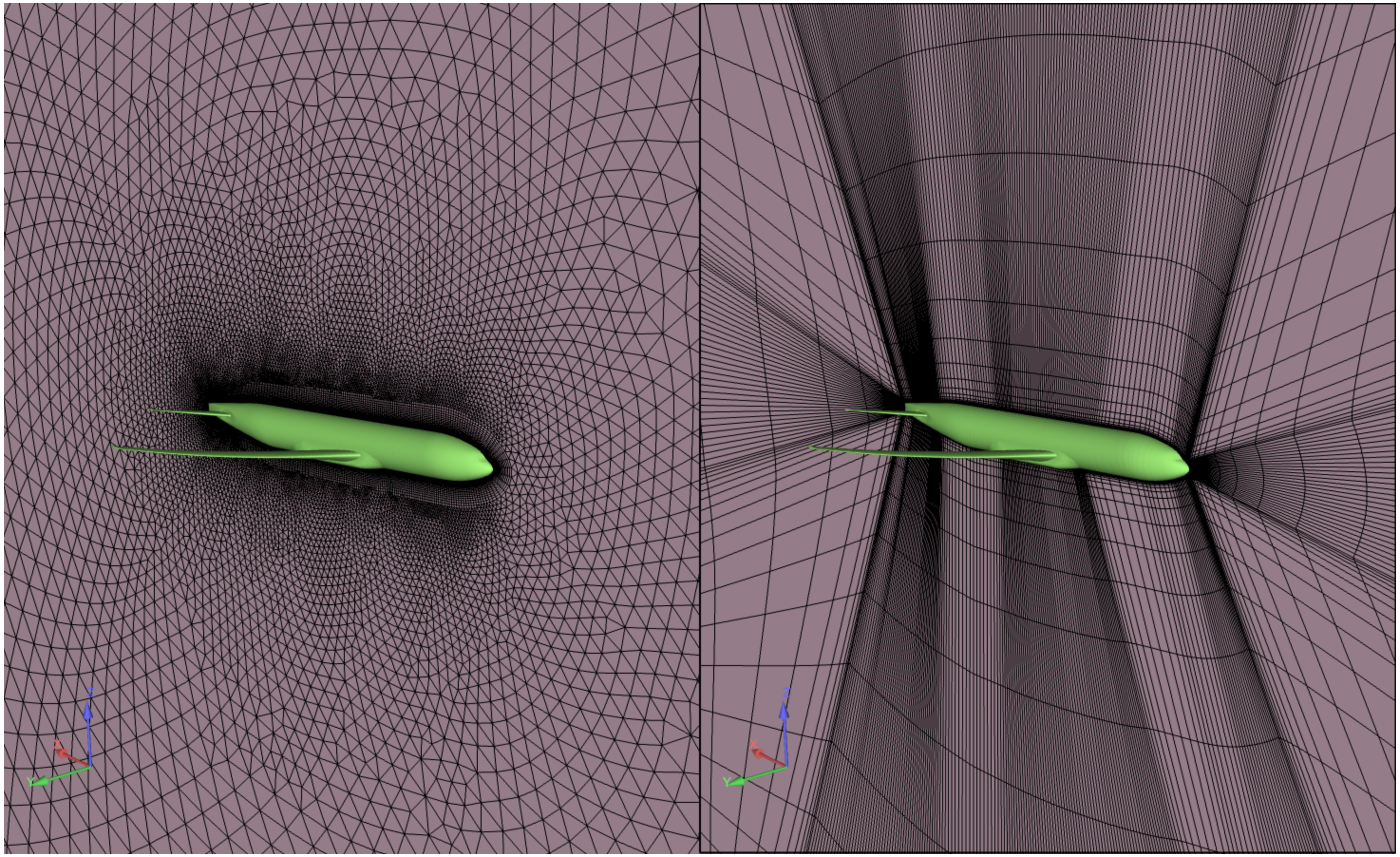
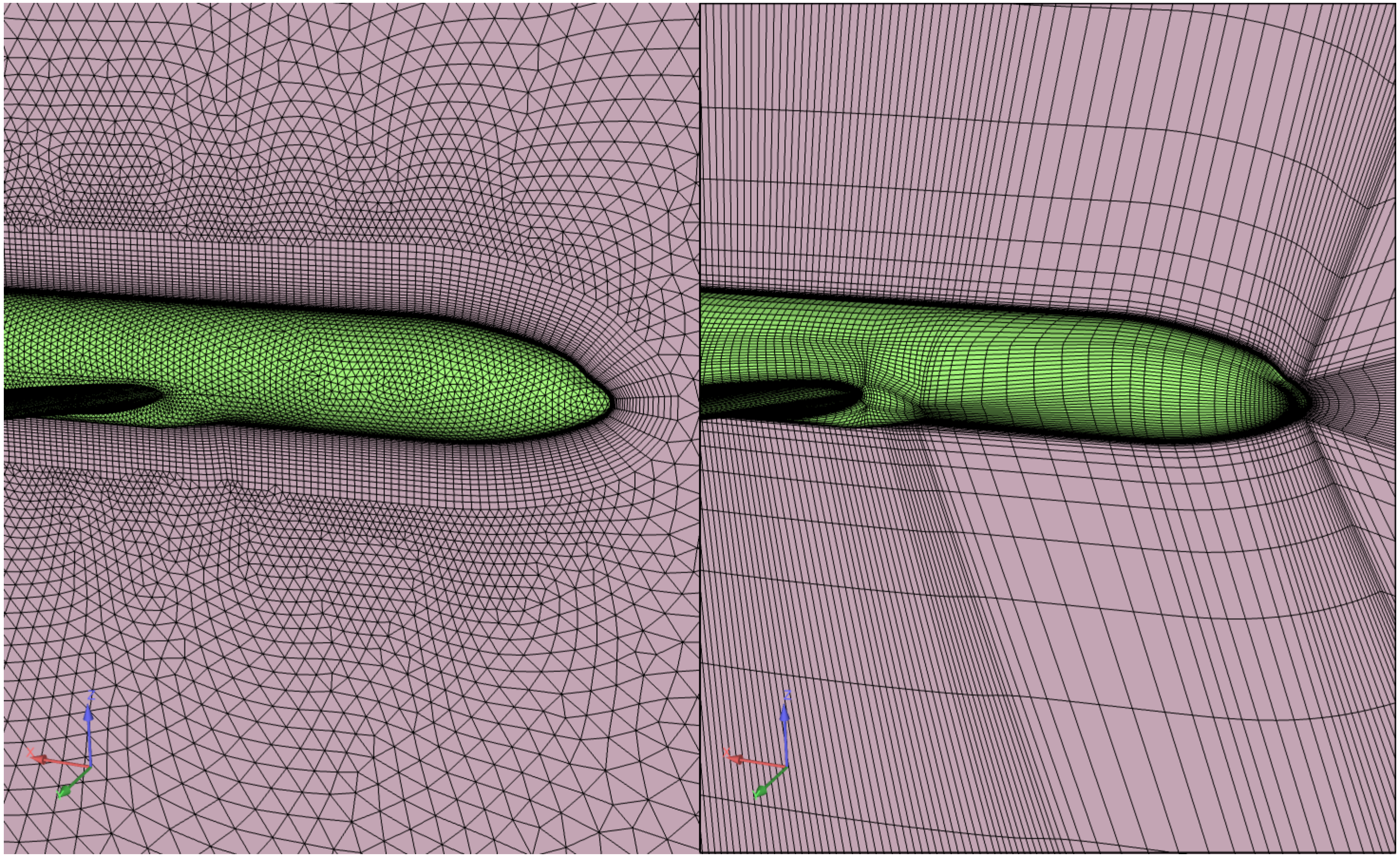
Turbulence modeling, CFD, and the mesh



- **Structured or unstructured meshing methods.**
  - The meshes generated using any of these methods will give similar results if good practices are followed.
  - However, generating meshes using unstructured methods is much easier.
  - This is not a consensus but structured meshes can generate better results.

Turbulence modeling, CFD, and the mesh



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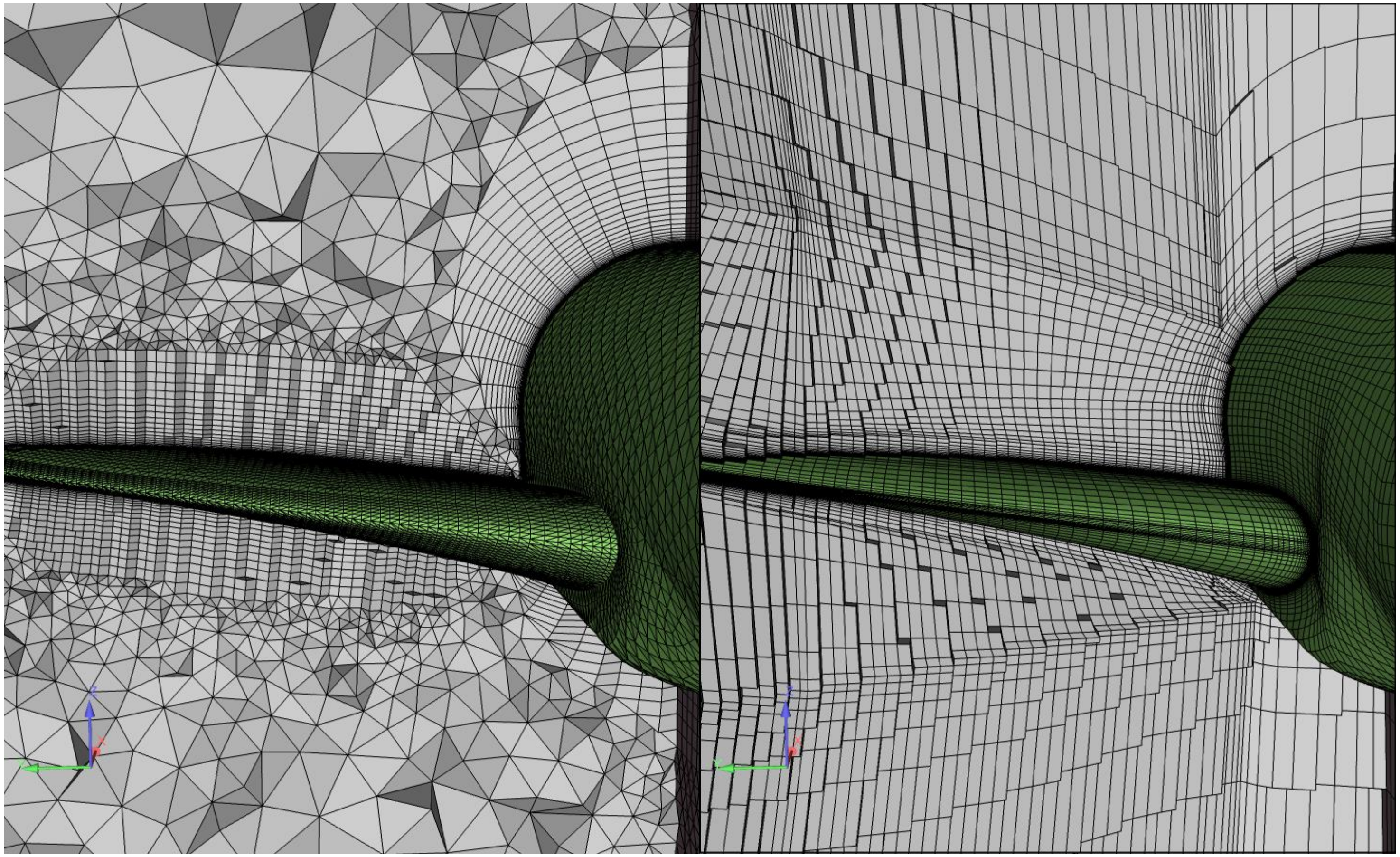
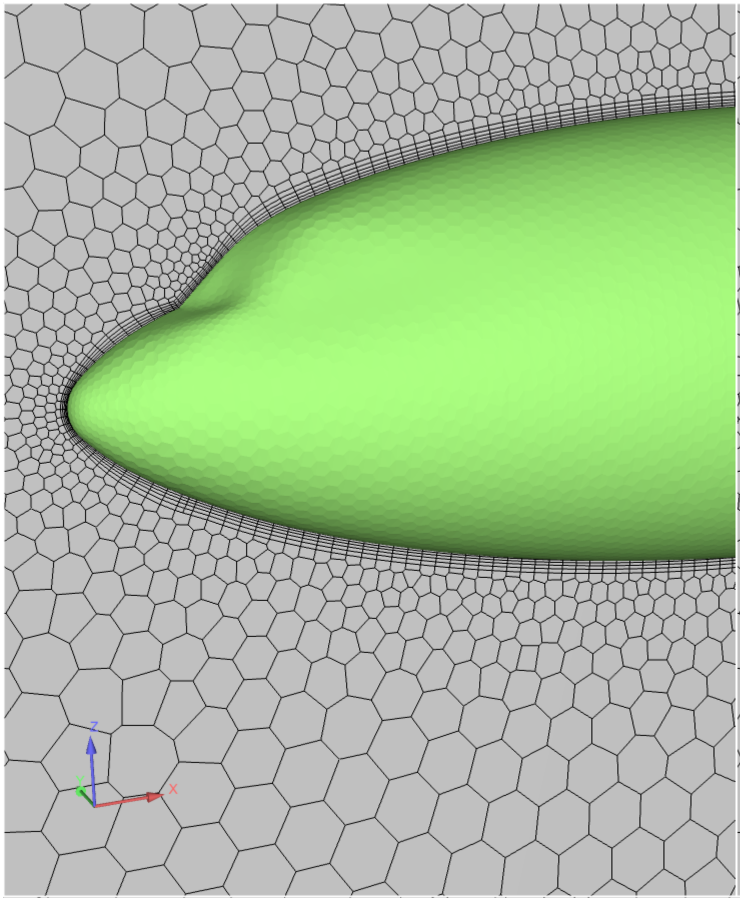
Turbulence modeling, CFD, and the mesh

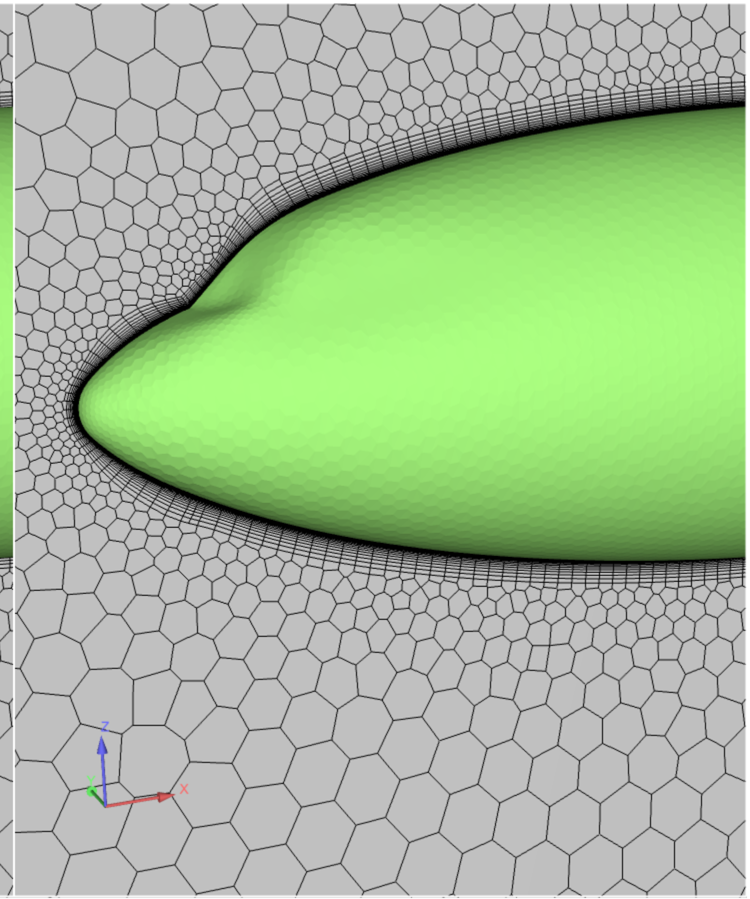


- **Structured or unstructured meshing methods.**
  - The meshes generated using any of these methods will give similar results if good practices are followed.
  - However, generating meshes using unstructured methods is much easier.
  - This is not a consensus but structured meshes can generate better results.

Turbulence modeling, CFD, and the mesh



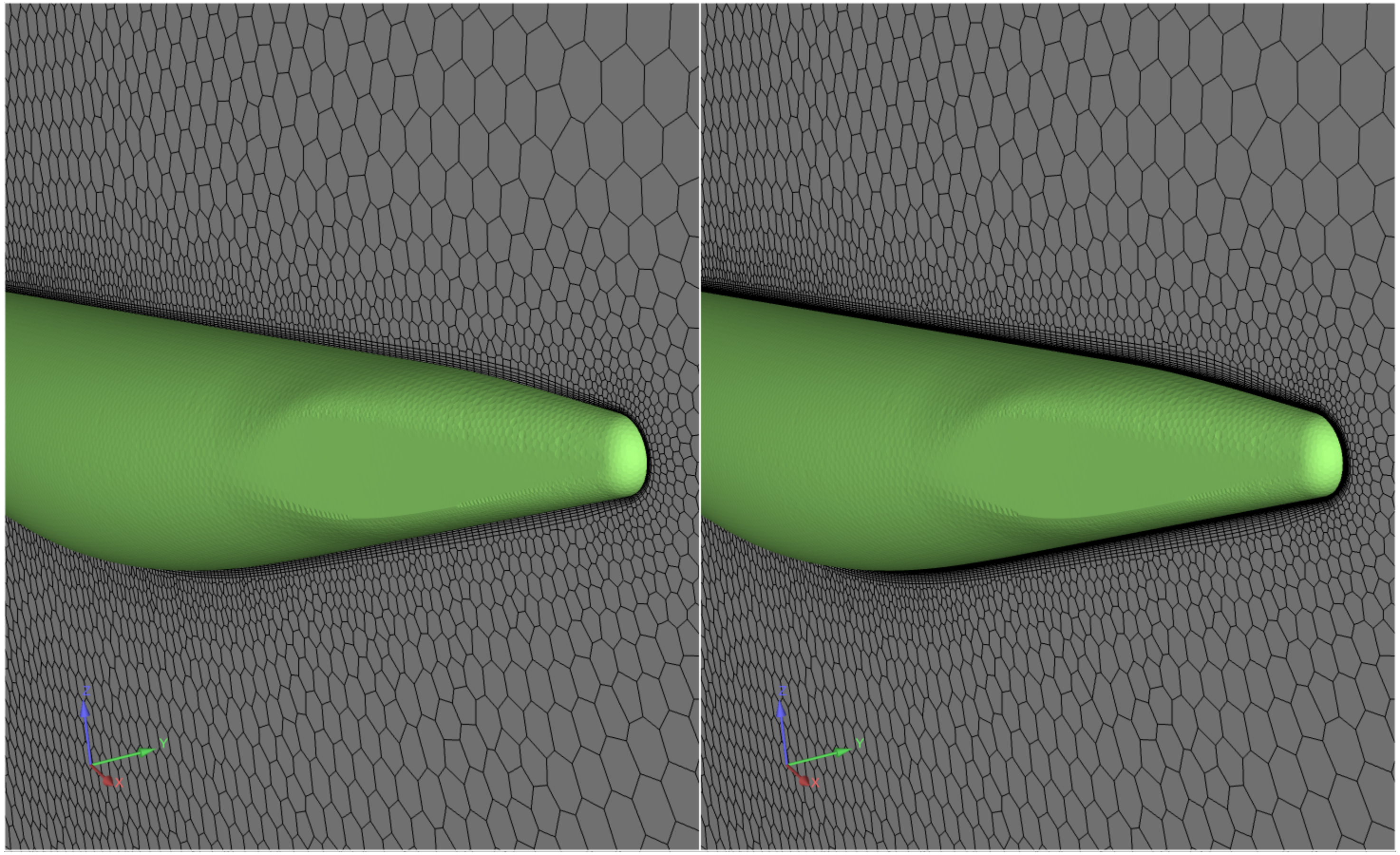
Wall modeling mesh



Wall resolving mesh

- For boundary layer meshing there is no doubt, you need to use hexahedral or prismatic cells.
- If you want to resolve the boundary layer, you need to cluster a lot cells close to the walls.
- In wall resolving meshes, 50% or even more of the total cell count might be located in the inflation layer zone

Turbulence modeling, CFD, and the mesh



Wall modeling mesh — Wall resolving mesh

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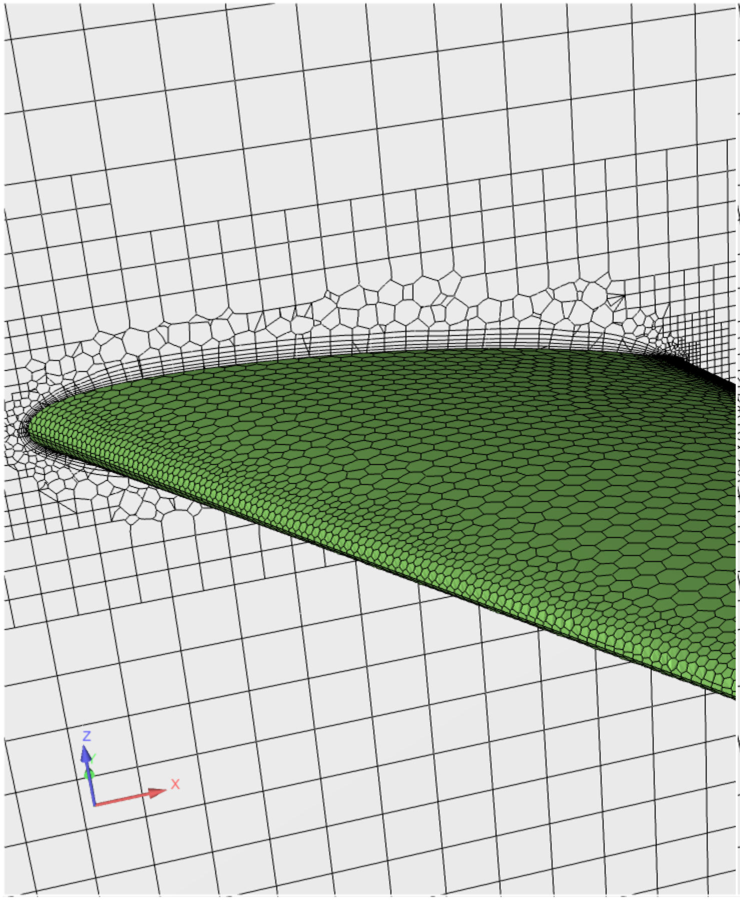
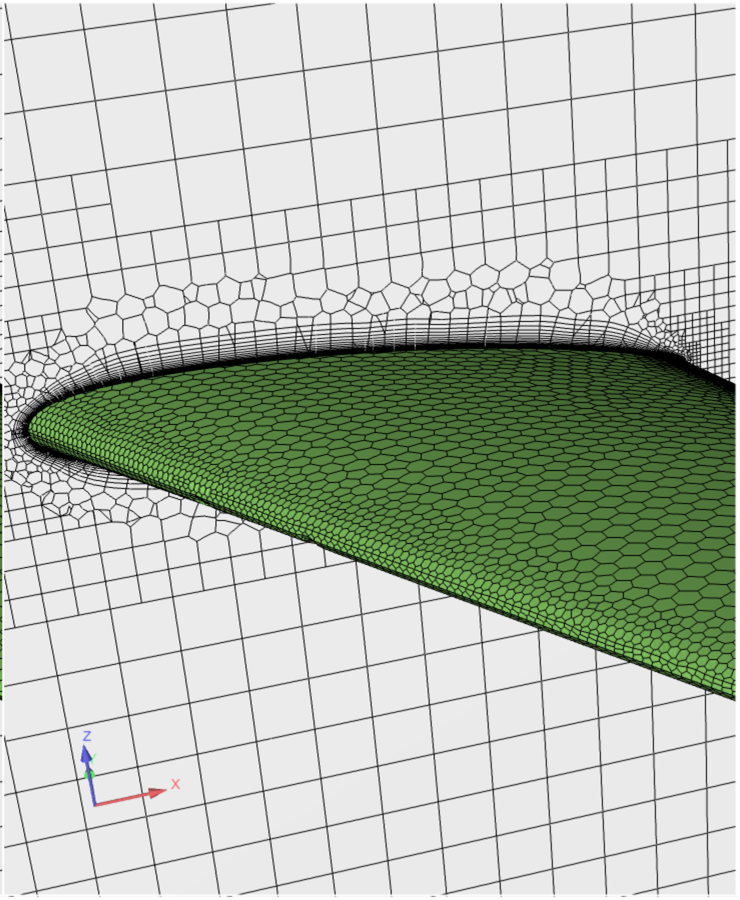
Turbulence modeling, CFD, and the mesh



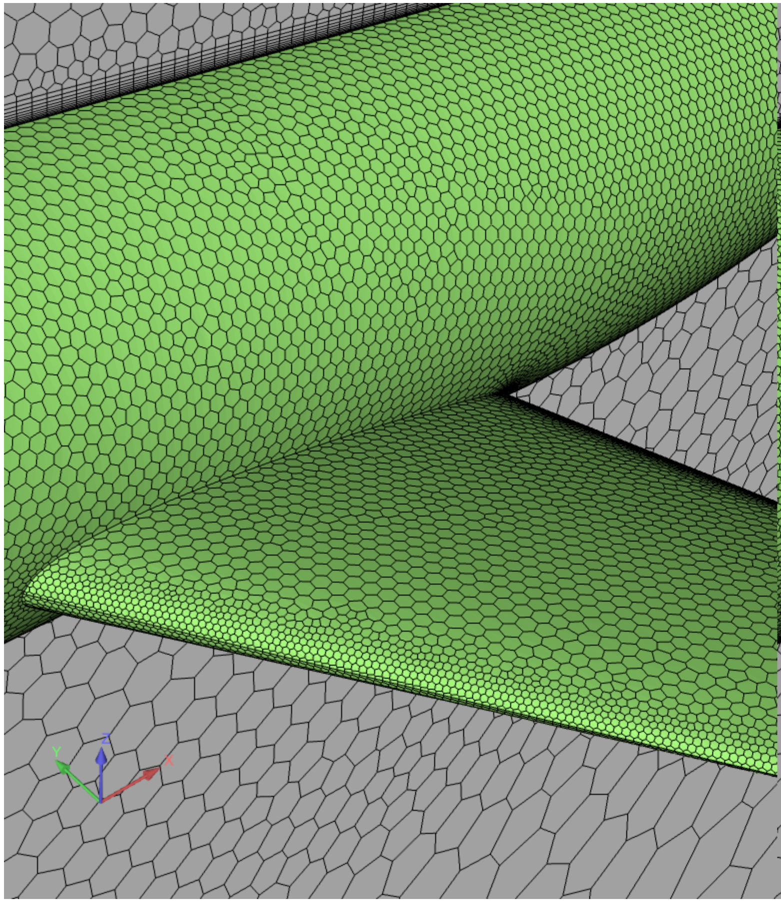
Wall modeling mesh

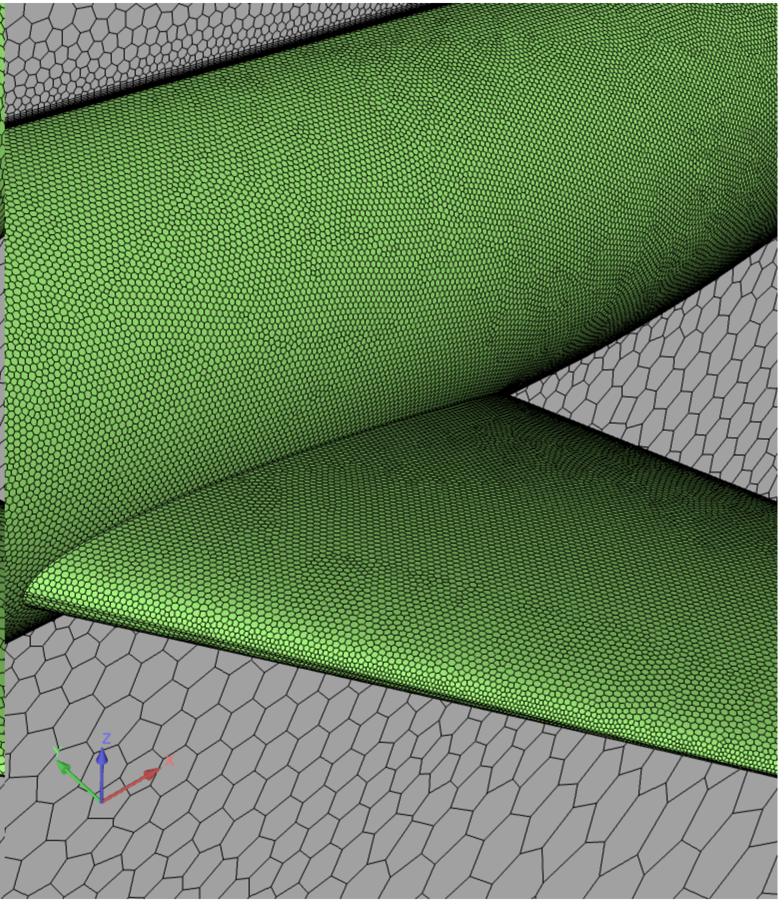


Wall resolving mesh
The final aspect ratio of this mesh is too large

- It is not only about the number of prismatic layers.
- When you use fine inflation layers you should also use a finer surface mesh.
- Otherwise, the aspect ratio of the prismatic layers will be too large and that can have a strong influence on the outcome

Turbulence modeling, CFD, and the mesh



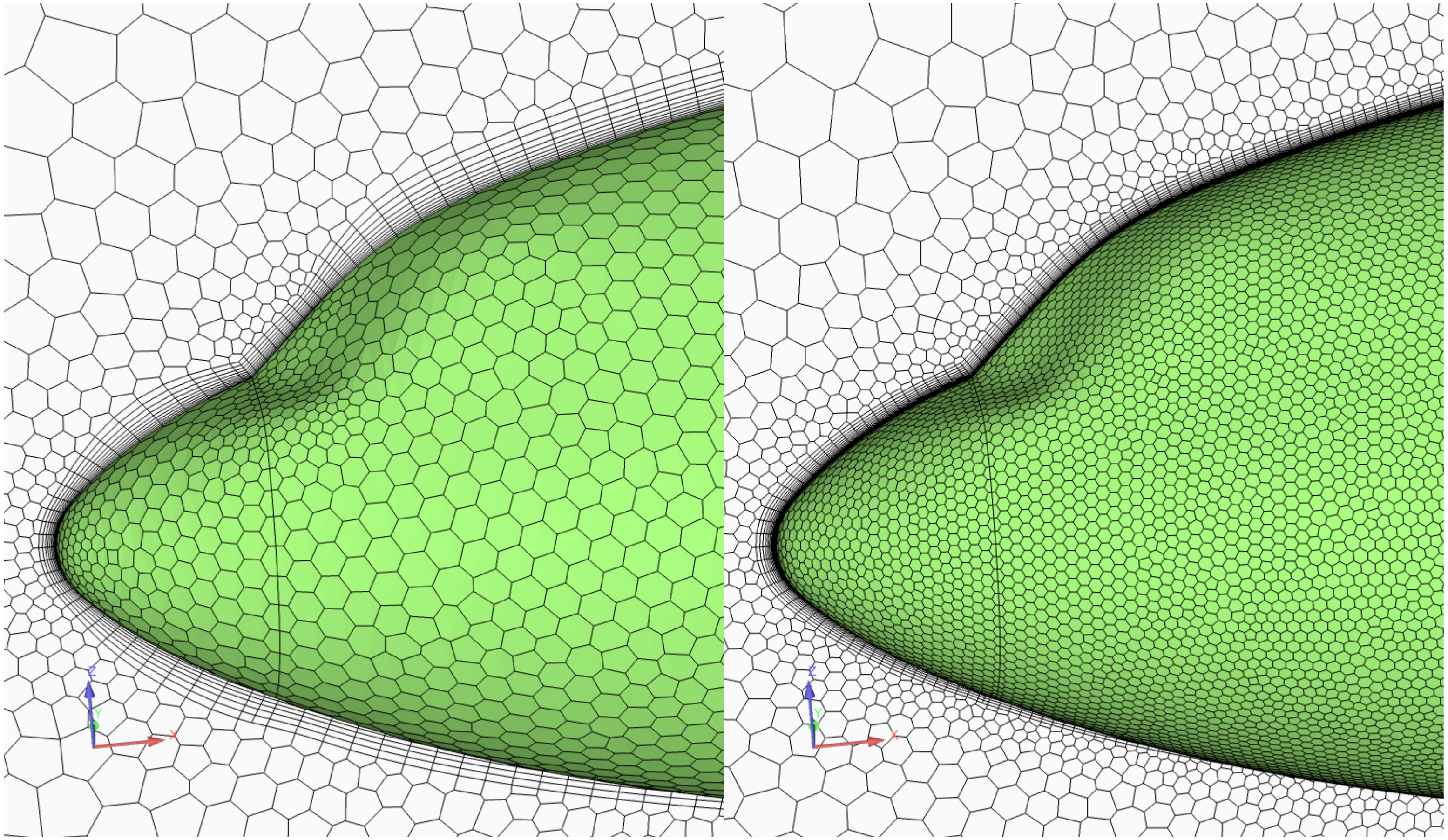


Coarse surface mesh

Fine surface mesh

- Besides the meshing requirements imposed by the turbulent scales, your mesh must also resolve the geometrical features, such as, surface curvature, surface mesh aspect ratio, gaps, and so on.
- Surface meshes with low aspect ratio are likely to fulfill all these requirements.

Turbulence modeling, CFD, and the mesh



Coarse surface mesh | Fine surface mesh

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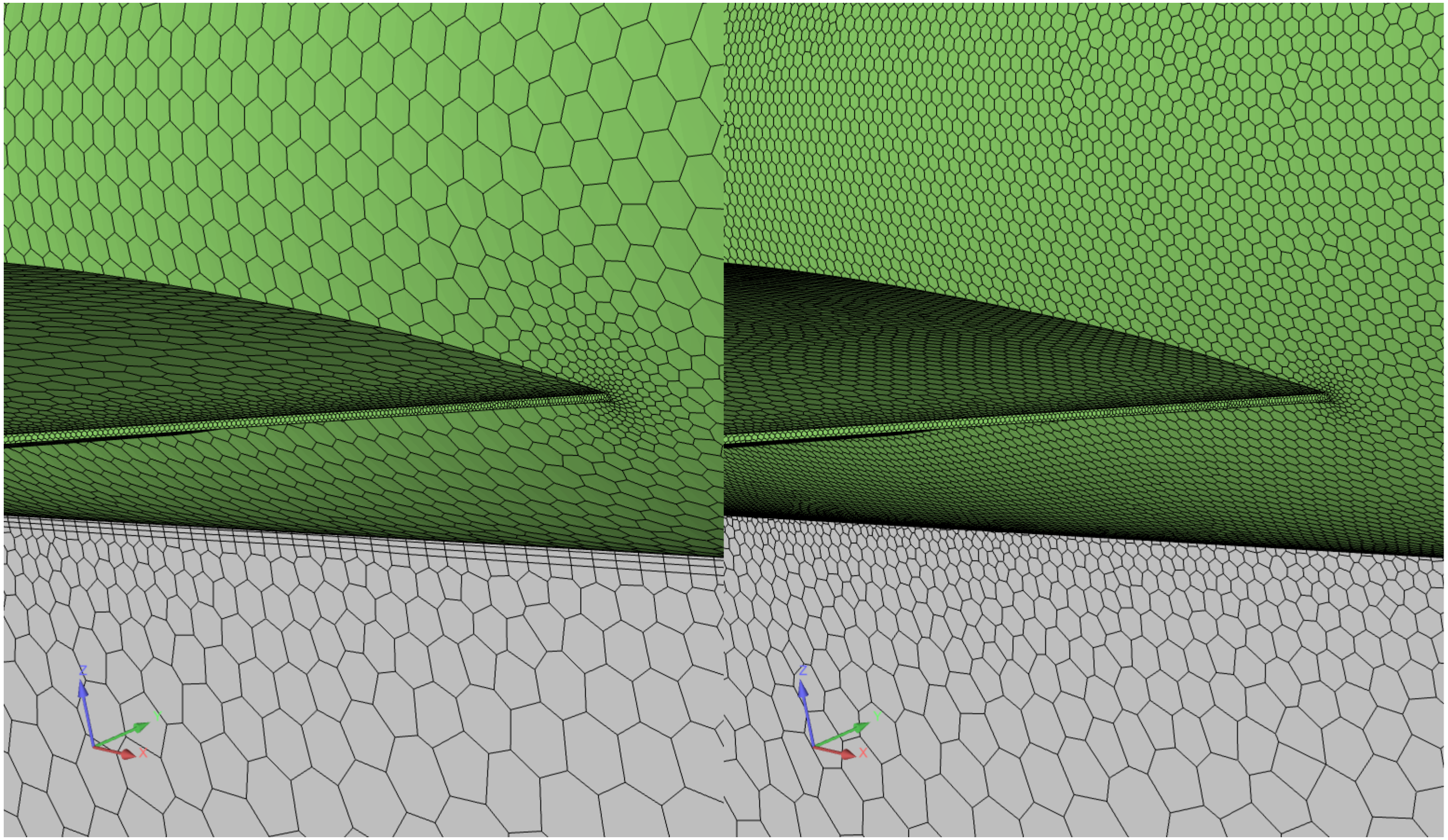
Turbulence modeling, CFD, and the mesh



Coarse surface mesh

Fine surface mesh

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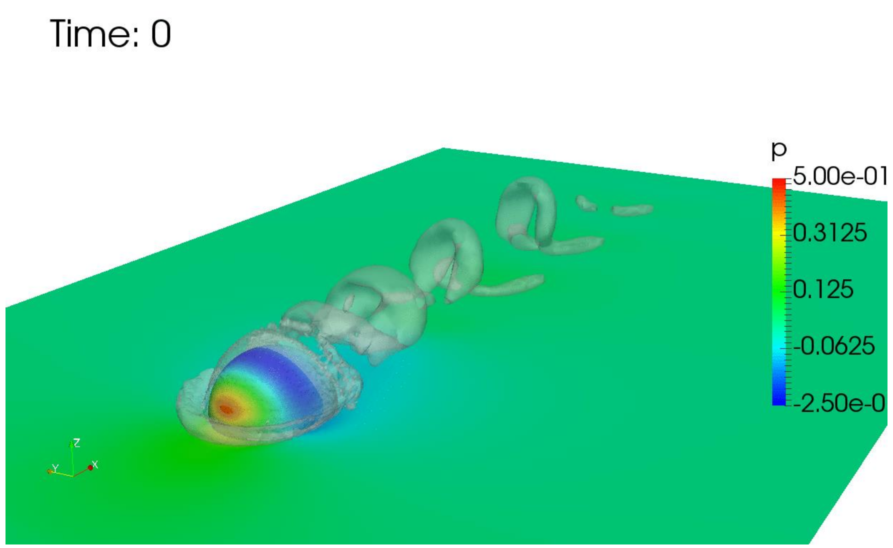
- Surface meshes with low aspect ratio are likely to fulfil all these requirements.

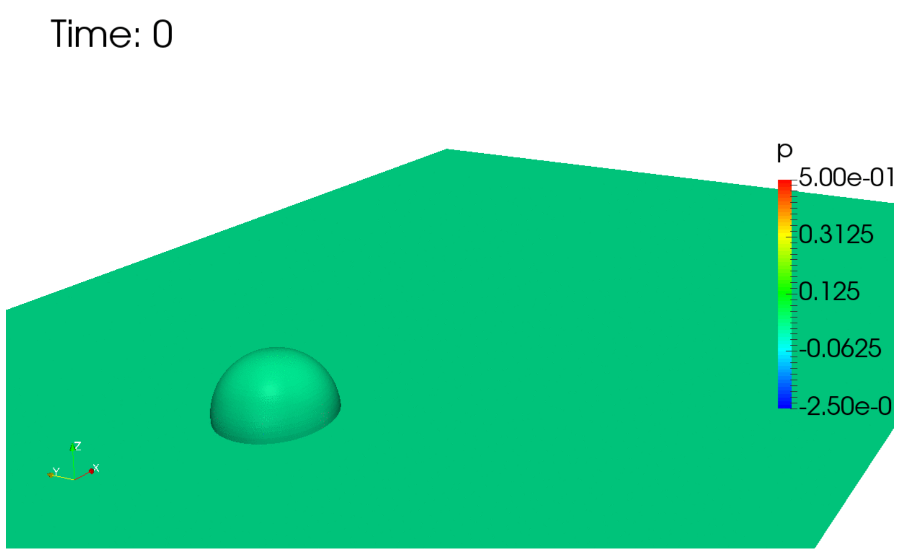
Turbulence modeling and the numerics – Space discretization



Coarse mesh

http://www.wolfdynamics.com/training/turbulence/image7.gif

- The vortices are dissipated due to numerical diffusion (low mesh resolution).
- This case uses as initial conditions the outcome of a steady simulation.
- Due to the low mesh resolution, it is quite difficult to onset the instability if you start from a uniform initial condition.



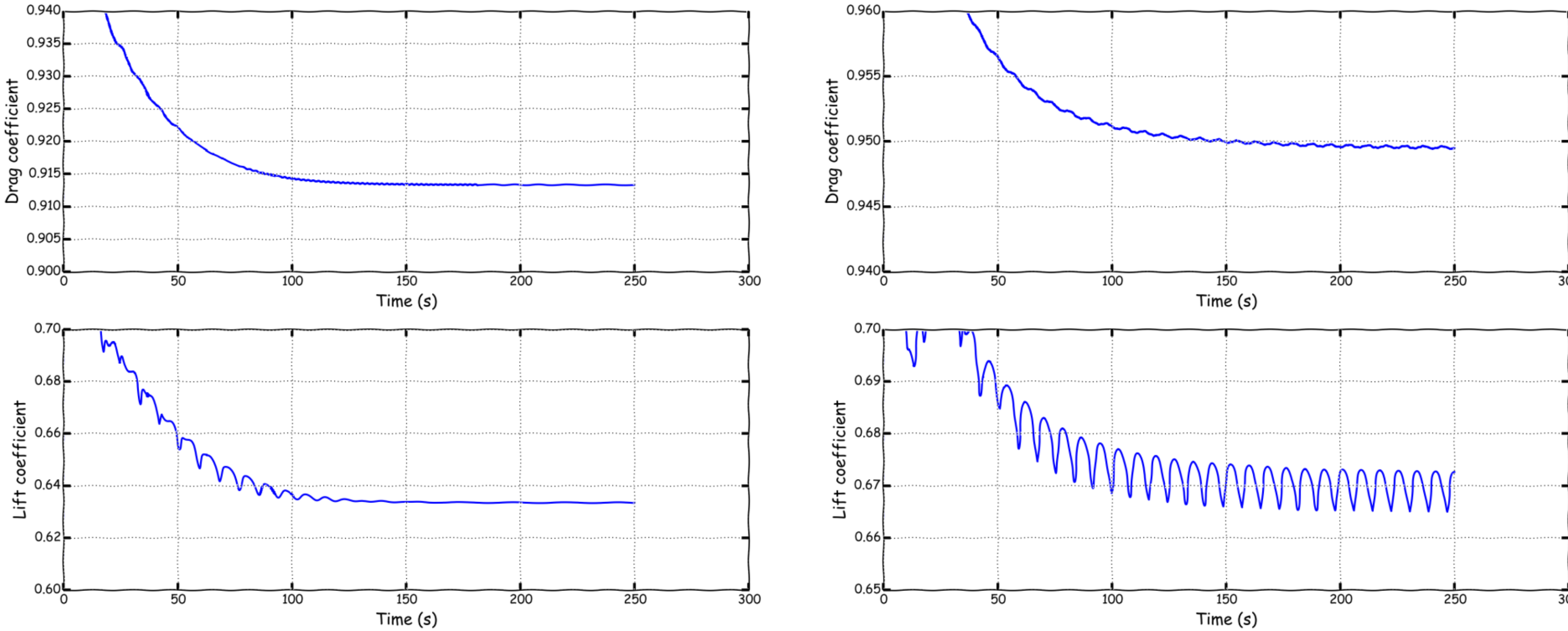
Fine mesh

http://www.wolfdynamics.com/training/turbulence/image8.gif

- The fine mesh captures the small spatial scales that the coarse mesh does not manage to resolve.
- Even with uniform initial conditions, the mesh captures the special scales without numerical dissipation.

Unsteady simulations – Vortices visualized using Q criterion.

Turbulence modeling and the numerics – Space discretization



Coarse mesh

- The coarse mesh does not capture small spatial scales; hence, it adds numerical diffusion to the solution.
- In this case, you will have the impression that you have arrived at a steady state.

Fine mesh

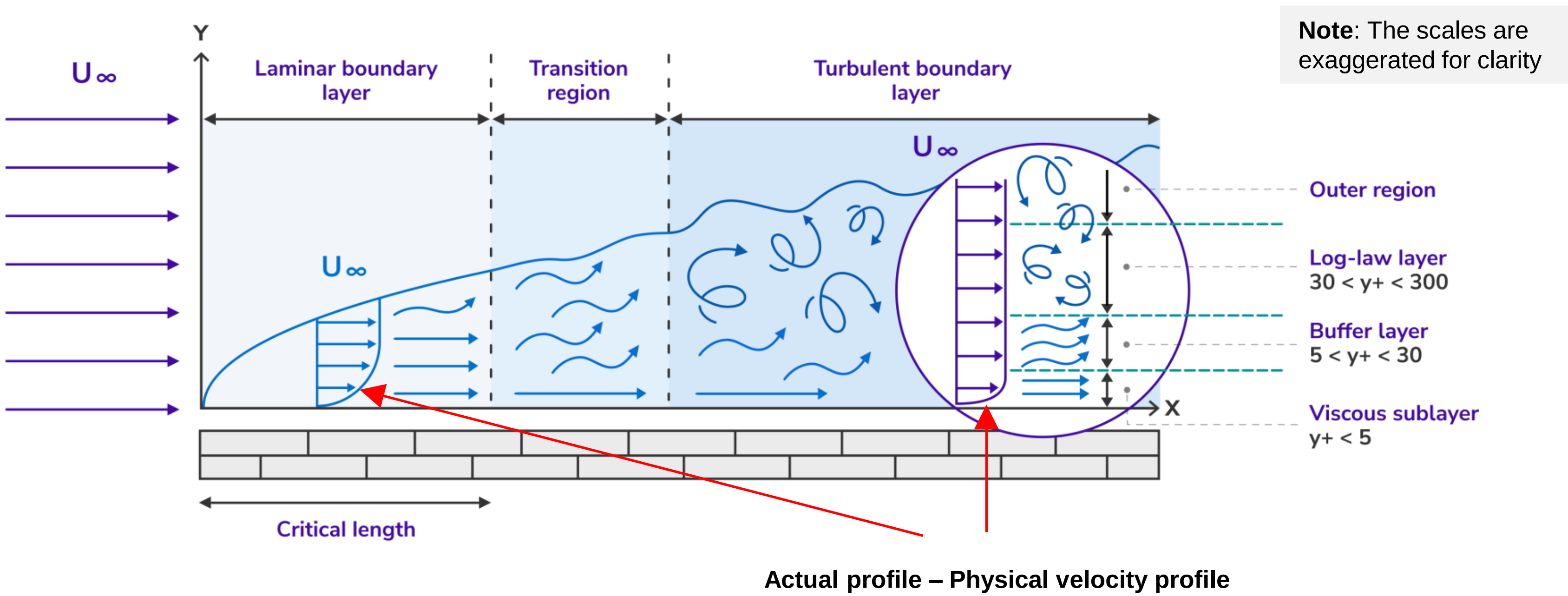
- The fine mesh captures the small spatial scales that the coarse mesh does not manage to resolve.
- This mesh capture well the unsteady behavior.

Unsteady simulations – Drag and lift coefficient signals

Roadmap to Lecture 3

1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales to integral scales~~
2. ~~Energy spectrum and energy cascade. Integral length scale and grid length scale~~
3. **Turbulence near the wall - Law of the wall**
4. A glimpse to a turbulence model

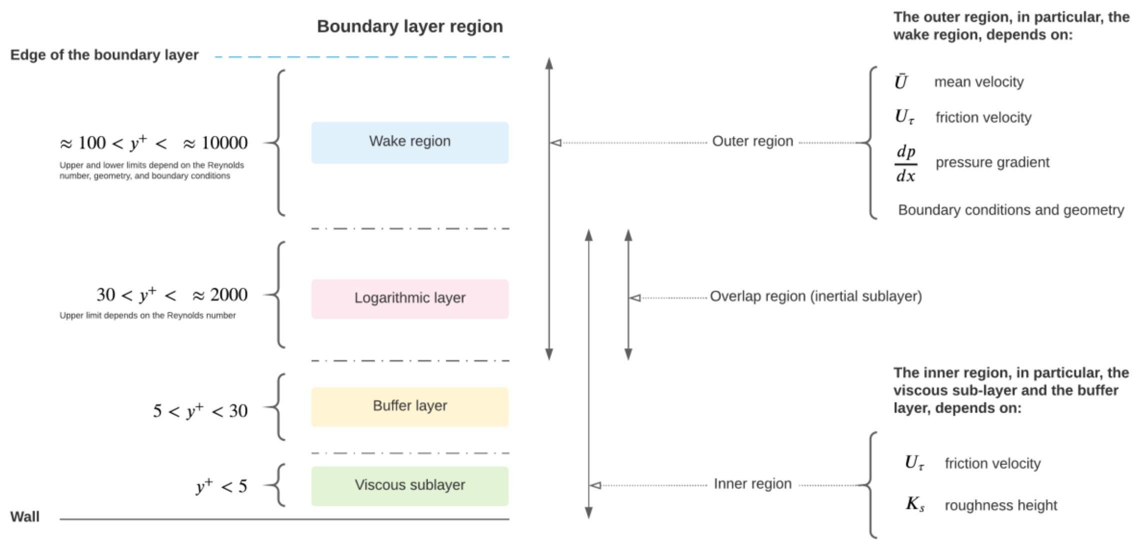
Turbulence near the wall – Boundary layer



Boundary layer (Laminar-Transitional-Turbulent flow)

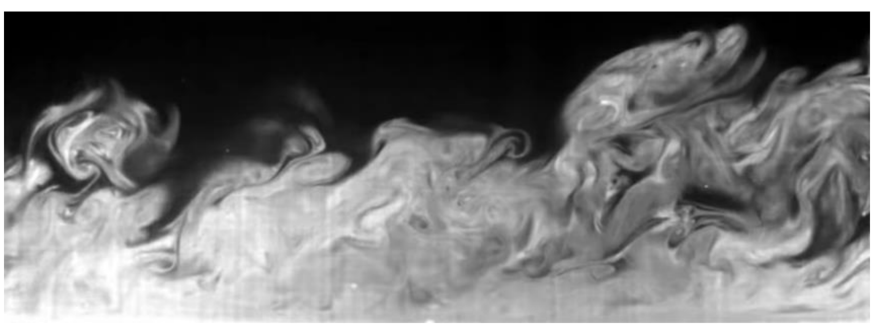
- Near the walls, in the boundary layer (BL), the velocity changes rapidly.
- In turbulence modeling in CFD, the most important zones are the viscous sublayer and the log-law layer.
- The buffer layer is the transition layer which we try to avoid as much as possible.
- Turbulence modeling in CFD requires different considerations depending on whether you solve the viscous sublayer, model the log-law layer, or solve the whole boundary layer (including the buffer zone).

Turbulence near the wall – The boundary layer and the Law of the wall

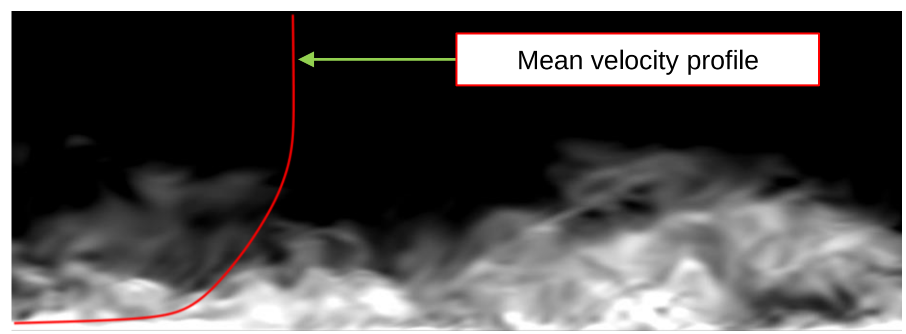


Regions in the turbulent boundary layer.
Adapted from references [1, 2].
The figure is not to scale. The figure does not scale in reference to the images below.

Turbulent boundary layer on a flat plate.



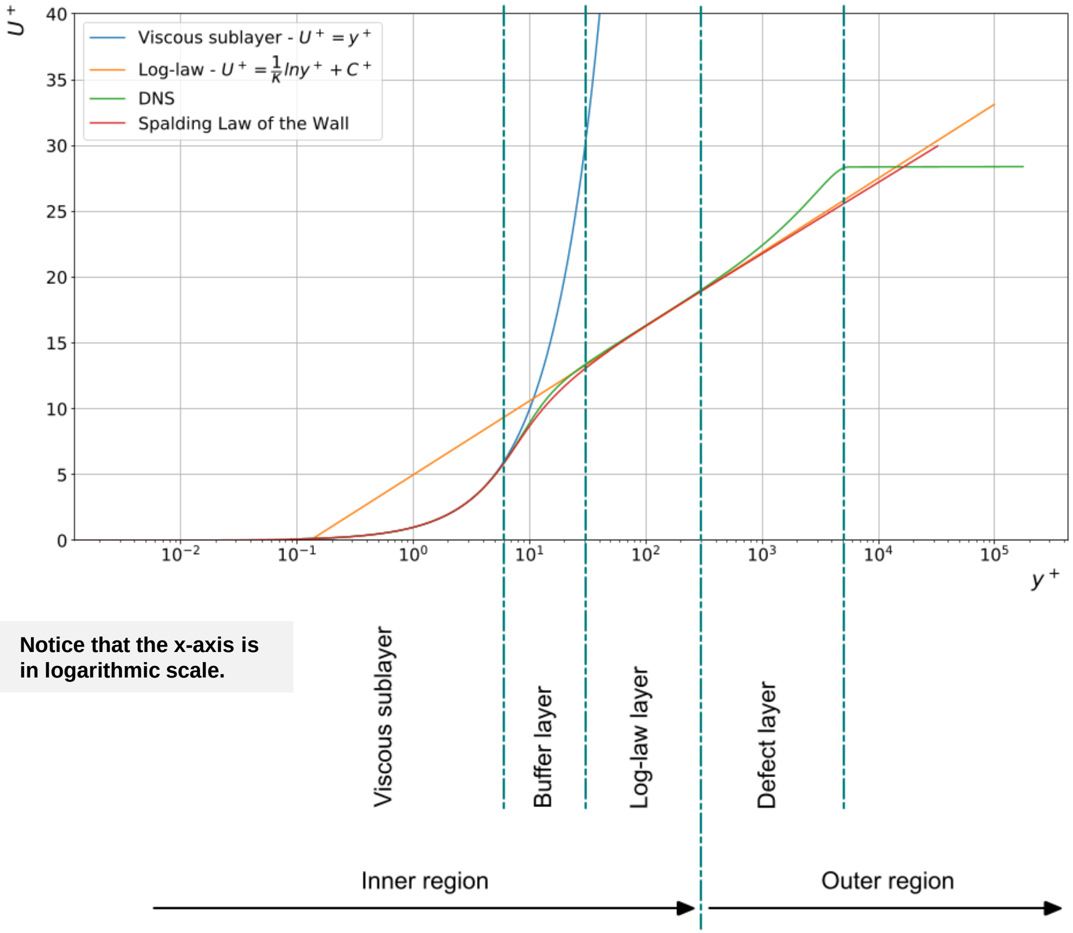
Experimental results [3].



DNS simulation. Instantaneous velocity field [4].

[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.
[2] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[3] Photo credit: https://arxiv.org/abs/1210.3881. Copyright on the images is held by the contributors. Apart from Fair Use, permission must be sought for any other purpose.
[4] J. Lee, T. Zaki. Transitional boundary layer dataset. http://turbulence.pha.jhu.edu/Transition_bl.aspx

Turbulence near the wall – Law of the wall



Notice that the x-axis is in logarithmic scale.

- The law of the wall is one of the cornerstones of fluid dynamics and turbulence modeling.
- It is based on the early works of Prandtl [1], Von Karman [2], Nikuratze [3], and Millikan [4].
- Many other authors have derived/confirmed the law of the wall using experimental or numerical measurements.
- By using dimensional analysis and taking the right assumptions, the following expression can be derived,

$$\frac{U}{u_\tau} = f\left(\frac{y u_\tau}{\nu}\right)$$

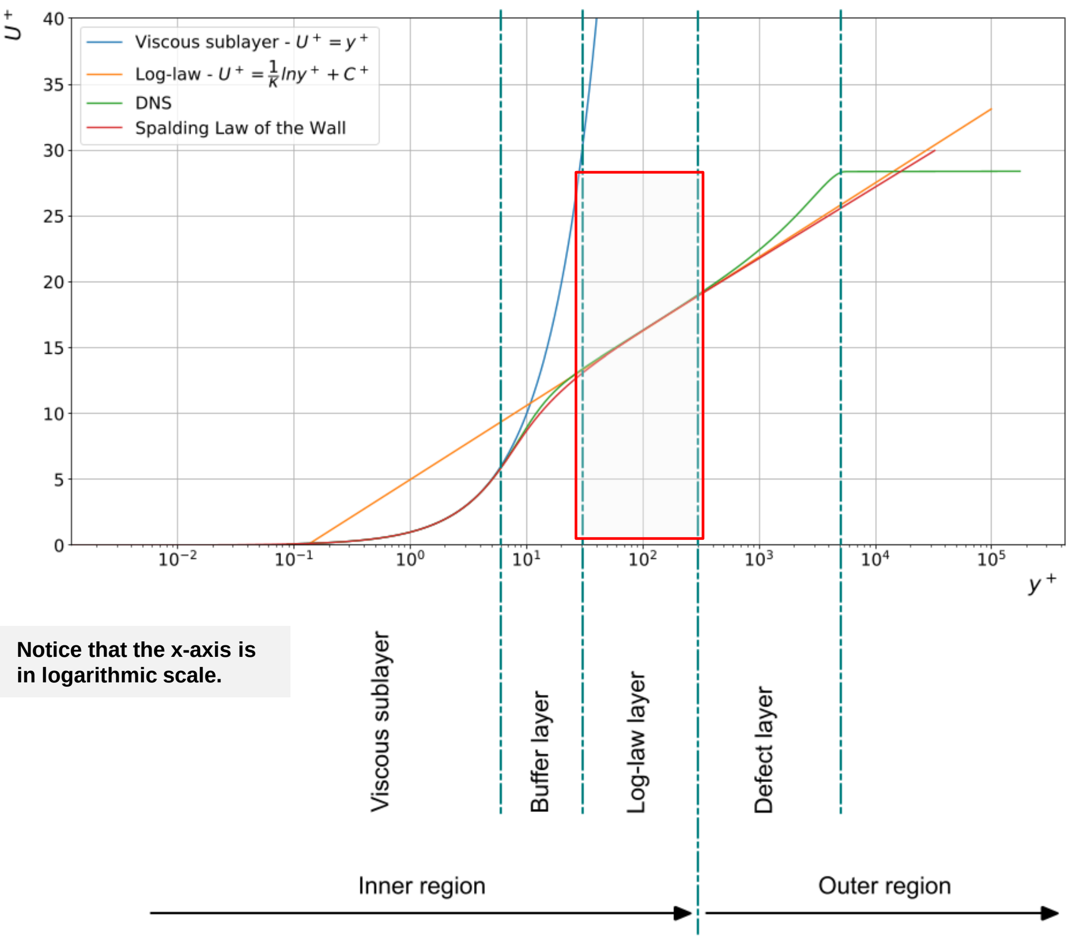
- Or by using non-dimensional groups (u^+ and y^+),

$$u^+ = f\left(y^+\right)$$

- The law of the wall basically describes the mean velocity distribution close to the wall, in the inner region of the boundary layer.
- Where viscous effects dominates.

[1] L. Prandtl. “Report on Investigation of Developed Turbulence”. Technical Memorandum 1231, NACA. 1949. Translation of the 1925 paper.
[2] T. von Karman. “Mechanical Similitude and Turbulence”. Technical Memorandum 611, NACA. 1931. Translation of the 1930 paper.
[3] J. Nikuradse. “Law of Flow in Rough Pipes”. Technical Memorandum 1292, NACA. 1950. Translation of the 1933 paper.
[4] C. Millikan. “A critical discussion of turbulent flows in channels and circular tubes”. Proc. Fifth Int. Congress for Applied Mechanics, Harvard and MIT, 1938.

Turbulence near the wall – Logarithmic law or log-law



Notice that the x-axis is in logarithmic scale.

- The logarithmic law, refers to the region of the inner-region of the boundary layer (or the overlap between the inner and outer regions) that can be described using a simple analytic function in the form of a logarithmic equation.
- This is one of the most famous empirically determined relationships in turbulent flows near solid boundaries.
- Measurements show that, for both internal and external flows, the streamwise velocity in the flow near the wall varies logarithmically with distance from the surface.
- The log-law, is stated as follows,

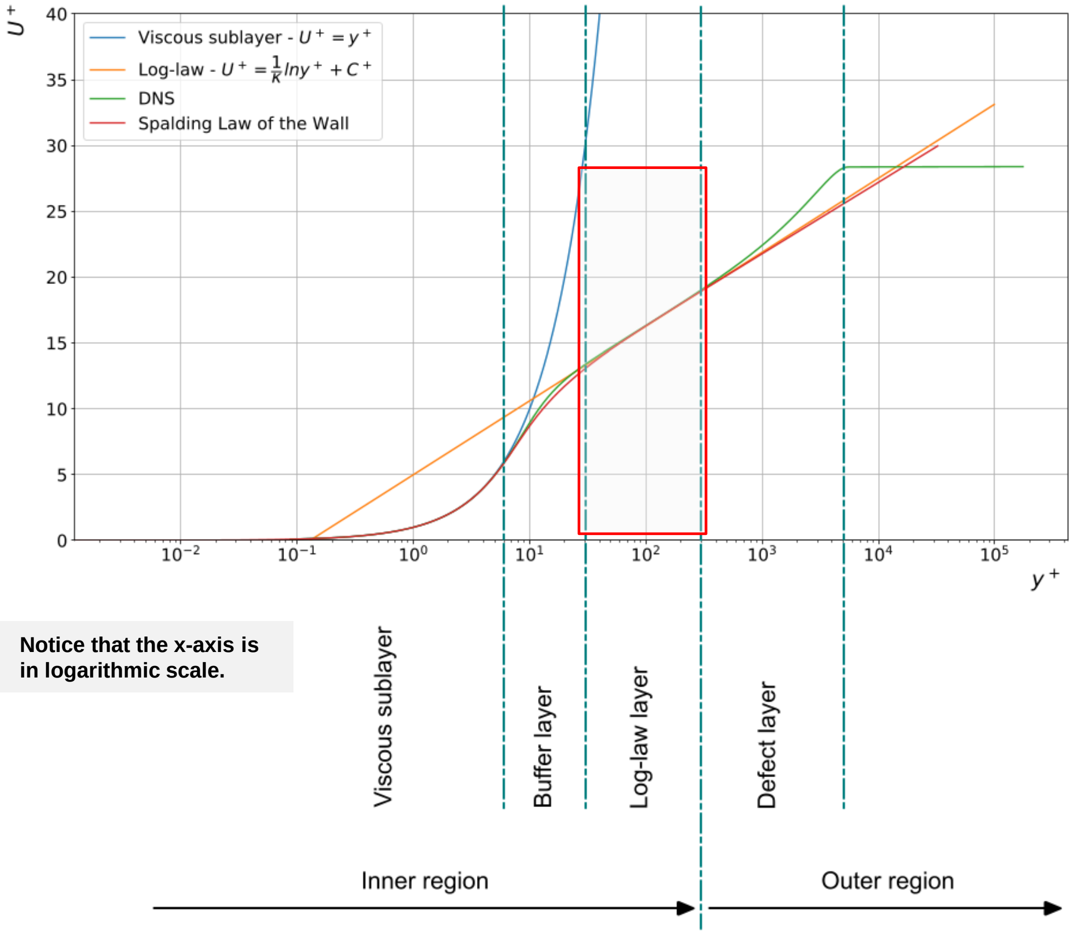
$$u^+ = \frac{1}{\kappa} \ln y^+ + C^+$$

- Where the most common values for the constants appearing in the previous function are,

$$\kappa \approx 0.41 \qquad C^+ \approx 5.0$$

- Reported values for the constant C+ can go anywhere from 4.5 to 5.5.
- Reported values of the Karman constant κ can go anywhere from 036 to 0.42.

Turbulence near the wall – Logarithmic law or log-law



Notice that the x-axis is in logarithmic scale.

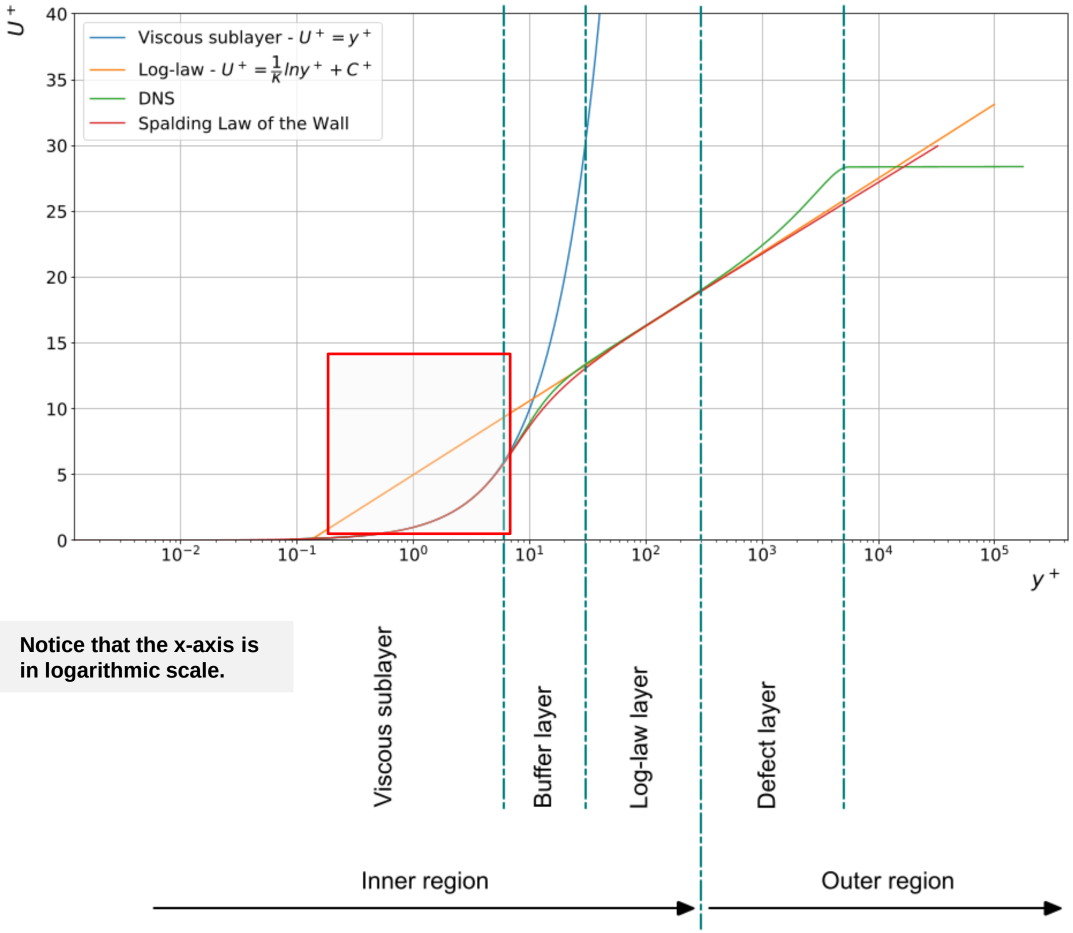
- It is interesting to mention that after the log-law was derived, it took some time to determine the constant values using experimental measurements.
- The logarithmic law or log-law is valid for values of y^+ ranging from,

$$30 < y^+ < 300$$

- The previous range of y^+ values is the most common one found in literature.
- In reality, this range depends on the Reynolds number.
- The upper limit can be as high as 2000 or more.
- For practical purposes in CFD, let us say that the lowest limit is 30, and the upper one is 600.
- The log-law can be derived using dimensional analysis [1, 2, 3].
- In the literature and based on the assumptions taken, you will find different roads to arrive to the log-law, it is up to you to choose one.
- I like to use a particular definition that uses the most physically correct assumptions (in my opinion).

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.
[3] J. O. Hinze. “Turbulence”. McGraw-Hill, New York, 1975.

Turbulence near the wall – The viscous sublayer



Notice that the x-axis is in logarithmic scale.

- The viscous sublayer, refers to the region of the inner-region of the boundary layer, very close to the wall where the flow is laminar.
- In this region the flow mean velocity can be described using a simple analytic function.
- The viscous sublayer law, is stated as follows,

$$u^+ = y^+$$

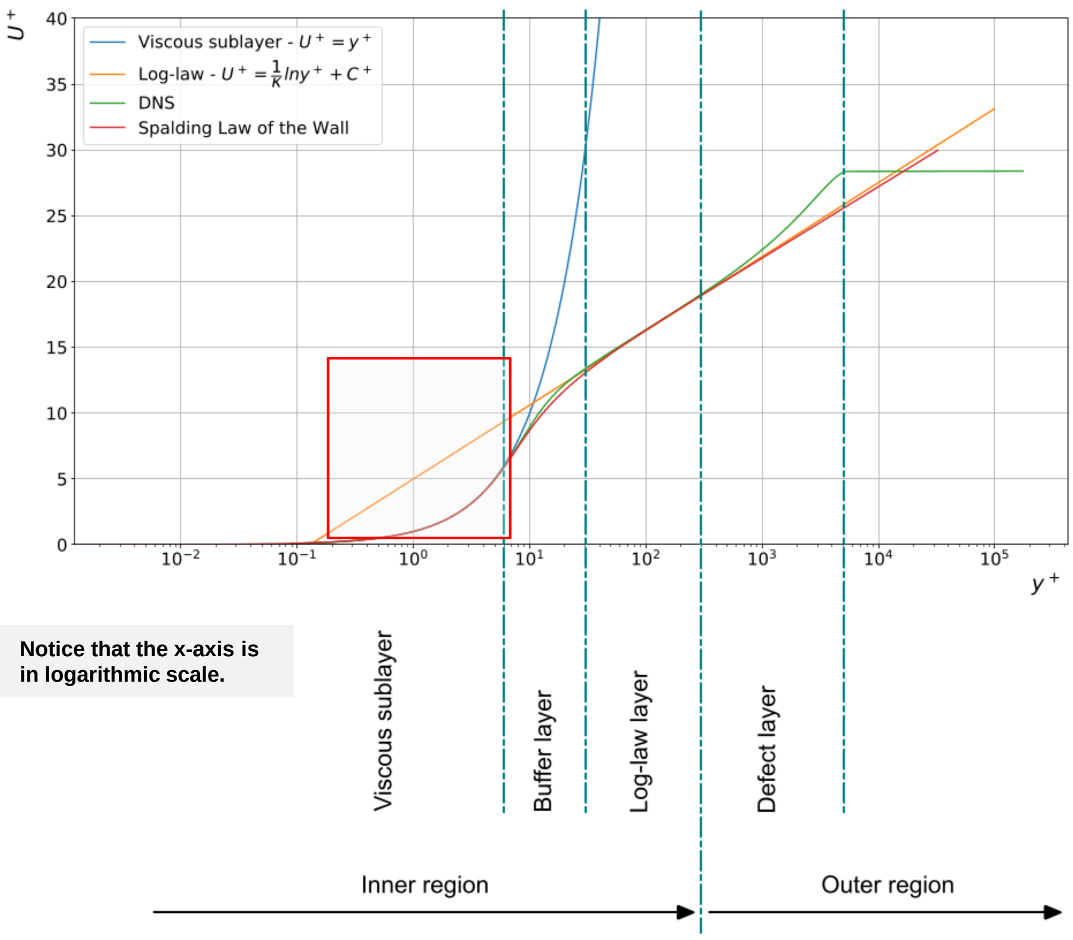
- Remember, this equation is only valid in the viscous sublayer, where the flow is laminar and viscous effects are very strong.
- In the viscous sublayer, the wall shear stresses can be computed using the following expression,

$$\tau_{wall} = \mu \frac{\partial u}{\partial y}$$

- According to the viscous sublayer expression, the behavior of the mean velocity is linear in this region.
- Again, the viscous sublayer law can be derived using dimensional analysis [1, 2, 3].

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.
[3] J. O. Hinze. “Turbulence”. McGraw-Hill, New York, 1975.

Turbulence near the wall – The viscous sublayer



Notice that the x-axis is in logarithmic scale.

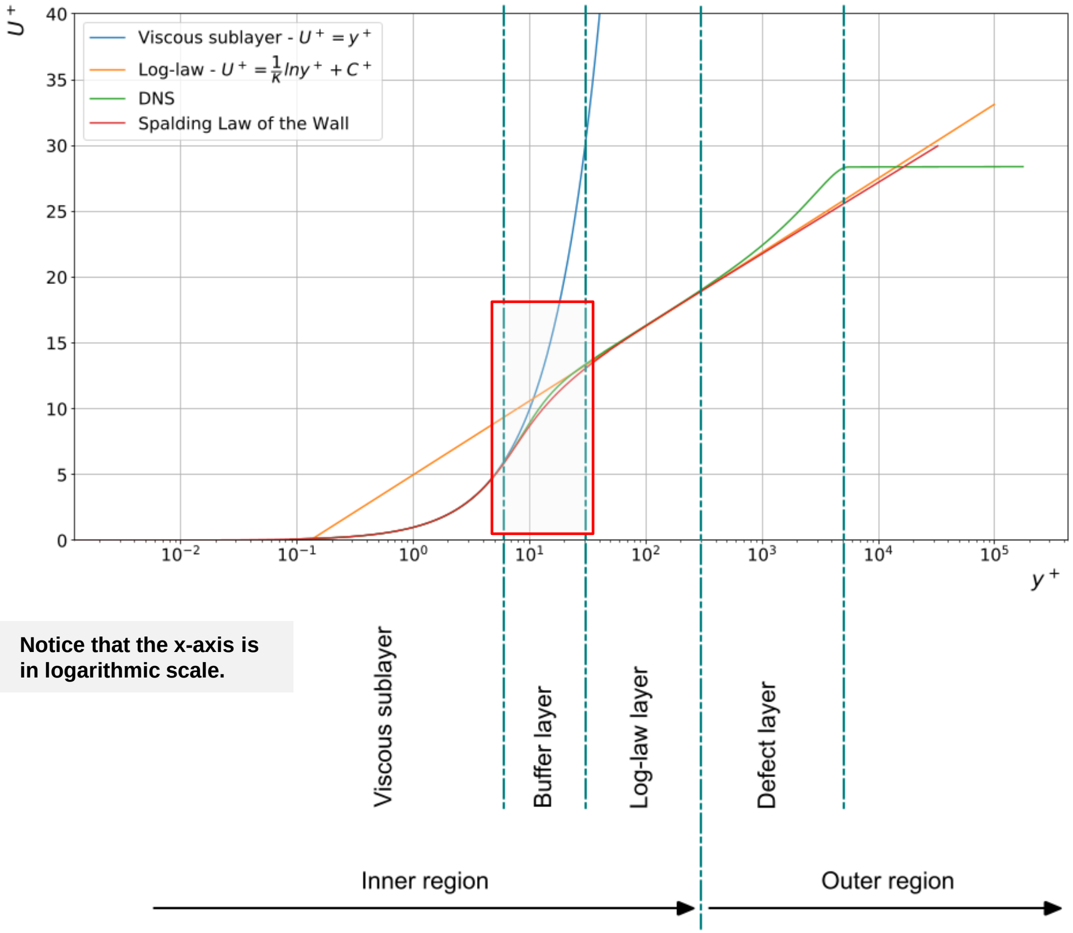
- The behavior of the viscous sublayer has been confirmed using experimental and numerical measurements.
- The viscous sublayer law is valid for values of y^+ ranging from,

$$y^+ < 5$$

- In the literature, you will find different values of the upper limit.
- But maybe this is the most agreed value of y^+ for the viscous sublayer.
- Reported values of y^+ in the viscous sublayer can be as high as 10, and as low as 3.
- The range of validity of the viscous sublayer law has been questioned by many authors [1,2,3,4].
- For practical purposes in CFD, and to avoid excessive computational load, let us say that the upper limit is 6.
- Yes, one unit can make a huge different in CFD.
- We will take a look at the influence of y^+ in the computational overhead later on.

[1] A. Cenedese, G. Romano, R. Antonia. A comment on the linear law of the wall for fully developed turbulent channel flow. Experiments in Fluids 25, 1998.
[2] A. Townsend. The Structure of Turbulent Shear Flow. Cambridge, Cambridge University Press, 1956.
[3] J. Sternberg. A theory for the viscous sublayer of a turbulent flow. Journal of Fluid Mechanics, 13(2), 1962.
[4] D. Spalding. A Single Formula for the Law of the Wall. J. Appl. Mech. Sep 1961.

Turbulence near the wall – The buffer layer



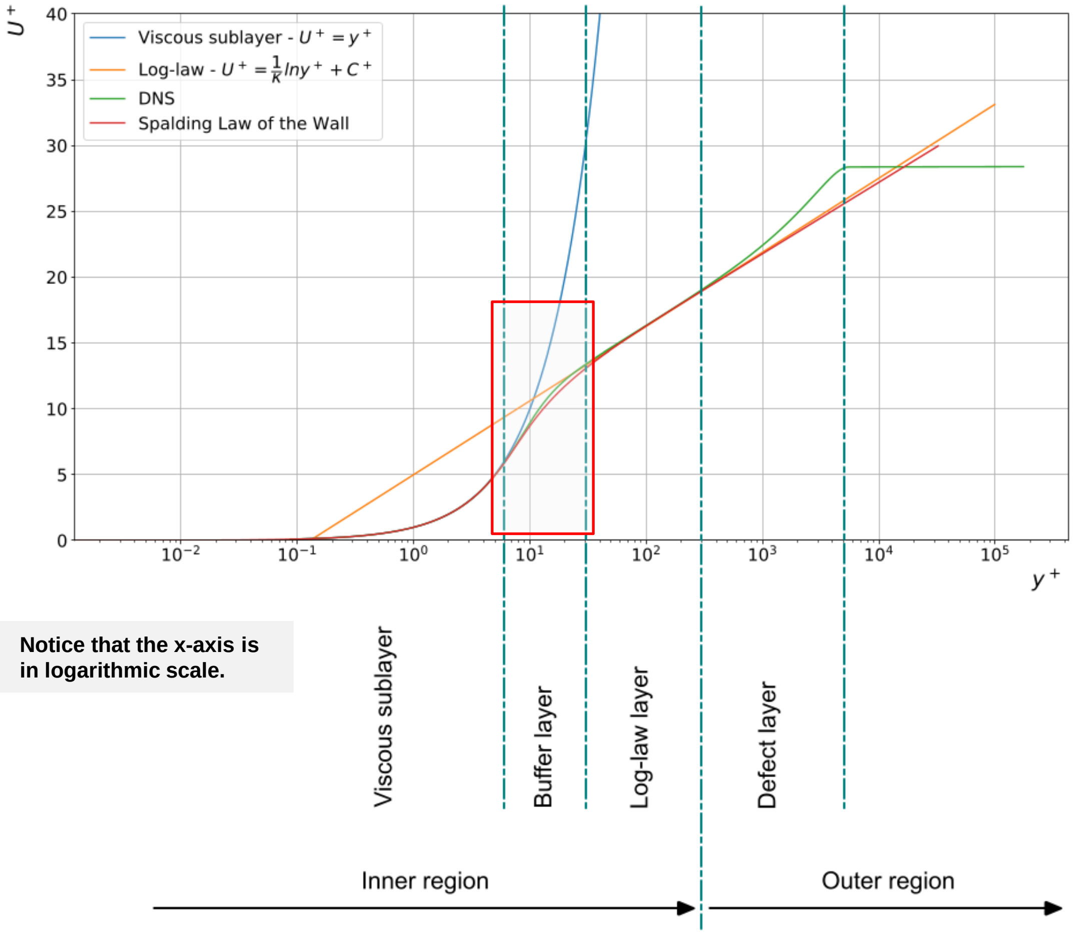
Notice that the x-axis is in logarithmic scale.

- In the buffer layer, where the flow transitions from laminar to turbulent, there is nothing conclusive.
- In CFD, we try to avoid as much as possible this region because there is no single function that can describe accurately this region.
- The buffer layer is enclosed in the following range of y^+ values,

$$5 < y^+ < 30$$

- As usual, in the literature you will find different ranges of y^+ values.
- But maybe this is the most common range.
- We will present in lecture 9 a few different correlations that can be used to approximate the buffer layer (but none of then is general).
- It is worth mentioning that the buffer layer is very energetic. The production of TKE peaks in this region.

Turbulence near the wall – The buffer layer

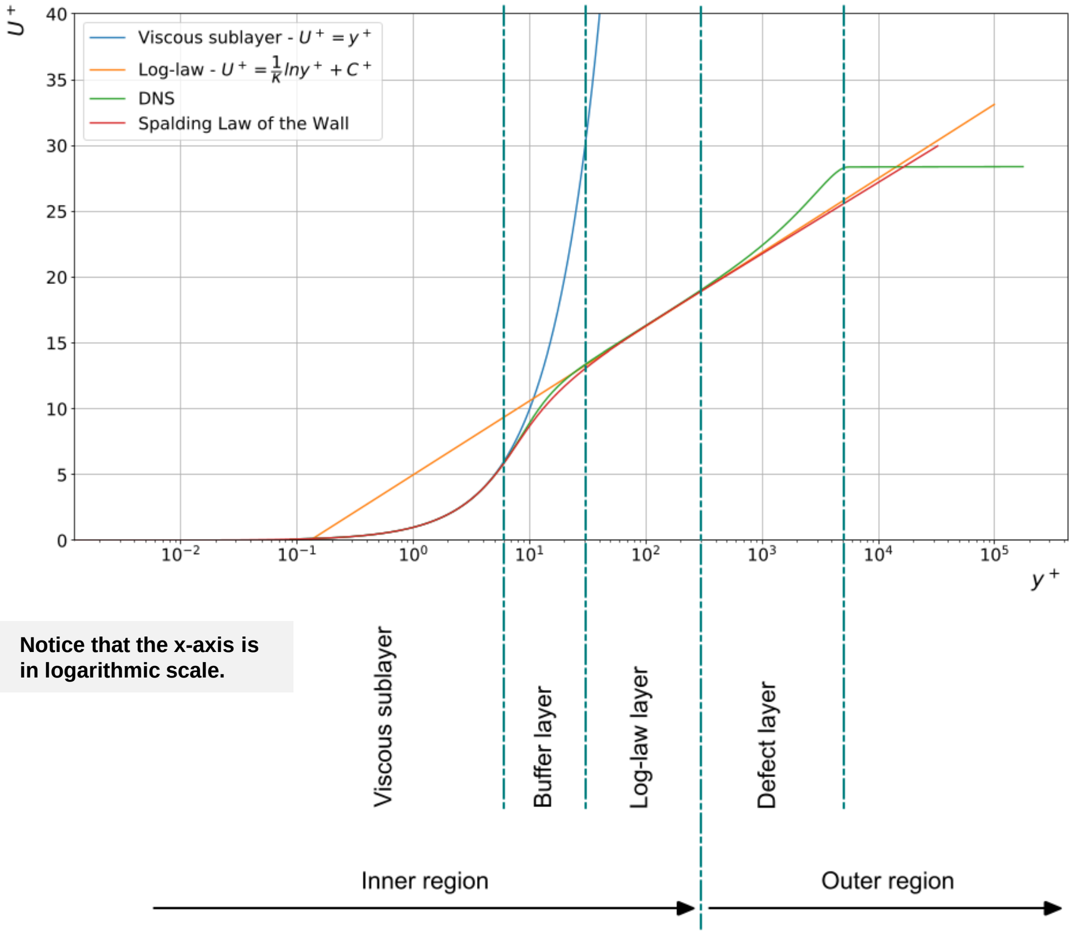


Notice that the x-axis is in logarithmic scale.

- If we solve the viscous sublayer (y+ < 5), inevitably this region will be covered by the mesh.
- What is important is not to place the first cell center next to the wall in this region (or cell node, depending on the method used).
- That is, depending on the wall treatment method used, try to place the first cell center outside the buffer region.
    - If you are using a wall resolving approach, place the first cell center at a distance normal to the wall equivalent to y+ < 5.
    - If you are using a wall modeling approach, place the first cell center at a distance normal to the wall equivalent to y+ > 40.
- If you are conducting simulations at relative low Reynolds number but still turbulent, the log-law layer and the buffer layer is very narrow.
- In such cases, it is recommended to always solve the viscous sublayer.

Turbulence near the wall - Law of the wall

Turbulence near the wall – Definition of y^+ and u^+



Notice that the x-axis is in logarithmic scale.

$$y^+ = \frac{\rho \times U_\tau \times y}{\mu} = \frac{U_\tau \times y}{\nu}$$

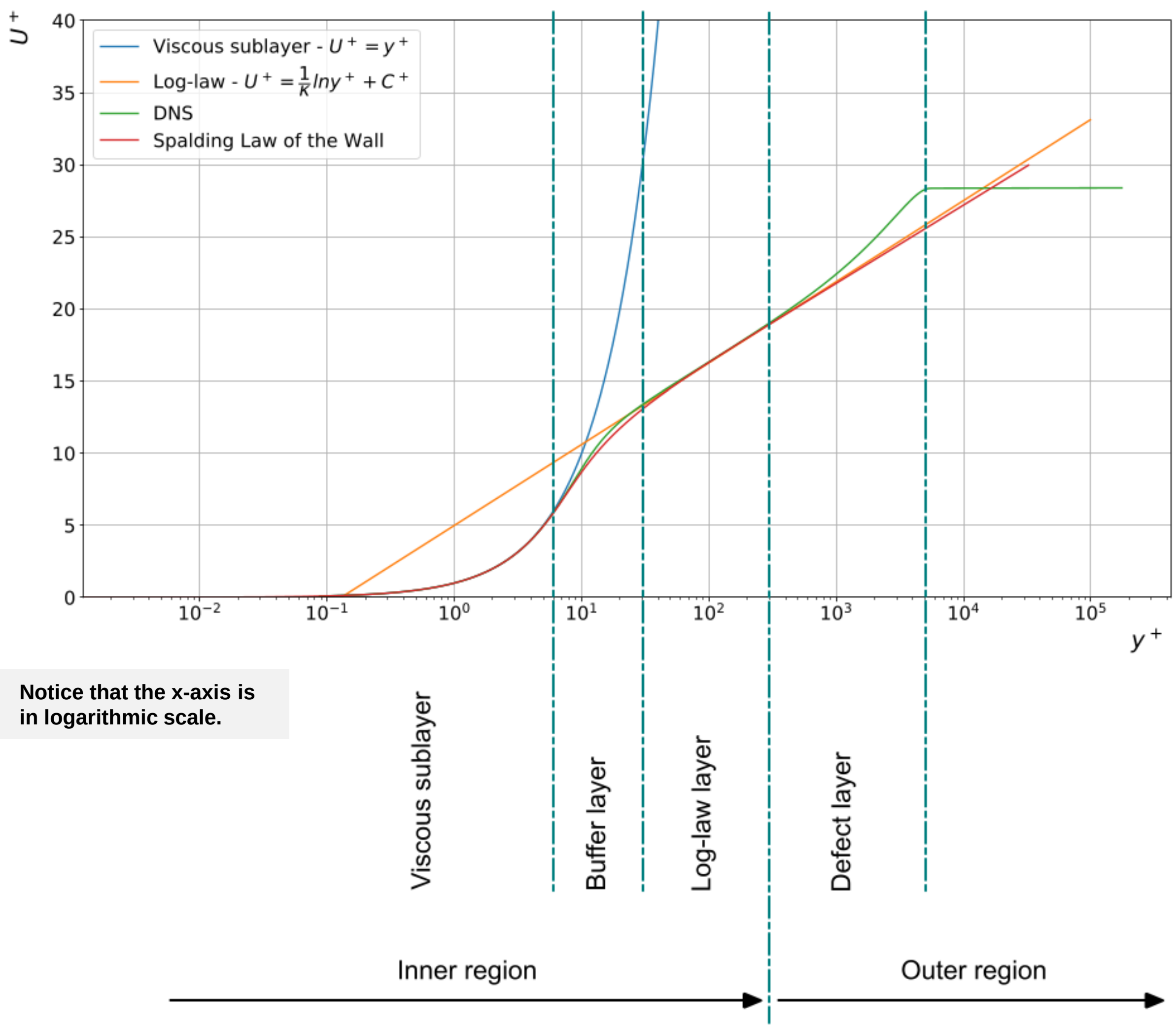
$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$u^+ = \frac{U}{U_\tau}$$

Where y is the distance normal to the wall, U_τ is the shear velocity, and u^+ relates the mean velocity to the shear velocity

- y^+ or wall distance units normal to the wall is a very important concept when dealing with turbulence modeling.
- Remember this definition as we are going to use it a lot.

Turbulence near the wall – Relations according to the y⁺ value



Notice that the x-axis is in logarithmic scale.

Viscous sublayer

$$y^+ < 5$$

$$u^+ = y^+$$

Buffer layer

$$5 < y^+ < 30$$

$$u^+ \neq y^+$$

$$u^+ \neq \frac{1}{\kappa} \ln y^+ + C^+$$

Log-law layer

$$30 < y^+ < 300$$

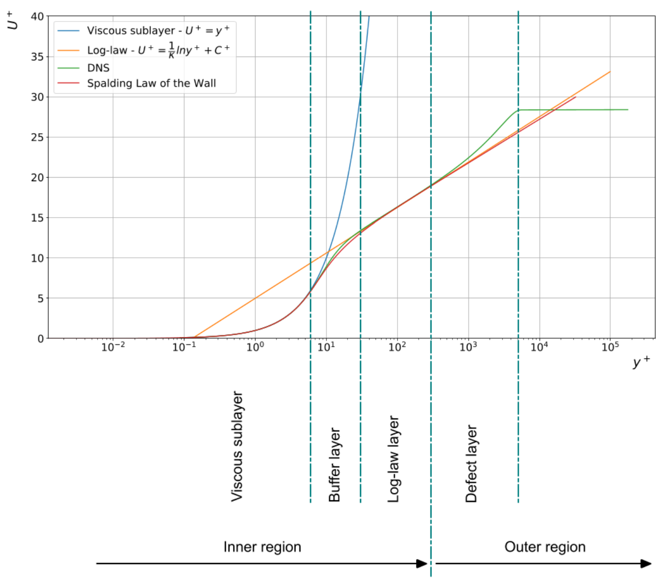
$$u^+ = \frac{1}{\kappa} \ln y^+ + C^+$$

$$\kappa \approx 0.41 \quad C^+ \approx 5.0$$

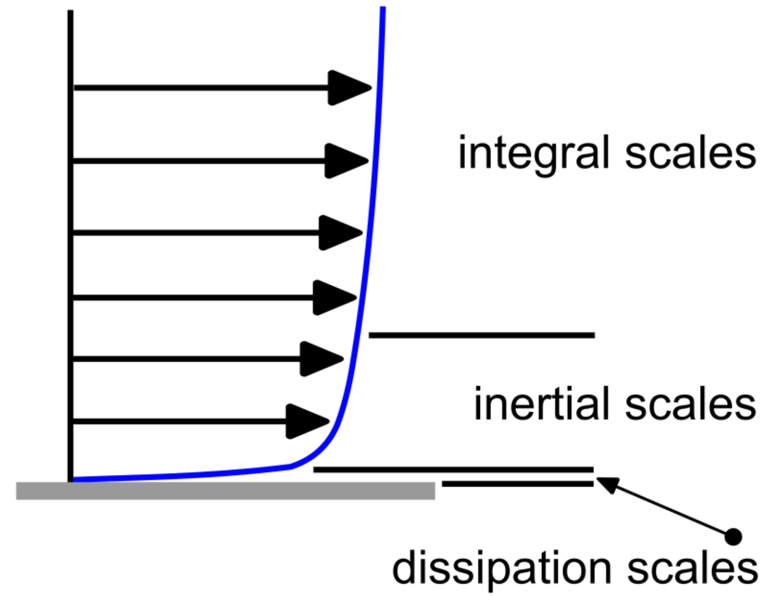
Note: the range of y^+ values might change from reference to reference but roughly speaking they are all close to these values.

Turbulence near the wall - Law of the wall

Non-dimensional profile against physical velocity profile



Non-dimensional profile

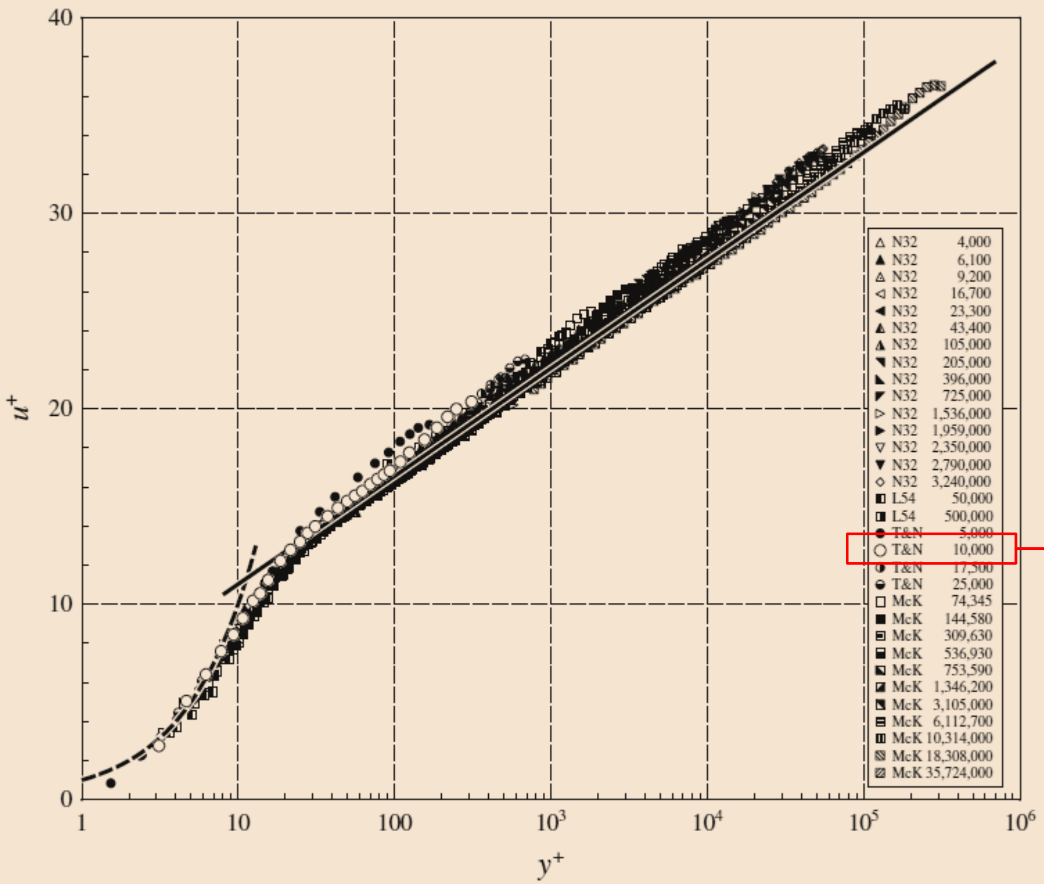


Physical velocity profile – Actual profile

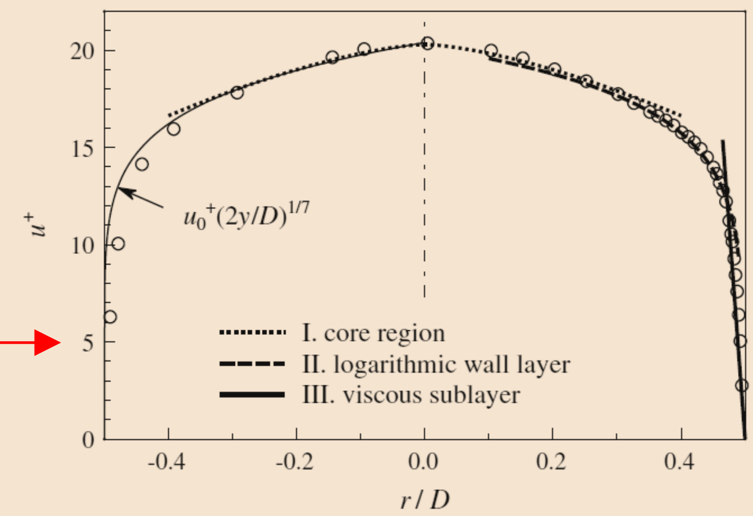
- The use of the non-dimensional velocity u^+ and the non-dimensional distance from the wall y^+, results in a predictable boundary layer profile for a wide range of flows.
- Under standard working conditions this profile is the same, however, under non-equilibrium conditions (production and dissipation of turbulent kinetic energy not balanced), rough walls, porous media, buoyancy, viscous heating, heat transfers, strong pressure gradients, and so on, the profile might be different.
- While the non-dimensional velocity profile is the same for many flows, the physical velocity profile is different.

Turbulence near the wall - Law of the wall

Turbulence near the wall – Experimental data



Dimensionless mean velocity profile u^+ as a function of the dimensionless wall distance y^+ for turbulent pipe flow with Reynolds numbers between 4000 and 3600000.

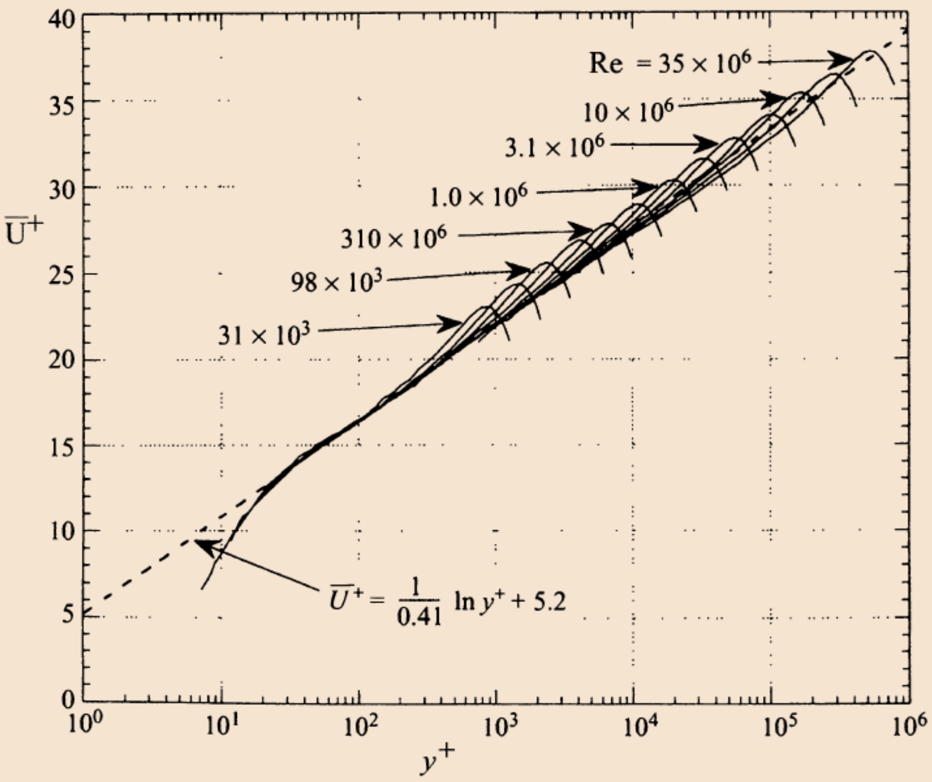


The dimensionless mean velocity profile u^+ of a turbulent flow in a pipe at Re = 10000 as a function of the distance r from the centerline of a pipe with diameter D, compared to experimental data.

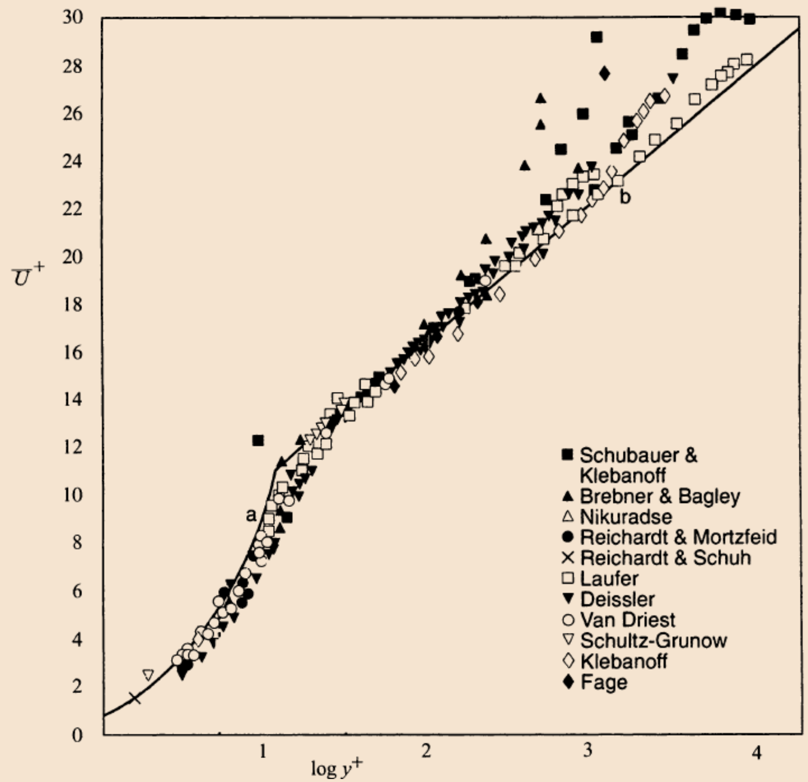
References:

F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

Turbulence near the wall – Experimental data



U+ vs. y+ plot for high-Reynolds number experiments [1].

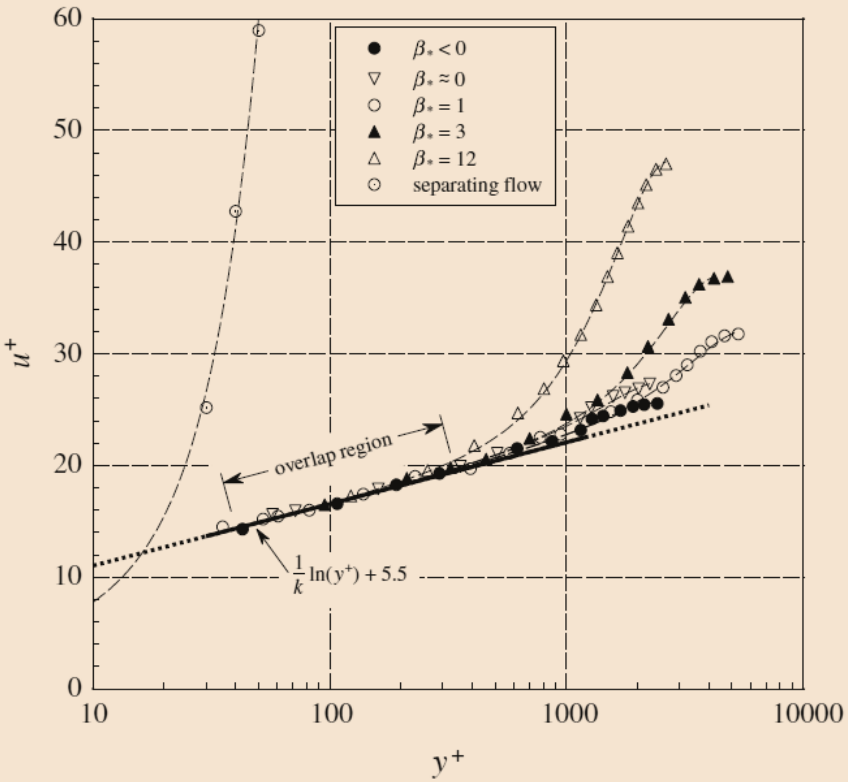


Log-law versus experiment at several Reynolds numbers [2].

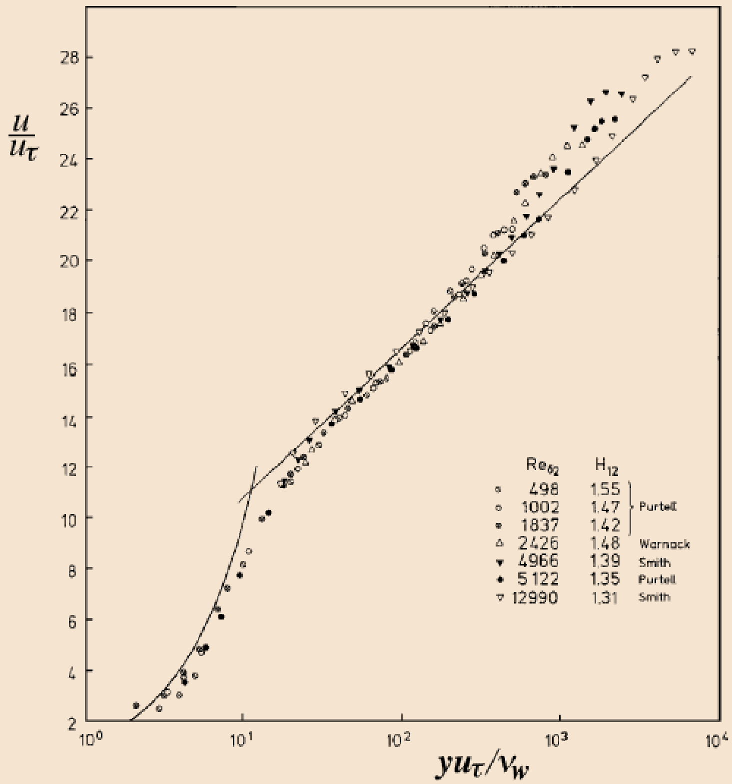
- The log-law is one of the most famous empirically determined relationships in turbulent flows near solid boundaries.
- The extension of the log-law layer depends on the system Reynolds number.

[1] M. Zagarola, A. Smits. Mean flow scaling in turbulent pipe flow. J. Fluid Mech. 373. 1998.
[2] A. S. Monin and A. M. Yaglom. Statistical Fluid Mechanics: The Mechanics of Turbulence, volume 1. M. I. T. Press, 1971.

Turbulence near the wall – Experimental data



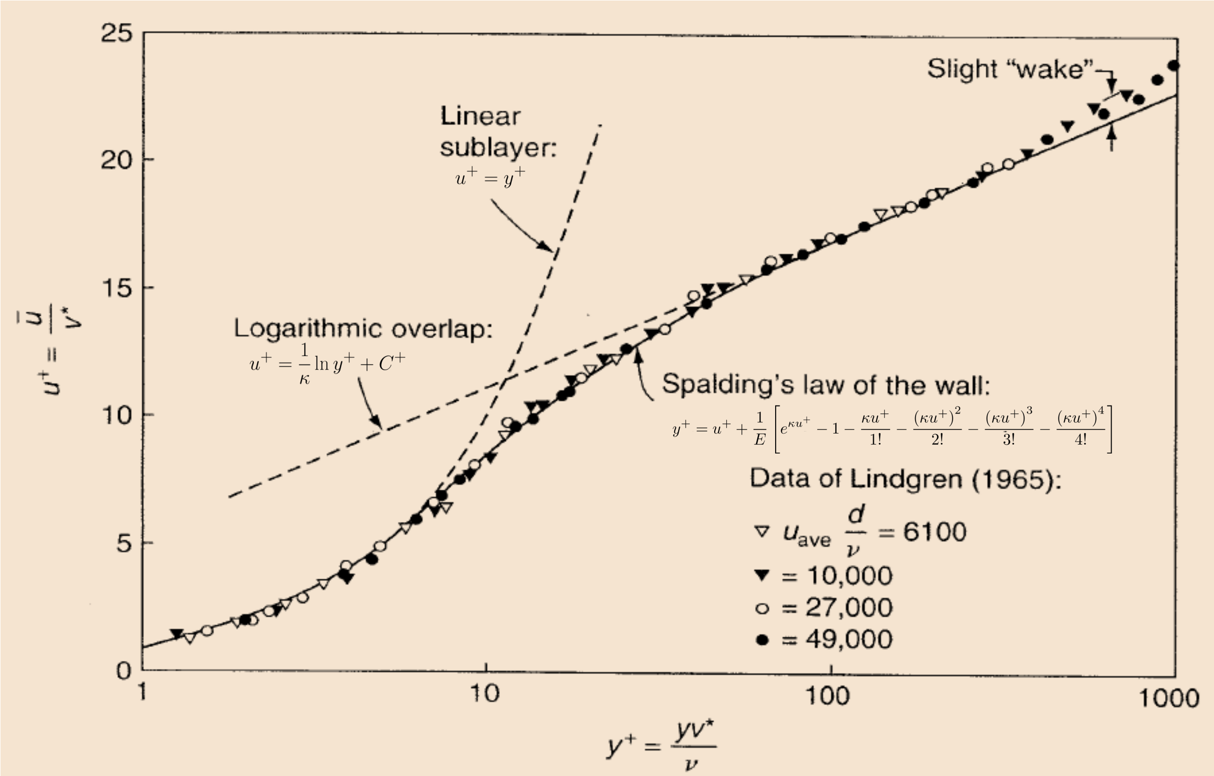
U+ as a function of y+ for various values of the pressure gradient, expressed in terms of the Clauser parameter β_* [1].



Development of the mean velocity in the inner layer scaling at low to medium Reynolds numbers [2].

[1] F. White. Fluid Mechanics. McGraw-Hill. 2011.
[2] H. Fernholz, P. Finley. Incompressible zero-pressure gradient turbulent boundary layers: An assessment of the data. Prog. Aerospace Sciences, 32, 1996.

Turbulence near the wall – Experimental data

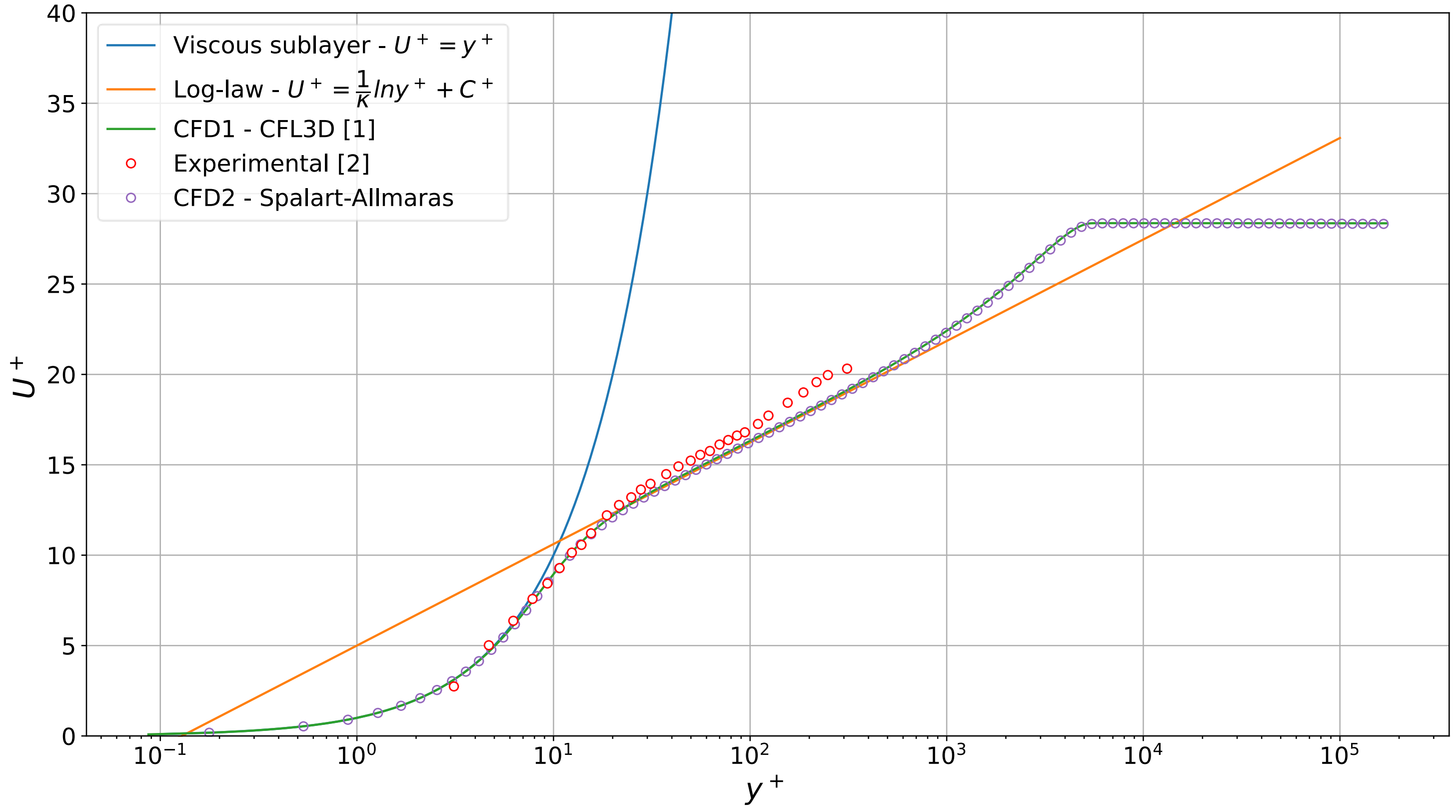


Comparison of Spalding's inner law expression with pipe-flow data of Lindgren [1].

- For decades, there were no mean-velocity data close enough to the wall.
- One of the first works to measure data very close to the wall and in the inner region is that of Lindgren in 1965 [1].
- The agreement of these measurements with Spalding's formula (we will talk about it later) is excellent.

[1] E. Lindgren. Experimental study on turbulent pipe flows of distilled water. Oklahoma State Univ. Report 1AD621071. 1965.

Turbulence near the wall – Relations according to y^+ value



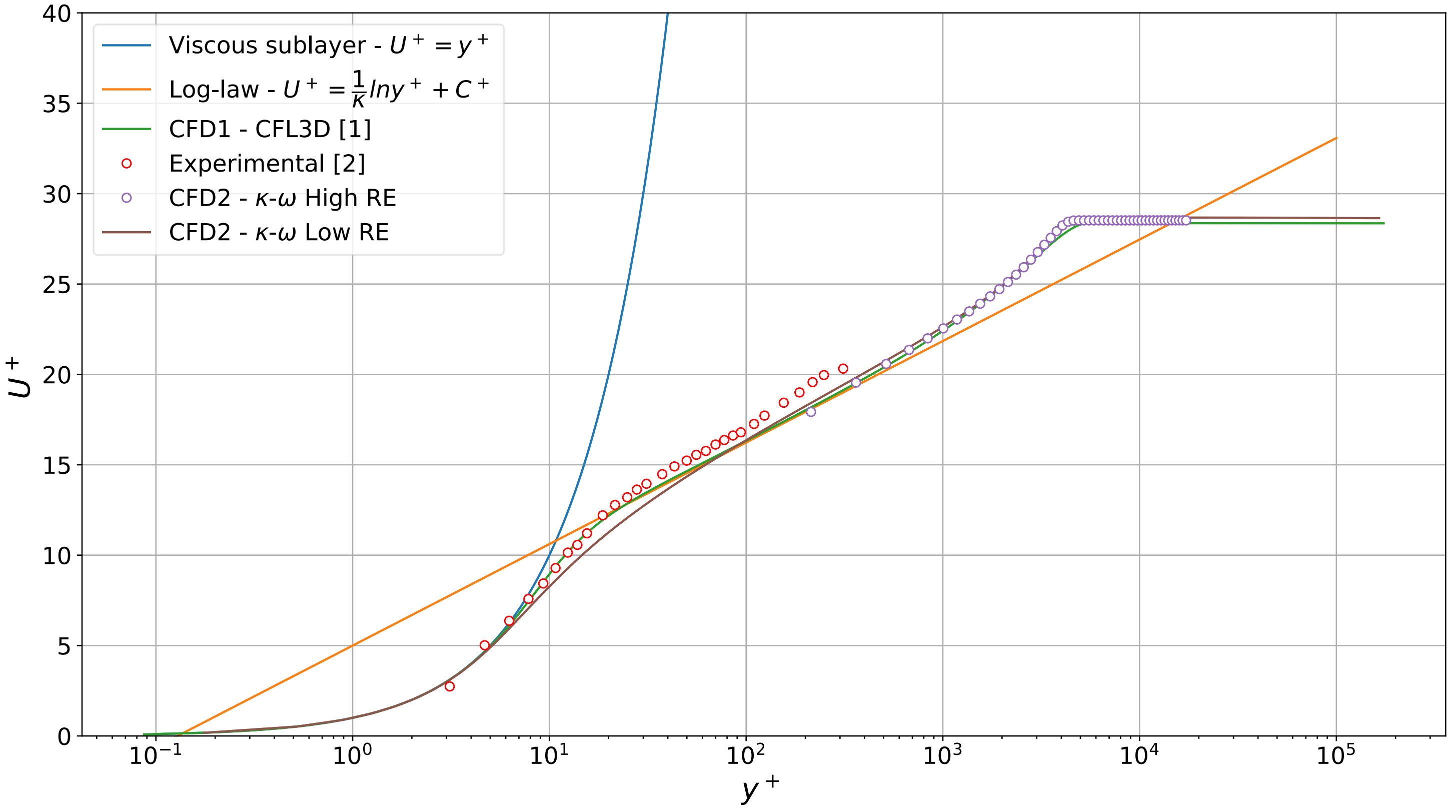
- Plot of the non-dimensional velocity profile.
- Comparison of experimental and numerical results.
- Notice that all cases plotted correspond to different physics and Reynolds numbers.

References:

[1] https://turbmodels.larc.nasa.gov

[2] J. M. J. den Toonder and F. T. M. Nieuwstadt. Reynolds number effects in a turbulent pipe flow for low to moderate Re. Physics of Fluids 9, 3398 (1997).

Turbulence near the wall – Relations according to y+ value



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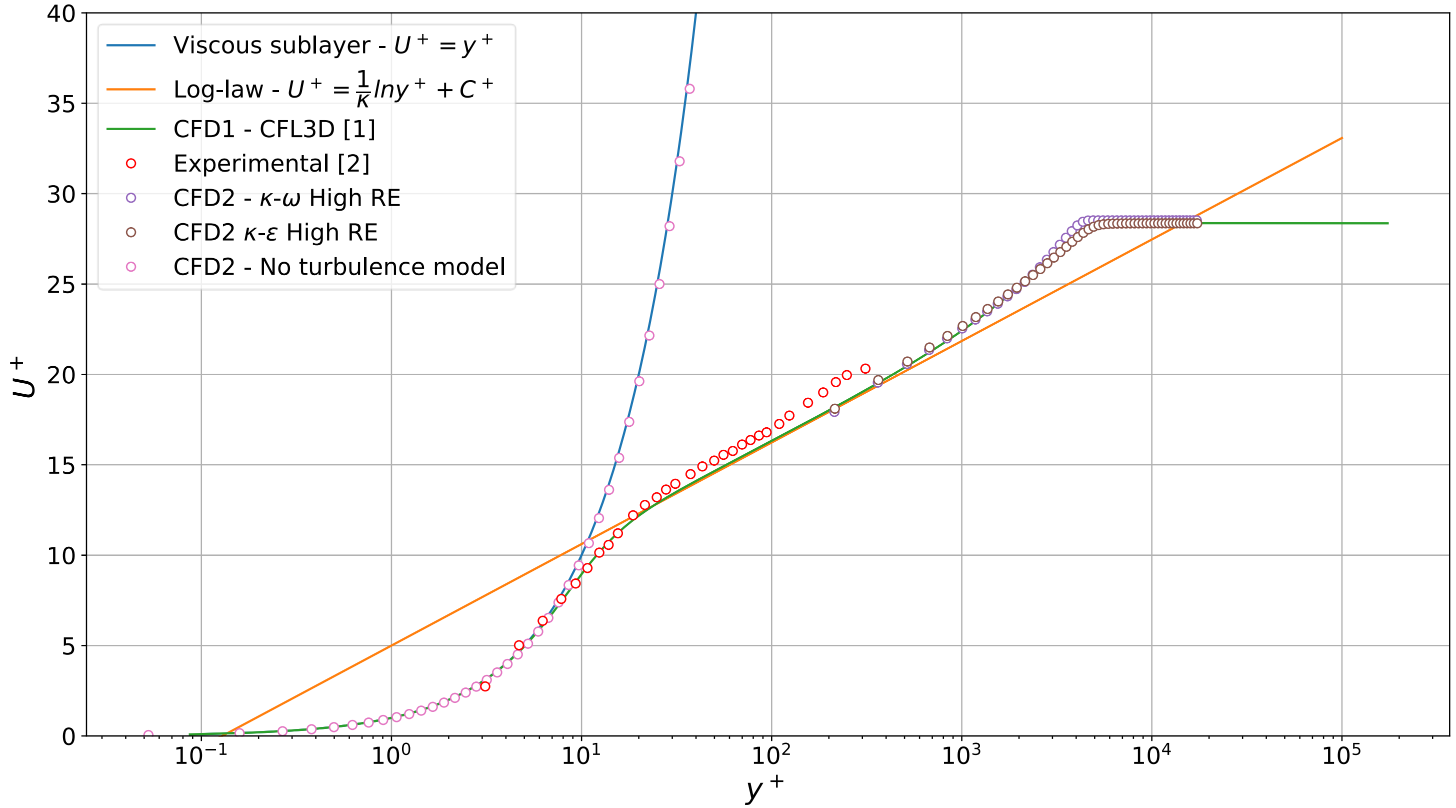
Turbulence near the wall – Relations according to y+ value



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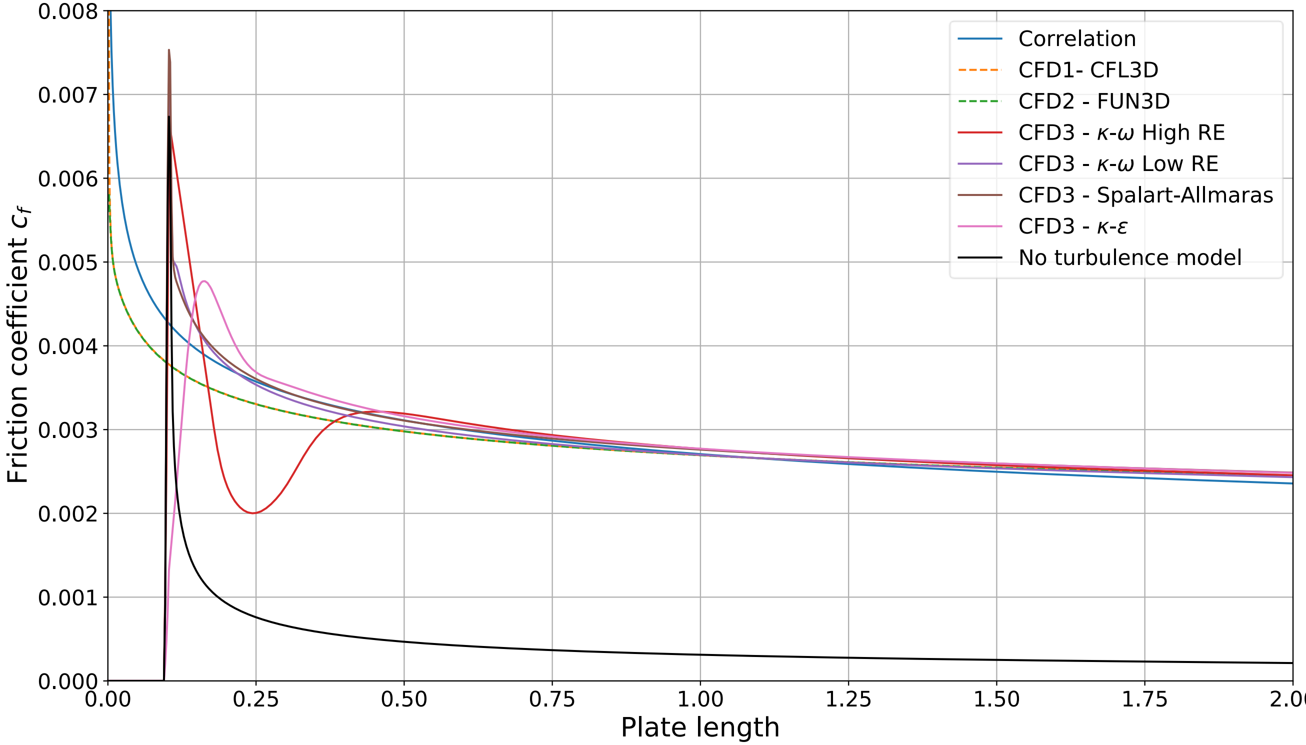
Turbulence near the wall – Relations according to y^+ value



- Plot of friction coefficient in function of length for the flat plate case [1].
- Notice that the case where we did not use turbulence model, the simulation highly under predicted the friction coefficient value.
- The importance of using a turbulence model.

References:
https://turbmodels.larc.nasa.gov/flatplate.html

Turbulence near the wall – Relations according to y^+ value

- From the non-dimensional u^+ vs y^+ plots, it is possible to fit a function that covers the entire laminar and turbulent regimes (including the buffer layer).
- One of the most widely known *"universal"* velocity profile is Spalding's law [1], which is essentially a fit of the laminar, buffer and logarithmic regions of the boundary layer,

$$y^+ = u^+ + \frac{1}{E}\left[e^{\kappa u^+} - 1 - \frac{\kappa u^+}{1!} - \frac{(\kappa u^+)^2}{2!} - \frac{(\kappa u^+)^3}{3!} - \frac{(\kappa u^+)^4}{4!}\right]$$

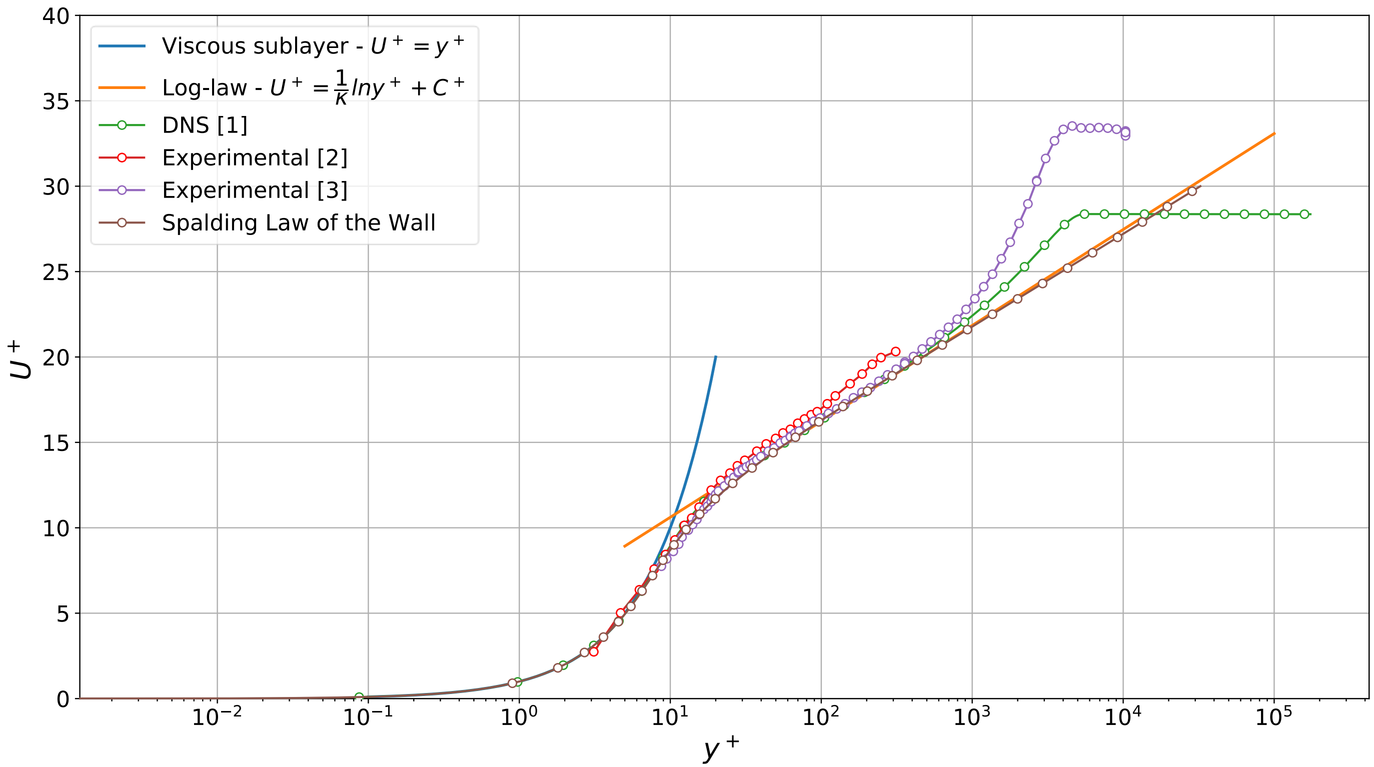
$$\frac{1}{E} = e^{-\kappa C^+}$$

- Here, κ is the von Karman constant and *E* is another constant needed to fit the well-known logarithmic law.
- Reported values of C^+ can go anywhere from 4.5 to 5.5.
- Reported values of κ can go anywhere from 0.36 to 0.42
- Reported values of *E* can go anywhere from 8.5 to 9.3.

[1] D. Spalding. A single formula for the law of the wall. J. of Applied Mechanics. 1961.

Turbulence near the wall – Relations according to y^+ value

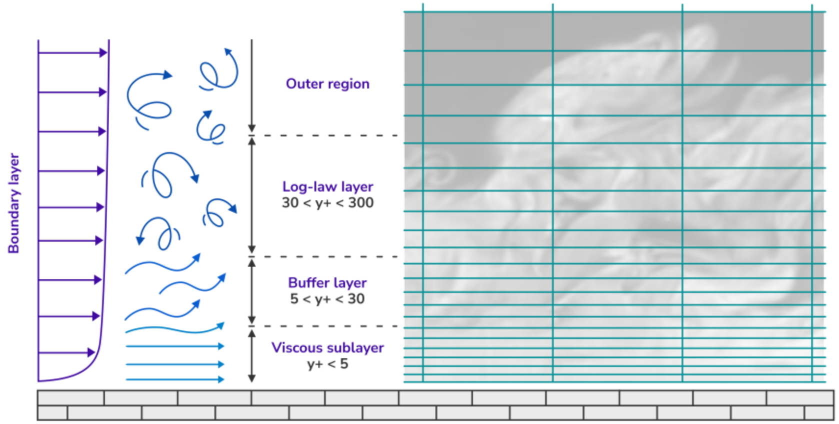
- Comparison of the Spalding's law against the Log-law, experimental results, and numerical results.
- The following values were used to plot Spalding's law, $\kappa = 0.42 \qquad E = 9.1$

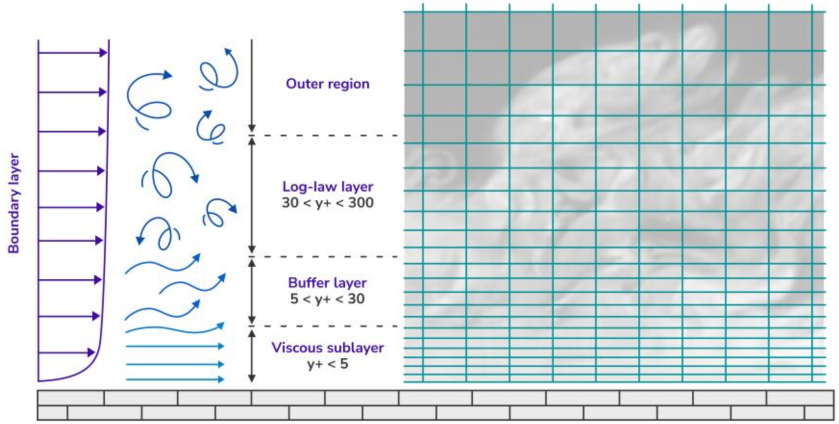


References:
[1] https://turbmodels.larc.nasa.gov
[2] J. M. J. den Toonder and F. T. M. Nieuwstadt. Reynolds number effects in a turbulent pipe flow for low to moderate Re. Physics of Fluids 9, 3398 (1997).
[3] https://www.flow.kth.se/flow-database/experimental-data-1.791818 (APG database)

Near-wall treatment and wall functions

- When dealing with wall turbulence, we need to choose a near-wall treatment.
- If you want to resolve the boundary layer, all the way down to the viscous sub-layer, you need very fine meshes close to the wall.
- In terms of y^+, you need to cluster at least 5 to 10 layers at $y^+ < 5$.
- You need to properly resolve the profiles of the transported quantities (U, k, epsilon, Reynolds stresses, temperature, species concentration and so on).
- Usually, this type of meshes will cluster 15 to 30 layers (or even more) close to the walls.
- This is the most accurate approach, but it is computationally expensive.

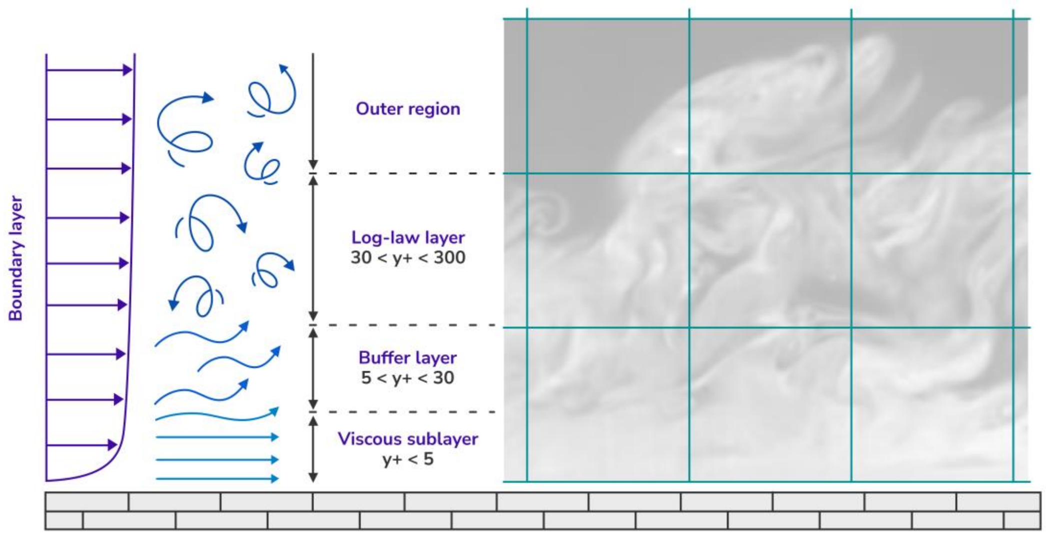




Resolving the streamwise direction is also important, for example, when dealing with transition to turbulence.

Near-wall treatment and wall functions

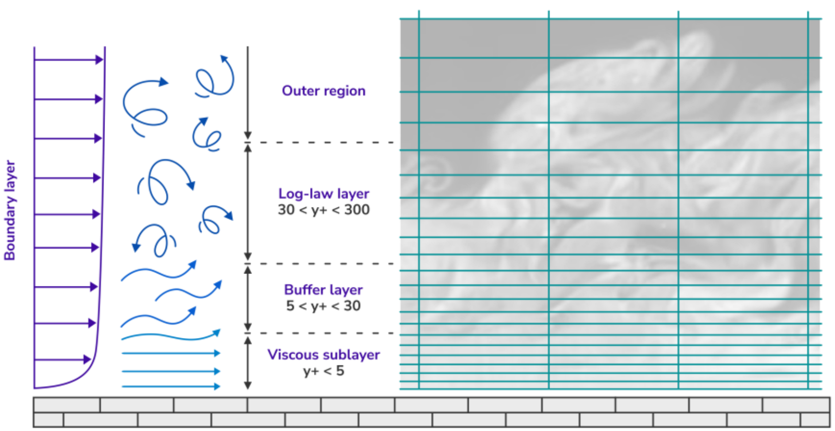
- When dealing with wall turbulence, we need to choose a near-wall treatment.
- If you are not interested in resolving the boundary layer up to the viscous sub-layer, you can use wall functions.
- In terms of y^+, wall functions will model everything below $y^+ < 30$ or the target y^+ value.
- This approach use coarser meshes, but you should be aware of the limitations of the wall functions.
- You will need to cluster at least 5 to 10 layers close to the walls in order to resolve the profiles of the transported quantities (U, k, epsilon, Reynolds stresses, temperature, species concentration and so on).
- As a general rule, when using wall functions, the first cell center should be located above y+ > 40-50 and below $y \approx 0.2\delta_{99}$ (boundary layer thickness).

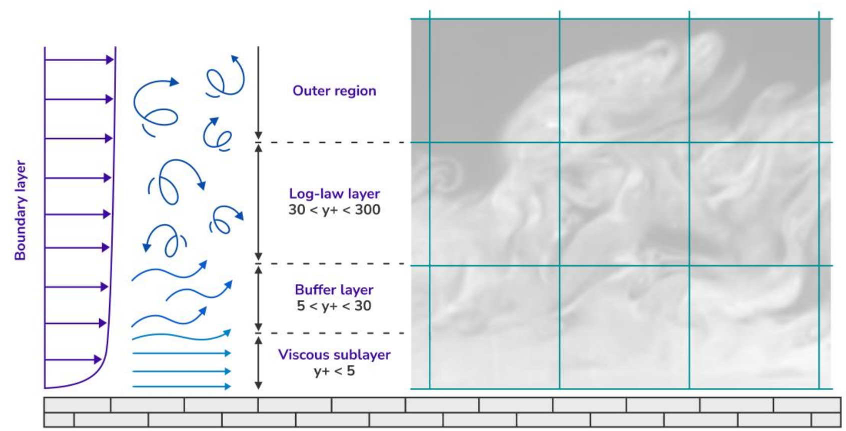


Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

- When dealing with wall turbulence, we need to choose a near-wall treatment.
- You can also use the y^+ insensitive wall treatment (sometimes known as continuous wall functions or scalable wall functions).
- This kind of wall functions are valid in the whole boundary layer.
- In terms of y^+, you can use this approach for values between $1 < y^+ < 300\text{-}600$ (the upper limit depends on the Reynolds number).
- This approach is very flexible as it is independent of the y^+ value, but is not available in all turbulence models
- Again, you should cluster enough cells close to the walls (at least 8-10 layers) to resolve the profile of the transported quantities (U, k, epsilon, Reynolds stresses, temperature, species concentration and so on).





Insensitive wall treatment will automatically switch between the wall modeling approach or the wall resolving approach according to the y+ value.

Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

- Generally speaking, wall functions is the approach to use if you are more interested in the mixing in the outer region, rather than the forces on the wall.
- If accurate prediction of forces or heat transfer on the walls are key to your simulation (aerodynamic drag, turbomachinery blade performance, heat transfer) you might not want to use wall functions.
- By following good standard practices, both approaches can give similar results.

Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

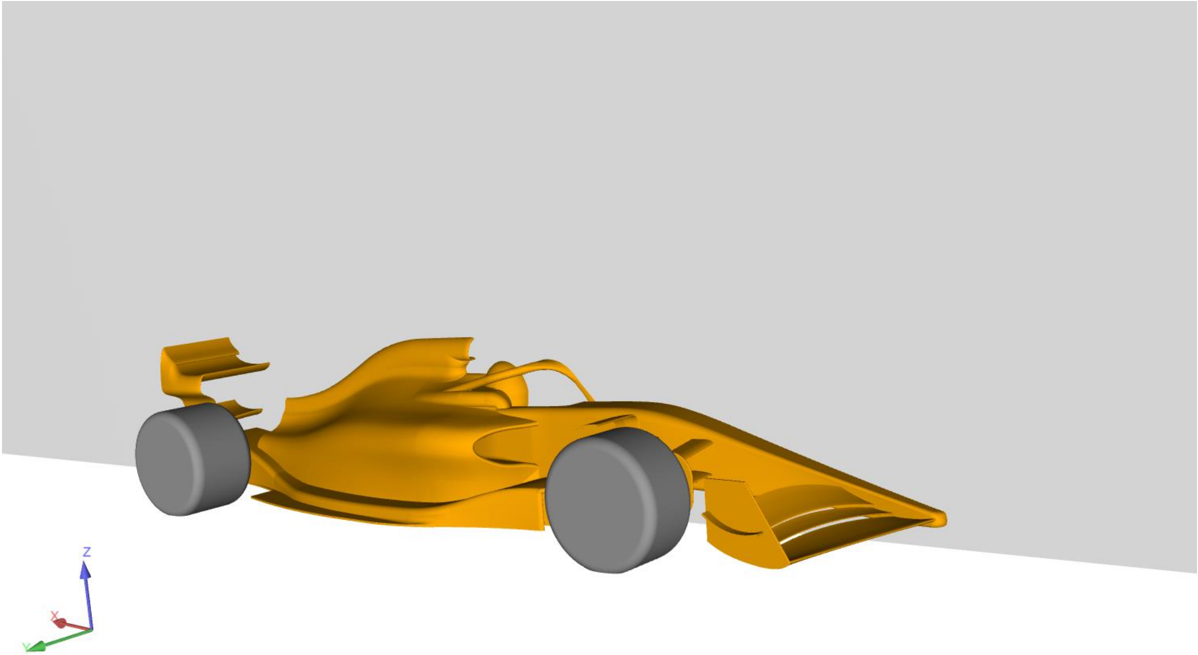
- The wall function approach is also known as high-RE (HRN).
- The wall resolving approach (where you do not use wall functions) is also known as low-RE (LRN).
- Wall functions should be avoided if $10 < y^+ < 30$.
    - This is the transition region (buffer layer), and wall functions do not perform very well here.
    - Still, nobody knows precisely what is going on in this region.
- If you are using the HRN approach and a cell-centered solver, the first cell center should be located in a region where $y^+ > 30$.
- The low-RE approach is computational expensive as it requires clustering a lot cells near the walls.
- To get good results with LRN, you will need to cluster at least 10 layers for $y^+ < 6$.
- But values up to $y^+ < 10$ are acceptable.
- It is primordial to properly solve the velocity profile.

Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

- If you do not have any restrictions in the near-wall treatment, use wall functions.
- Wall functions can be used in RANS, DES and LES.
- You can also use them with moving walls.
- If you are doing LES, it is highly recommended to use wall functions. Otherwise, your meshing requirements will be very similar to DNS.
- In practice, maintaining a fixed value of y^+ in the cells next to the walls throughout the domain is very challenging (if not impossible). In these cases, you should monitor the average value.
- Maintaining a value of $y^+ > 30$ when using wall functions during grid refinement studies can be difficult and problematic.
- Grid refinement studies are a common practice in CFD and a recommended best practice. Therefore, wall treatments that are insensitive to y^+ values are preferred.
- Be careful, some turbulence models in combination with a given wall function approach will give you inaccurate results when $y^+ < 30$.

Influence of near-wall treatment in cell count

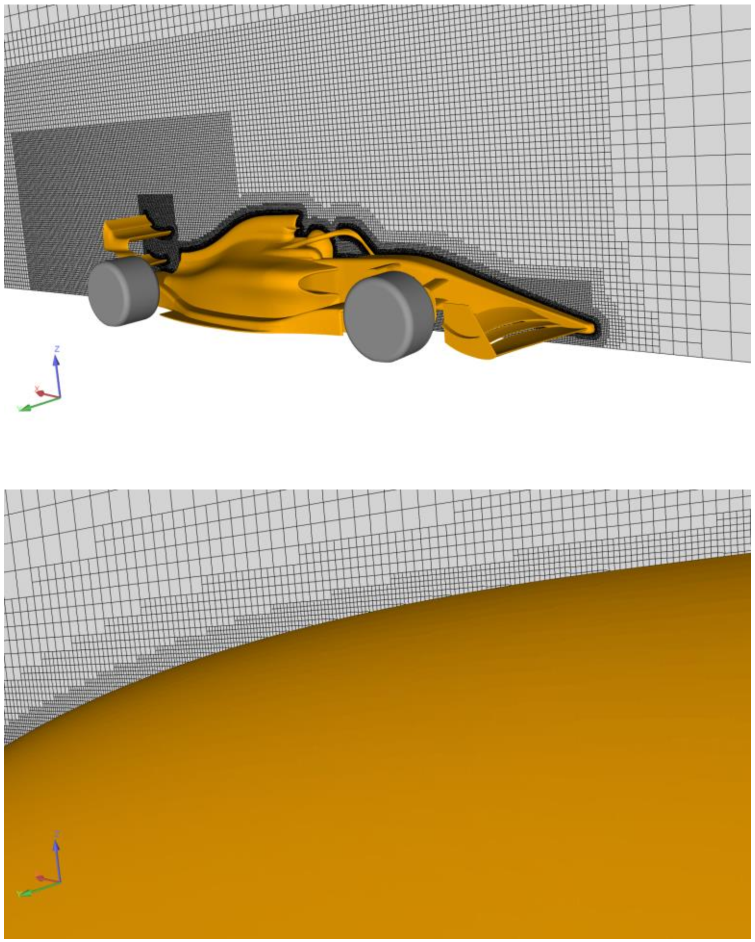


	Mesh 1	Mesh 2
Number of cells	57 853 037	111 137 673

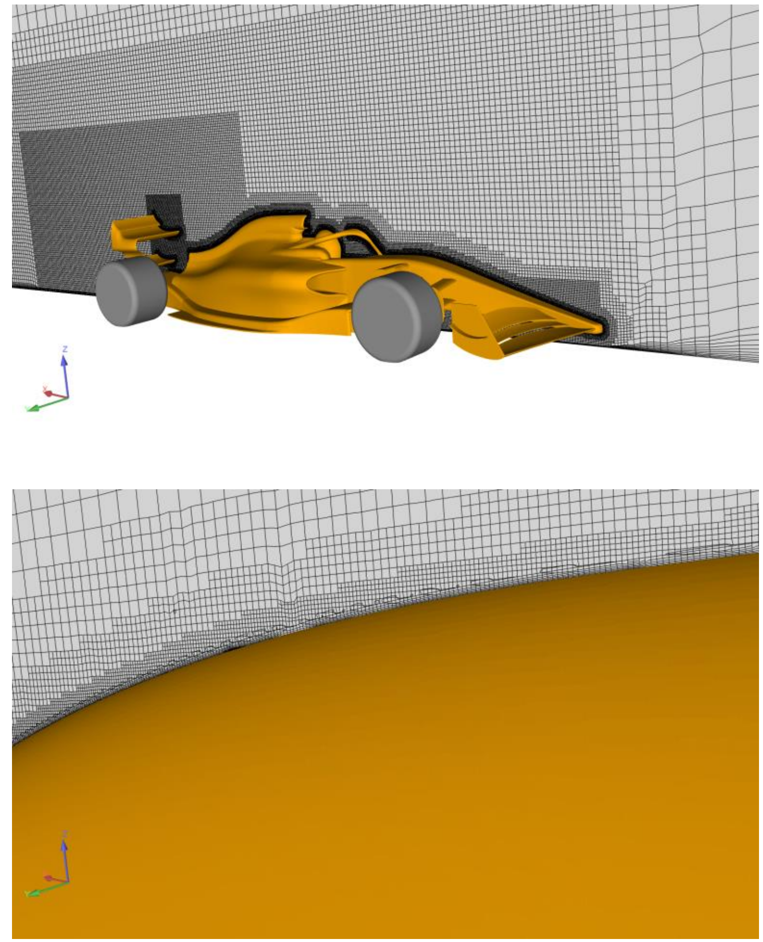
- Can you guest the difference between the meshes?

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



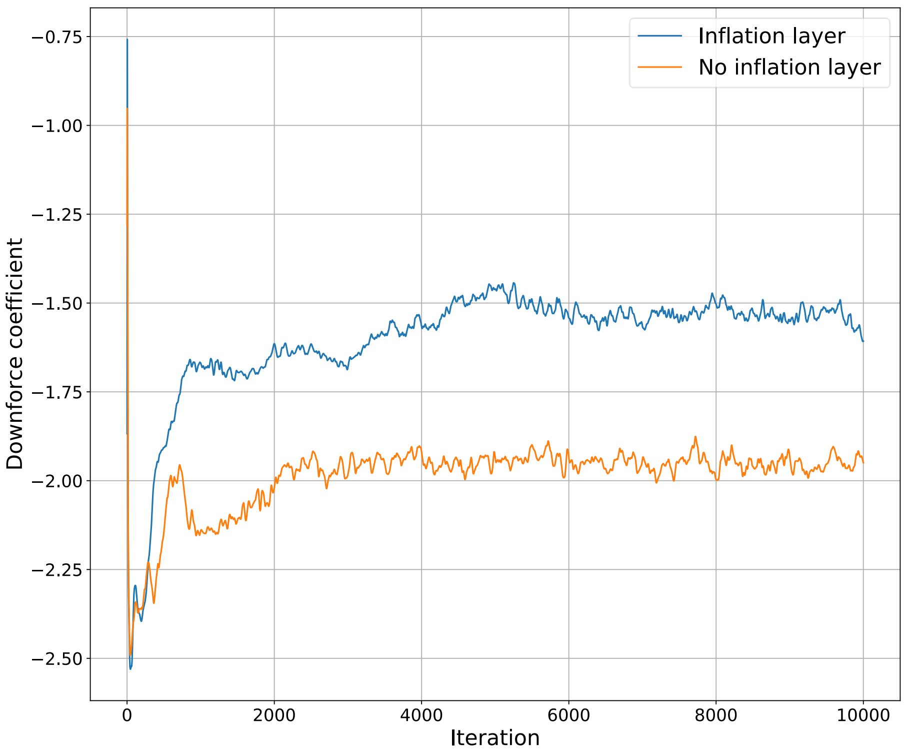
Average y^+ approximately 60

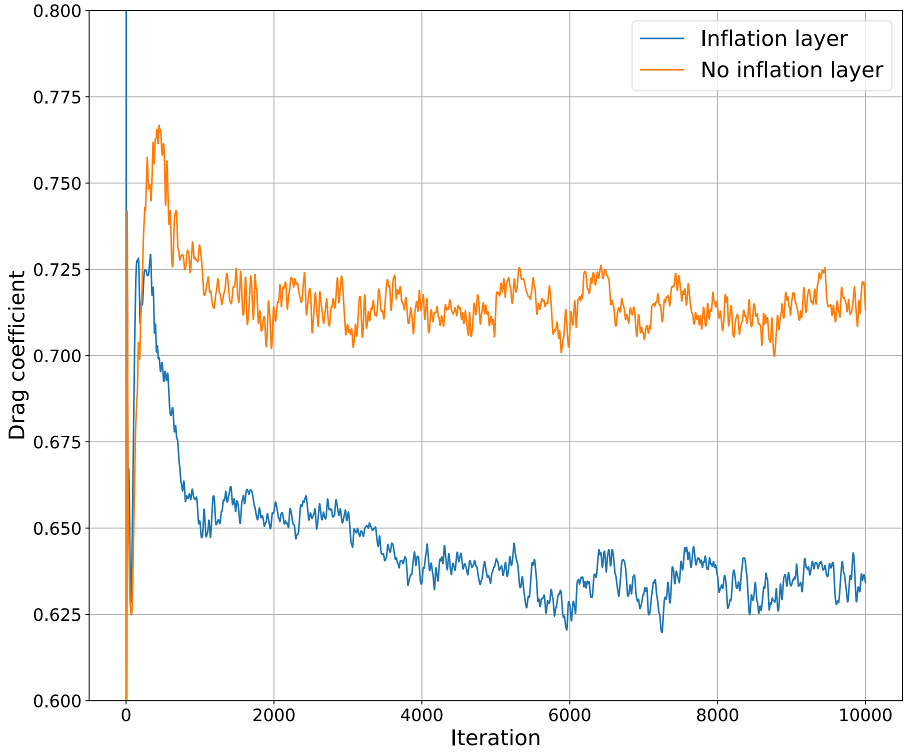


Average y^+ approximately 7

- By only adding the inflation layer to resolve the boundary layer we almost doubled the number of cells.

Influence of near-wall treatment in cell count

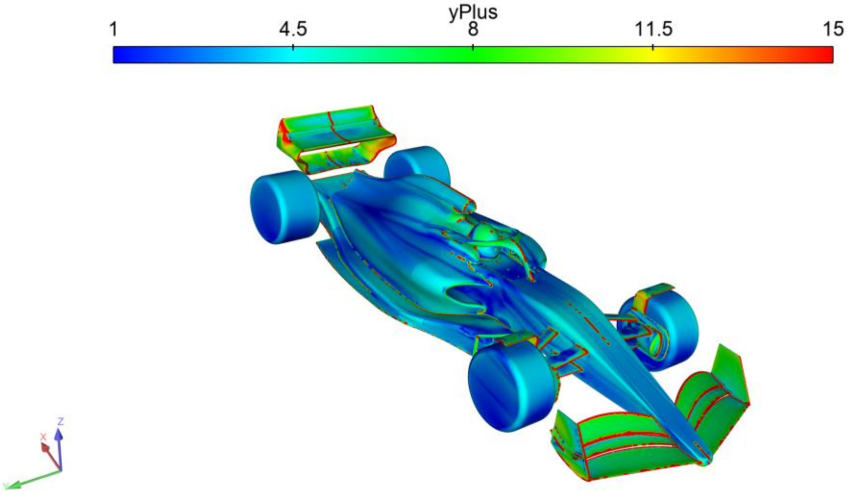




- As you can note, two different near the wall treatments give very different results.
- You should be very critical when analyzing these results.
- Not necessary the finer mesh gives the best results.

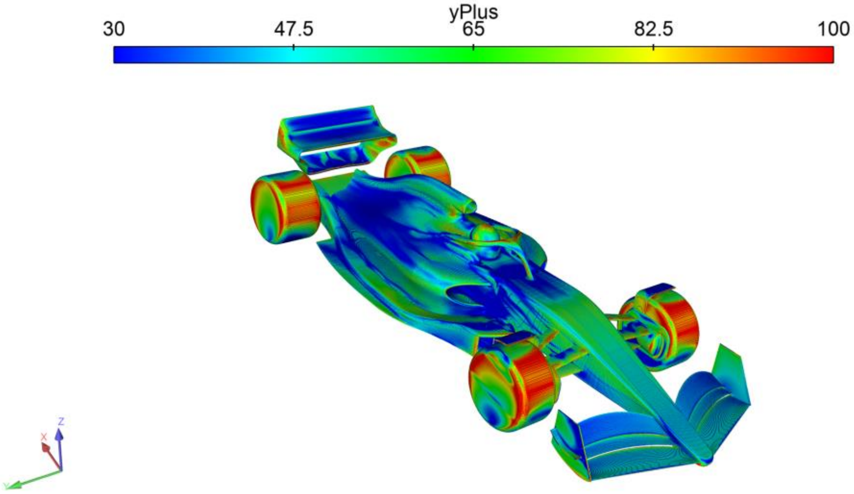
Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)

	Average y+
FWD. Wing	14
RWD. Wing	12
Body	4

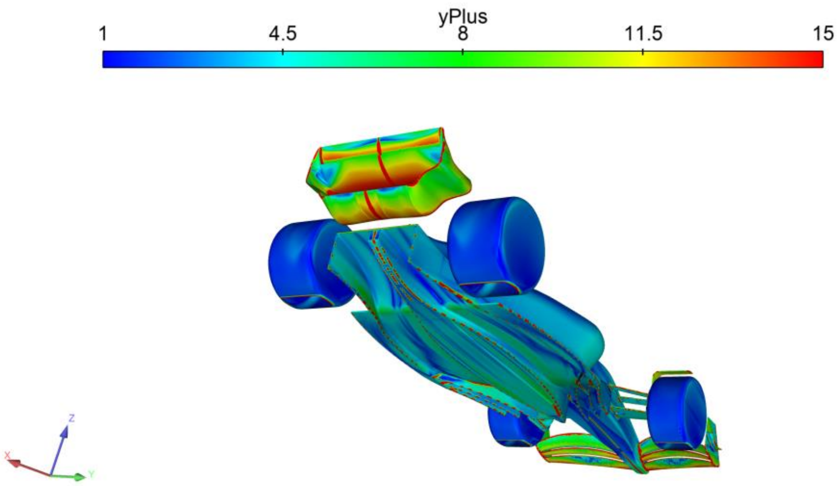


Wall modeling approach (High-RE)

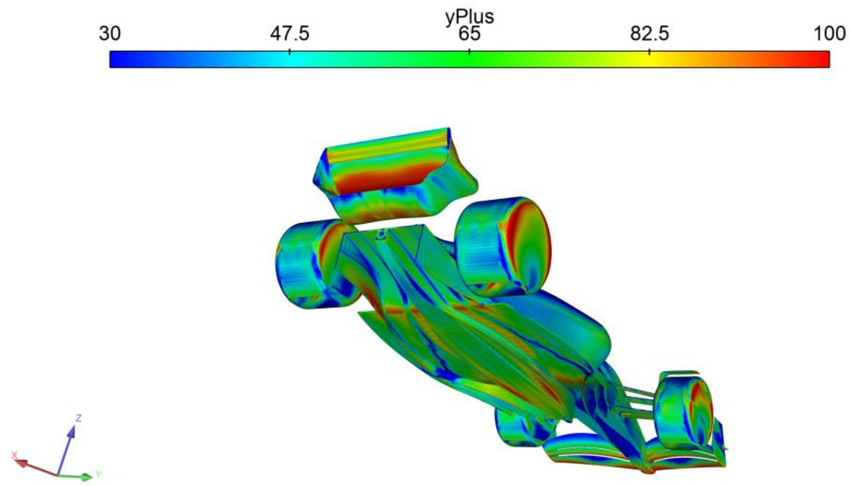
	Average y+
FWD. Wing	56
RWD. Wing	62
Body	46

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)



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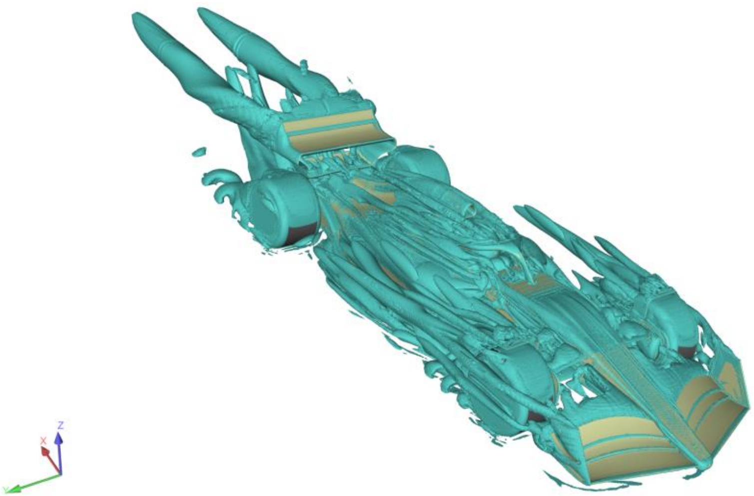
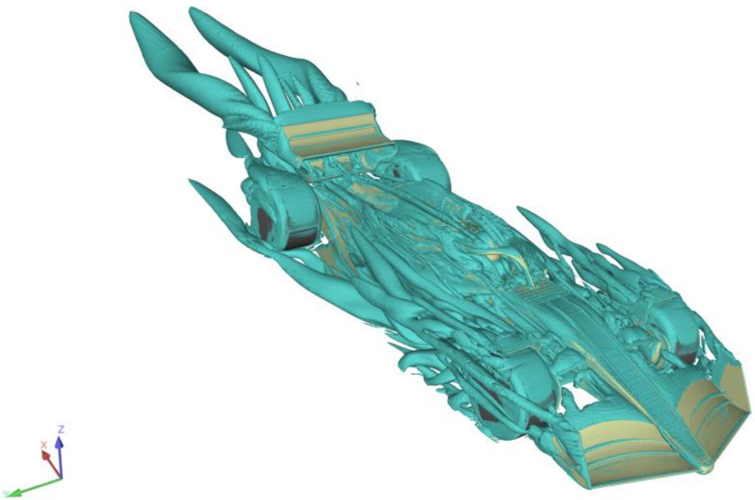


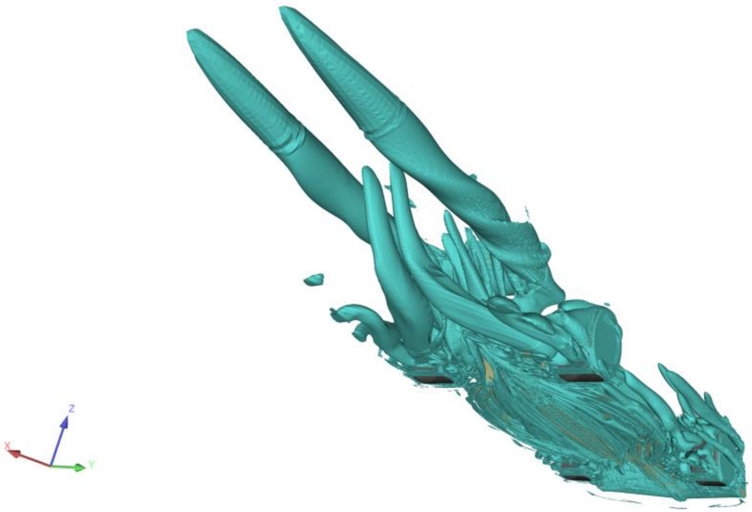
Wall resolving approach (Low-RE)



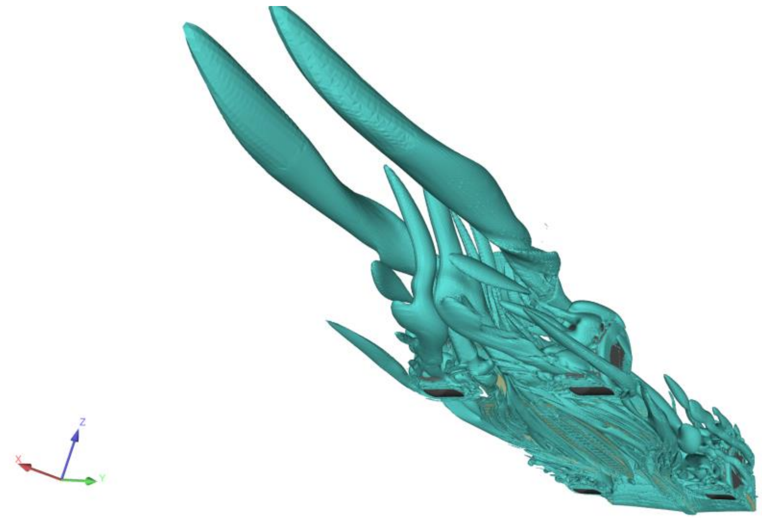
Wall modeling approach (High-RE)

- Qualitative post-processing.
- The vortical structures are visualized using the Q-criterion.
- By the way, if you switch off the turbulence model, it is likely probable that your results will be garbage (unless you have an extremely fine mesh that resolves all scales).

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)



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Roadmap to Lecture 3

1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales to integral scales~~
2. ~~Energy spectrum and energy cascade. Integral length scale and grid length scale~~
3. ~~Turbulence near the wall - Law of the wall~~
4. **A glimpse to a turbulence model**

Turbulence modeling – Starting equations

Exact NSE

$$\left\{\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\ &\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \tau + \mathbf{S_u} \\ &\frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) = -\nabla \cdot q - \nabla \cdot (p\mathbf{u}) + \boldsymbol{\tau}\boldsymbol{:}\nabla \mathbf{u} + \mathbf{S}_{e_t} \end{aligned}\right.$$

$$+$$

Additional equations to close the system (thermodynamic variables)

Additionally, relationships to relate the transport properties

Additional closure equations for the turbulence models

- Turbulence models equations cannot be derived from fundamental principles.
- All turbulence models contain some sort of empiricism.
- Some calibration to observed physical solutions is contained in the turbulence models.
- Also, some intelligent guessing is used.
- A lot of uncertainty is involved!

Turbulence modeling – Starting equations

- Let us write down the governing equations for an incompressible flow.
- When conducting DNS simulations (no turbulence models involved), this is our starting point,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- When using RANS turbulence models, these are the governing equations,

If we retain this term we talk about URANS equations
and if we drop it we talk about RANS equations

Reynolds stress tensor
This term requires modeling

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} - \frac{1}{\rho} \nabla \cdot \tau^R$$

Turbulence modeling – Starting equations

- The differences between the exact Navier-Stokes equations (or DNS equations) and RANS equations, are the overbar over the primitive variables and the appearance of the Reynolds stress tensor.
- The overbar over the primitive variables in the RANS equations means that the quantities have been averaged (time average, spatial average or ensemble average).
- By averaging the exact Navier-Stokes equations, we are removing the instantaneous fluctuations.
- The instantaneous fluctuations are difficult and expensive to solve.
- We will explain how to derive the RANS equations in Lecture 5.
- In the RANS equations, the Reynolds stress tensor requires modeling.
- To model the Reynolds stress tensor, we need to add additional closure equations (turbulence modeling).
- There are many turbulence models available, and none of them is universal.
- Therefore, it is essential to understand their range of applicability and limitations.

Turbulence modeling – Boussinesq hypothesis

- The most widely approach used to model the Reynolds stress tensor τ^R is to use the Boussinesq hypothesis.
- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean velocity gradient such that,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_t\bar{\mathbf{S}}^R - \frac{2}{3}\rho k\mathbf{I} = \mu_t\left[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right] - \frac{2}{3}\rho k\mathbf{I}$$

$$\bar{\mathbf{S}}^R = \frac{1}{2}\left[\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right]$$

Computed from the turbulence model

- Each turbulence model will compute the turbulent viscosity in a different way.

Turbulence modeling – Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can write down the solvable RANS equations starting from the exact RANS equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}\overbrace{(\mu + \mu_t)}\nabla \bar{\mathbf{u}}\right]$$

Effective viscosity

Turbulent viscosity

Normal stresses arising from the Boussinesq approximation

Note:

$$\nu = \frac{\mu}{\rho}$$

- Note that all the field variables are computed in term of mean quantities.
- By using the Reynolds averaging we removed the instantaneous fluctuations.
- The problem now reduces to computing the turbulent eddy viscosity μ_T or ν_T in the momentum equation.
- This is the closure problem in turbulence modeling.

A glimpse to a turbulence model

$k-\epsilon$ Turbulence model equations

- Let us take a glimpse to a very popular turbulence model, the standard $k-\epsilon$ turbulence model [1,2,3].
- It is called $k-\epsilon$ because it solves two additional equations for modeling the turbulent flow, namely,
    - The turbulent kinetic energy k.
    - The turbulent dissipation rate ϵ.

- Remember, as we are introducing additional equations, we need to define boundary conditions and initial conditions for the new variables.
- These variables are used to model the Reynolds stress tensor (which contains the velocity fluctuations).
- Specifically, the variables are used to computed the turbulent viscosity arising from the modeling approach.
- In Lecture 6, we will study many turbulence models (including this one).

References:
[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[2] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.
[3] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.

A glimpse to a turbulence model

$k-\epsilon$ Turbulence model equations

- The closure equations of the $k-\epsilon$ turbulence model are,

Constant

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

Constant   Constant   Constant

- These new variables are not physical properties.
- They kind of represent the generation and destruction of turbulence.
- These quantities are used to compute the turbulent viscosity ν_T.

A glimpse to a turbulence model

$k-\epsilon$ Turbulence model equations

- In the standard $k-\epsilon$ model [1,2,3], the turbulent viscosity is computed as follows,

Constant

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- As we have done so far, if you check the dimensional groups, you will find that this combination of variables results in the right viscosity units.
- The turbulent viscosity is introduced to take into account the increased mixing and shear stresses due to the turbulent motion and is not a physical property of the fluid. It is a property of the flow.
- The turbulent viscosity is used in the Boussinesq hypothesis to model Reynolds stress tensor appearing in the RANS equations.
- Have in mind that there are many methods to model Reynolds stress tensor.

References:
[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[2] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.
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A glimpse to a turbulence model

$k - \epsilon$ Turbulence model equations

- Let us recall the base units of the derived quantities used in the $k - \epsilon$ turbulence model.

Derived quantity	Symbol	Dimensional units	SI units
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Specific dissipation rate	ϵ	L^2T^{-3}	m^2/s^3
Kinematic viscosity (laminar)	ν	L^2T^{-1}	m^2/s
Kinematic viscosity (turbulent)	ν_T	L^2T^{-1}	m^2/s
Dynamic viscosity (laminar)	μ	$ML^{-1}T^{-1}$	kg/m-s
Dynamic viscosity (turbulent)	μ_T	$ML^{-1}T^{-1}$	kg/m-s

- Remember, the turbulent eddy viscosity is not a fluid property, it is a property needed by the turbulence model.
- In turbulence modeling we will use many quantities that appear to be magical.
- To get an idea of what these quantities are for, always check the dimensional groups.

A glimpse to a turbulence model

$k-\epsilon$ Turbulence model – Dimensional arguments

- Let us study the logic behind the $k-\epsilon$ turbulence model by revisiting dimensions analysis.
- Entirely based on dimensional arguments, the molecular and turbulent viscosities can be expressed as follows,

$$\nu \sim \text{Velocity scale} \times \text{Length scale}$$

- The Reynolds stress tensor is correlated to the velocity fluctuations. Therefore, it makes sense to use the turbulent kinetic energy k as velocity scale,

$$\text{Velocity scale} \sim \sqrt{k} \qquad \text{where} \qquad k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

- The turbulent dissipation rate ϵ (or the dissipation of turbulent kinetic energy per unit time) can be expressed as follows,

$$\epsilon \sim \frac{k^{3/2}}{l_0}$$

A glimpse to a turbulence model

$k - \epsilon$ Turbulence model – Dimensional arguments

- After computing the turbulent kinetic anergy and the turbulent dissipation rate, we can compute the integral length scales l_0 as follows,

$$l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- We can view l_0 as the product of the velocity scale $\sqrt{k}$ and the eddy turnover time τ_0,

$$\tau_0 \sim \frac{l_0}{k^{1/2}} \sim \frac{k}{\epsilon}$$

- We can say that l_0 represents the approximate distance that eddies move during the events contributing to turbulent transport.
- Using the definitions of velocity scale, integral length scale, and turbulent dissipation rate, the turbulent viscosity can be computed as follows,

$$\nu_t \sim k^{1/2} l_0 \sim \frac{k^2}{\epsilon}$$

A glimpse to a turbulence model

$k - \epsilon$ Turbulence model – Dimensional arguments

- In the $k - \epsilon$ turbulence model, we use as the relevant velocity and length scales the following quantities,

$$\text{Velocity scale} \sim \sqrt{k} \qquad\qquad l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- Then, the turbulent viscosity is computed as follows,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- The turbulent kinetic energy and the turbulent dissipation rate are computed from the evolution of the eddies.
- Henceforth, we need to formulate appropriate transport equations for the turbulent kinetic energy and the turbulent dissipation rate.
- The constant C_μ is specific to the turbulence model.

A glimpse to a turbulence model

$k-\epsilon$ Turbulence model – Budget of the transported quantiles

- For example, the exact equation of the turbulent kinetic energy can be written as follows,

$$\underbrace{\frac{\partial k}{\partial t}}_{1}+\underbrace{\bar{u}_j\frac{\partial k}{\partial x_j}}_{2}=\underbrace{\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}}_{3}-\underbrace{\epsilon}_{4}+\underbrace{\frac{\partial}{\partial x_j}\left(\nu\frac{\partial k}{\partial x_j}\right)}_{5}-\underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{p'u'_j}\right)}_{6}-\underbrace{\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u'_iu'_iu'_j}\right)}_{7}$$

1. Transient rate of change term.
2. Convective term.
3. Production term arising from the product of the Reynolds stress and the velocity gradient.
4. Dissipation rate.
5. Rate of viscous stress diffusion (molecular diffusion).
6. Turbulent transport associated with the eddy pressure and velocity fluctuations.
7. Diffusive turbulent transport resulting from the triple correlation of velocity fluctuations.

- By exact equation we mean the equation with no models or approximations.
- Each term appearing in this equation contributes to the overall balance of the turbulent kinetic energy.
- In this balance, dissipation is important and is computed using a separate transport equation.
- As for the exact TKE equation, the exact turbulent dissipation rate equation is composed of different terms that contribute to the overall balance.

$k - \epsilon$ Turbulence model – Budget of the transported quantiles

- The budget (or balance) of the turbulence kinetic energy and the dissipation rate for a model problem is illustrated in the figure below [1].

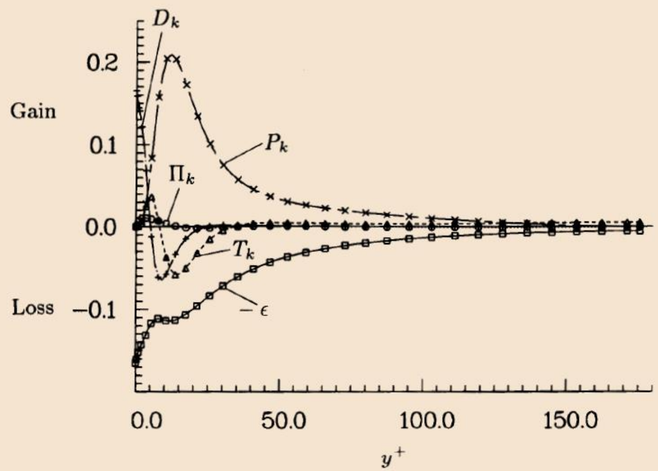

Figure 5. Terms in the budget of the turbulence kinetic energy, k, in wall coordinates. P_k = Production; T_k = Turbulent transport; D_k = Viscous diffusion; ϵ_k = Dissipation rate; Π_k = Velocity pressure-gradient term.

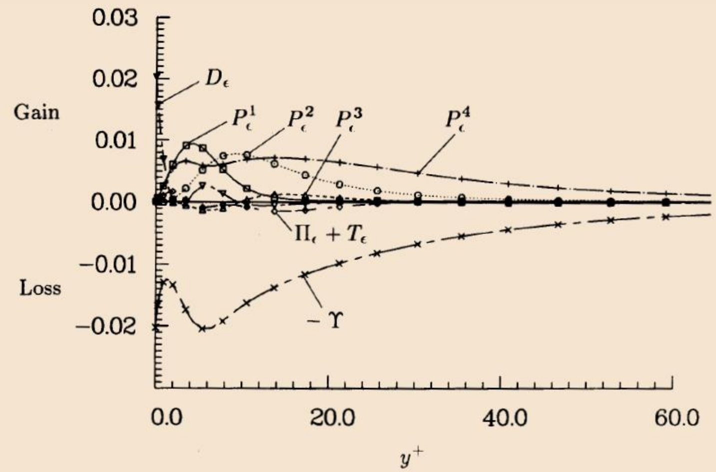

Figure 6. Terms in the budget of the dissipation rate of the turbulence kinetic energy, ϵ, in wall coordinates. P_ϵ^1 = Production by mean velocity gradient; P_ϵ^2 = Mixed production; P_ϵ^3 = Gradient production; P_ϵ^4 = Turbulent production; T_ϵ = Turbulent pransport; D_ϵ = Viscous diffusion; Υ = Dissipation rate; Π_ϵ = Pressure transport.

- From figure 5,
  - Production of TKE peaks in the buffer layer (about y^+ 15).
  - Dissipation peaks in the viscous sublayer.
  - As we get away from the walls ($y^+ > 60$), production and dissipation are in balance

[1] N. Mansour, J. Kim, P. Moin. Reynolds-Stress and Dissipation Rate Budgets in a Turbulent Channel Flow. NASA TM 89451. 1987

A glimpse to a turbulence model

Overview of the main turbulence modeling approaches.

MODELING APPROACH

RANS

Reynolds-**A**veraged **N**avier-**S**tokes equations

URANS

Unsteady **R**eynolds-**A**veraged **N**avier-**S**tokes equations

- Many more acronyms that fit between **RANS/URANS** and **SRS**.
- Some of the acronyms are used only to differentiate approaches used in commercial solvers.

PANS, **SAS**, **RSM**, **EARSM**, **PITM**, **SBES, ELES**

DES

Detached **E**ddy **S**imulations

LES

Large **E**ddy **S**imulations

DNS

Direct **N**umerical **S**imulations

SRS

Scale-**R**esolving **S**imulations

Increasing computational cost

Increasing modelling and mathematical complexity