Turbulence and CFD models: Theory and applications

Roadmap to Lecture 9

- 1. Favre averaging
- 2. Wall functions for heat transfer
- 3. Wall functions Additional observations
- 4. Surface roughness
- 5. Non-linear eddy viscosity models

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- When dealing with compressible flows (or variable density flows), besides the velocity and pressure fluctuations, we must also account for density and temperature fluctuations.
- If we use the Reynolds decomposition and time-averaging additional fluctuating correlations arise.
- To illustrate this, let us consider the conservation of mass equations for a compressible flow,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Now, let us use the Reynolds decomposition for the primitive variables (density and velocity),

$$\phi(\mathbf{x},t) = \overline{\phi}(\mathbf{x}) + \phi'(\mathbf{x},t)$$
 or $\phi(\mathbf{x},t) = \phi(\mathbf{x},t) + \phi'(\mathbf{x},t)$

• Substituting the Reynolds decomposition into the continuity equation yields,

$$\frac{\partial(\bar{\rho}+\rho')}{\partial t} + \nabla \cdot (\bar{\rho}\mathbf{u}+\rho'\mathbf{u}+\bar{\rho}\mathbf{u}'+\rho'\mathbf{u}') = 0$$

 By time averaging the previous equation and using the Reynolds averaging rules, we arrive to the Reynolds-averaged continuity equation for compressible flows,

$$\frac{\partial(\bar{\rho})}{\partial t} + \nabla \cdot \left(\bar{\rho}\bar{\mathbf{u}} + \overline{\rho'\mathbf{u}'}\right) = 0$$

- To achieve closure, we need to somehow approximate the correlation between the fluctuating quantities ($\rho' \mathbf{u}'$).
- The situation is more complicated for the momentum and energy equations, where triple correlations involving the density fluctuations appear.

- To simplify the problem, we introduce the density-weighted averaging procedure suggested by Favre [1].
- That is, we introduce the mass-weighted average of the quantity ϕ , as follows,

$$\tilde{\phi} = \frac{1}{\bar{\rho}} \lim_{T \to \infty} \frac{1}{T} \int_{t}^{t+T} \rho(\mathbf{x}, t) \phi(\mathbf{x}, t) dt = \frac{\overline{\rho \phi}}{\bar{\rho}}$$

For example, the mass-weighted average of the velocity field is given as follows,

Favre averaging
$$\tilde{\mathbf{u}} = \frac{\overline{\rho \mathbf{u}}}{\overline{\rho}}$$
 reynolds averaging or $\overline{\rho} \tilde{\mathbf{u}} = \overline{\rho \mathbf{u}}$ (1)

- In our notation, the overbar denotes Reynolds averaging and the tilde denotes Favre averaging.
- If we expand the RHS of equation 1 (that is, we substitute the Reynolds decomposition and use the Reynolds averaging rules), we obtain,

$$\bar{\rho}\tilde{\mathbf{u}} = \bar{\rho}\bar{\mathbf{u}} + \rho'\mathbf{u}'$$

[1] A. Favre. Equations des Gaz Turbulents Compressibles. Journal de Mecanique. 1965.

• If we substitute the following relation,

$$\bar{\rho}\tilde{\mathbf{u}} = \bar{\rho}\bar{\mathbf{u}} + \overline{\rho'\mathbf{u}'}$$

• Into the Reynolds averaged compressible continuity equation,

$$\frac{\partial(\bar{\rho})}{\partial t} + \nabla \cdot \left(\bar{\rho}\bar{\mathbf{u}} + \overline{\rho'\mathbf{u}'}\right) = 0$$

• We obtain the following equation,

$$\frac{\partial(\bar{\rho})}{\partial t} + \nabla \cdot (\bar{\rho}\tilde{\mathbf{u}}) = 0$$

- This is the Favre averaged compressible continuity equation.
- Which looks very similar to the incompressible Reynolds averaged equations.
- As for the Reynolds averaged equations, we simply eliminated the correlation of the fluctuating quantities.
- We can proceed in a similar fashion for the momentum and energy equations.

- Like the Reynolds decomposition, we can introduce a Favre decomposition of the variable ϕ , as follows,

$$\phi = \tilde{\phi} + \phi'' -$$

Notice that this fluctuating quantity also includes the effects of density fluctuations

• And to form a Favre average, we simply multiply by the density and do time average (in the same way as in Reynolds average),

$$\overline{
ho\phi} = ar{
ho} ilde{\phi} + \overline{
ho\phi''}$$

- And recall that the mass-weighted average of the field ϕ is given as follows,

$$\tilde{\phi} = \frac{\overline{\rho\phi}}{\bar{\rho}}$$

- It is important to mention that density and pressure are not mass-weighted averaged.
- For density and pressure, we use Reynolds averaging (that is, Reynolds decomposition).
- The rest of the field variables can be mass-weighted averaged, namely, u, v, w, T, h, H, e.

• At this point, the rules of Favre averaging are summarized as follows,

$$\begin{split} \bar{\tilde{\phi}} &= \tilde{\phi} \\ \overline{\rho \tilde{\phi}} &= \bar{\rho} \tilde{\phi} \\ \overline{\rho \phi''} &= 0 \\ \overline{\phi''} &= -\frac{\overline{\rho' \phi'}}{\bar{\rho}} \neq 0 \\ \overline{\rho \phi \psi} &= \bar{\rho} \tilde{\phi} \tilde{\psi} + \overline{\rho \phi'' \psi''} \\ \widetilde{\rho \phi} &= \bar{\rho} \tilde{\phi} \end{split}$$

- Plus, the Reynolds averaging rules.
- Finally, if the density is constant, $\ \tilde{\phi}=\bar{\phi}$ and $\phi''=\phi'$, and we recast the incompressible RANS equations.

• The Favre-averaged governing equations can be written as follows (using index notation),

$$\begin{aligned} \frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i} \left(\bar{\rho} \tilde{u}_i \right) &= 0\\ \frac{\partial \bar{\rho} \tilde{u}_i}{\partial t} + \frac{\partial \bar{\rho} \tilde{u}_j \tilde{u}_i}{\partial x_j} &= -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j} \left[\bar{\tau}_{ji} - \overline{\rho u_j'' u_i''} \right]\\ \frac{\partial}{\partial t} \left[\bar{\rho} \left(\tilde{e} + \frac{\tilde{u}_i \tilde{u}_i}{2} \right) + \frac{\overline{\rho u_i'' u_i''}}{2} \right] &+ \frac{\partial}{\partial x_j} \left[\bar{\rho} \tilde{u}_j \left(\tilde{h} + \frac{\tilde{u}_i \tilde{u}_i}{2} \right) + \tilde{u}_j \frac{\overline{\rho u_i'' u_i''}}{2} \right] =\\ \frac{\partial}{\partial x_j} \left[-q_{Lj} - \overline{\rho u_j'' h''} + \overline{\tau_{ji} u_j''} - \overline{\rho u_j'' \frac{1}{2} u_i'' u_i''} \right] + \frac{\partial}{\partial x_j} \left[\tilde{u}_i \left(\bar{\tau}_{ij} - \overline{\rho u_i'' u_j''} \right) \right]\\ P &= \bar{\rho} R \tilde{T} \end{aligned}$$

- These are the exact FANS equations (Favre-averaged Navier-Stokes).
- As for the RANS equations, we need to add approximations to derive the solvable equations.

- Let us review the most commonly used approximations for compressible flows.
 - The Reynolds-Stress tensor (or Favre-Stress tensor), can be modeled as follows,

$$\bar{\rho}\tau_{ij}^F = \overline{-\rho u_i'' u_i''} = \mu_T \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} + \frac{2}{3}\frac{\partial \tilde{u}_k}{\partial x_k}\delta_{ij}\right) \left(-\frac{2}{3}\bar{\rho}k\delta_{ij}\right)$$

This is the Boussinesq assumption but formulated for compressible flows.

•

- Let us review the most commonly used approximations for compressible flows.
 - Turbulent heat-flux vector,

$$q_{Tj} = \overline{\rho u_j'' h''} = -\frac{\mu_T c_p}{P r_T} \frac{\partial \tilde{T}}{\partial x_j} = -\frac{\mu_T}{P r_T} \frac{\partial \tilde{h}}{\partial x_j}$$

• Molecular diffusion and turbulent transport,

$$\overline{\tau_{ji}u_i''} - \overline{\rho u_j''\frac{1}{2}u_i''u_i''} = \left(\mu + \frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}$$

• These two terms are modeled using the gradient diffusion hypothesis.

- It is important to mention that for compressible flows, the non-zero divergence of the Favre averaged velocity modifies the mean strain rate term in the RHS of the Reynolds-Stress tensor.
- Therefore, the Reynolds-Stress tensor is manipulated in such a way to guarantee that its trace is equal to -2k.
- This implies that the second eddy viscosity is equal to,

$$\frac{2}{3}\mu_T$$

• The turbulent kinetic energy can be computed a follows,

$$\bar{\rho}k = \frac{1}{2}\overline{\rho u_i'' u_i''}$$

- The viscous stress tensor τ_{ij} and the molecular heat flux q_L (or laminar) are computed as follows,

$$\tau_{ij} = \mu \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial \tilde{u}_k}{\partial x_k} \delta ij \qquad q_L = -\kappa \frac{\partial T}{\partial x_j}$$

- As it can be seen, the Favre averaging is very similar to the Reynolds averaging.
- To derive the solvable equations, we must approximate all the terms that involve correlations of fluctuating quantities.
- Similar to the incompressible equations, we can derive the exact transport equations for the turbulent kinetic energy and Reynolds stresses.
- Favre averaging is used for compressible flows, mixture of gases, species concentration, and combustion.
- Favre averaging eliminates density fluctuations from the averaged equations.
- However, it does not remove the effect the density fluctuations have on the turbulence.
- Favre averaging is a mathematical simplification, not a physical one.
- Remember, when using Favre averaging, the density and pressure are not mass-weighted averaged.
 - For density and pressure, we use Reynolds averaging (that is, Reynolds decomposition).
 - The rest of the field variables can be mass-weighted averaged, namely, u, v, w, T, h, H, e.

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- As for the momentum (or viscous) wall functions, there is also a treatment for the thermal wall functions.
- Depending on the value of y*, the value of the non-dimensional temperature T* (equivalent to the concept of u*), can be computed as follows,



Logarithmic law for the turbulent region where effects of turbulence dominate conduction

• The following nondimensional temperature relation (in the log-law region),

$$T^* = Pr_t \left[\frac{1}{\kappa} \ln\left(Ey^*\right) + P\right]$$

- It is used to relate the temperature at the cell center T_P to the heat flux at the wall q_w .
- This is similar to what we did with the viscous wall function, where we related U_P to τ_w .

• When using thermal wall functions, we are interested in computing the thermal diffusivity used to approximate the wall heat transfer q_w,



• We need to relate the temperature at the cell center T_P to the heat flux at the wall q_w.



- Recall that in the momentum boundary layer, the velocity at the walls is zero.
- In the thermal boundary layer, the temperature at the wall can be higher or lower than the freestream temperature.
- Therefore, we can have very different temperature profiles growing from the walls.
- The temperature at the walls also has an influence of the wall shear stresses.



Momentum and thermal boundary layer



- Just as there is a viscous boundary layer in the velocity distribution (or momentum), there is also a thermal boundary layer.
- Thermal boundary layer thickness is different from the thickness of the viscous sublayer (momentum) and is fluid dependent.
- The thickness of the thermal sublayer for a high Prandtl number fluid (*e.g.*, water) is much less than the momentum sublayer thickness.
- For fluids of low Prandtl numbers (*e.g.*, air), it is much larger than the momentum sublayer thickness.
- For Prandtl number equal 1, the thermal boundary layer is equal to the momentum boundary layer.

- The normalized temperature plot (T⁺ vs. y⁺), depends on the Prandtl number.
- Different values of Prandtl number will result in different plots, with different intersection points between the viscous sublayer and the log-law region.
- The intersection point of the viscous sublayer and the log-law region is known as the nondimensional thermal sublayer thickness or y_l^* .



- The thermal diffusivity coefficient α is computed as follows,

$$\alpha_w = \begin{cases} \alpha & y^* < y_l^* \\ \frac{u_\tau y_p}{\Pr_t \left[\frac{1}{\kappa} ln \left(Ey^* \right) + P \right]} & y^* > y_l^* \end{cases}$$

- Remember, the value of y_l^* depends on the Prandtl number.
- At this point, the heat flux q_w at the wall can be computed as follows,

$$q_w = \rho c_p \alpha_w \frac{T_w - T_p}{y_p - y_w}$$

- The turbulent Prandtl number Pr_t is usually equal to 0.85, but it highly depends on the flow properties and the flow physics.

• Let us revisit the nondimensional temperature function T^* ,

$$T^* = Pr_t \left[\frac{1}{\kappa} \ln\left(Ey^*\right) + P\right] \quad y^* > 30$$

- The term *P* appearing in this relation, has a strong dependence on the Prandtl number (molecular and turbulent).
- One way to approximate this term is by using Jayatilleke function [1],

$$P = 9.24 \left[\left(\frac{Pr}{Pr_t} \right)^{3/4} - 1 \right] \left[1 + 0.28e^{-0.007Pr/Pr_t} \right]$$

• Kader [2] and Patankar and Spalding [3] also proposed alternative formulations for computing P.

[2] B. Kader. Temperature and concentration profiles in fully turbulent boundary layers. Int. J. Heat Mass Transfer. 24(9). 1981.

^[1] C. Jayatilleke. The influence of Prandtl number and surface roughness on the resistance of the laminar sublayer to momentum and heat transfer. Prog. Heat Mass Transfer, 1, 193-329. 1969.

^[3] S. Patankar and D. Spalding. A calculation procedure for heat, mass and momentum transfer in three dimensional parabolic flows. Int. J. Heat Mass Transfer, 15(10). 1972.

- Let us summarize the steps needed to compute the wall heat flux q_w using wall functions,
 - Compute y*.
 - Compute the Prandtl number,

$$Pr = \frac{\nu}{\alpha}$$

- Compute the intersection point y_l^* .
- Compute the thermal diffusivity at the wall.

$$\alpha_w = \begin{cases} \alpha & y^* < y_l^* \\ \frac{u_\tau y_p}{Pr_t \left[\frac{1}{\kappa} ln \left(Ey^*\right) + P\right]} & y^* > y_l^* \end{cases}$$

• Compute the heat flux at the wall.

$$q_w = \rho c_p \alpha_w \frac{T_w - T_p}{y_p - y_w}$$

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- When studying the boundary layer, heat transfer rate, compressibility, pressure gradient, mach number, and surface roughness (among many factors), they all have an effect on the velocity profile.
- It is not obvious what effect each of these parameters have on the velocity profile.
- Hereafter, we present a few observations or general guidelines (use with care).



• Effect of adverse pressure gradient on the law of the wall.



The boundary layer thickens, and the skin friction decreases as the adverse pressure gradient is increased.

• Effect of compressibility on the law of the wall.



Increasing compressibility causes the skin friction to decrease.

• Effect of heat transfer on the law of the wall.



- Notice that depending on the Prandtl number, the boundary layer can be thicker or thinner.
- A cold wall creates a thinner boundary layer and increases the skin friction
- A hot wall thickens the boundary layer and decreases the skin friction.

• Effect of surface roughness on the law of the wall.



- Surface roughness increases the skin friction.
- It shifts the velocity profile downwards.

- When dealing with high-speed flows and heat transfer, the coefficients appearing in the law of the wall relations and closure relations, have a strong dependence on the Mach number and the heat transfer rate.
- No need to mention that the physical properties also depend on the Mach number and heat transfer rate.

$$u^* = \begin{cases} y^* & y^* < 11.225\\ \frac{1}{\kappa} \ln(y^*) + C & y^* > 11.225 \end{cases}$$

$$T^* = \begin{cases} Pr \, y^* & y^* < 5\\ Pr_t \left[\frac{1}{\kappa} \ln \left(Ey^* \right) + P \right] & y^* > 30 \end{cases}$$



Figure 5.3: Computed and measured velocity profiles for compressible flat-plate boundary layers: — Wilcox (2006) k- ω ; - - - Chien k- ϵ ; \circ Coles; \Box Watson.

 In the case of pressure gradients and flow separation (non-equilibrium conditions), the law of the wall can be sensitized to pressure gradients effects [1].

$$\frac{\tilde{U}C_{\mu}^{1/4}k^{1/2}}{\tau_w/\rho} = \frac{1}{\kappa} ln\left(E\frac{\rho C_{\mu}^{1/4}k^{1/2}y}{\mu}\right)$$

• Where,

$$\tilde{U} = U - \frac{1}{2} \frac{dp}{dx} \left[\frac{y_v}{\rho \kappa k^{1/2}} ln\left(\frac{y}{y_v}\right) + \frac{y - y_v}{\rho \kappa k^{1/2}} + \frac{y_v^2}{\mu} \right]$$
$$y_v = \frac{\mu y_v^*}{\rho C_\mu^{1/4} k_p^{1/2}} \qquad \qquad y_v^* = 11.225$$

- This correction is recommended for use in complex flows involving separation, reattachment, and impingement where the mean flow and turbulence are subjected to pressure gradients and rapid changes.
- In such flows, improvements can be obtained, particularly in the prediction of wall shear (skinfriction coefficient) and heat transfer (Nusselt or Stanton number).

[1] S. Kim, D. Choudhury. A Near-Wall Treatment Using Wall Functions Sensitized to Pressure Gradient. In ASME FED Vol. 217, Separated and Complex Flows. ASME. 1995.

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• The non-dimensional temperature T*, can be further improved by adding the viscous heating contribution [1],

$$T^* = T_c^* + \frac{D}{q}$$

• Where,

$$T_{c}^{*} = \begin{cases} Pr \, y^{*} & y^{*} < y_{l}^{*} \\ Pr_{T} \left[\frac{1}{\kappa} \ln \left(Ey^{*} \right) + P \right] & y^{*} > y_{l}^{*} \end{cases}$$
$$D = \begin{cases} \rho u^{*} \frac{1}{2} Pr U_{p}^{2} & y^{*} < y_{l}^{*} \\ \rho u^{*} \frac{1}{2} \left[Pr_{t} U_{p}^{2} + \left(Pr - Pr_{t} \right) U_{c}^{2} \right] & y^{*} > y_{l}^{*} \end{cases}$$

- Where U_c is the mean velocity at the intersection point y_l^* .
- This correction is recommended for highly compressible flows, where heating by viscous dissipation can have a strong influence in the temperature distribution in the near-wall region.

^[1] J. R. Viegas, M. W. Rubesin, C. C. Horstman. On the Use of Wall Functions as Boundary Conditions for Two-Dimensional Separated Compressible Flows. Technical Report AIAA-85-0180. 1985.

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- So far, we have only considered smooth walls.
- In reality, every wall and material is characterized by small irregularities, which we refer to as roughness.
- The roughness has a characteristic height *h*, as shown in the figure below.
- Wall roughness increases the wall shear stress and heat transfer rate.



Examples of wall roughness. (a) Surface of an aluminum pipe: $h_{rms} \approx 0.16 \ \mu m$. (b) Scanned surface (1.4 x 1.1 mm²) of a non-rusted commercial steel pipe: $h_{rms} \approx 5 \ \mu m$. (c) Scales of the great white shark: $h_{rms} \approx 0.1 \ mm$. (d) Aerial views of tropical forest in Gabon (h $\approx 0.1-10 \ m$). (d) Barcelona landscape (h $\approx 10-100 \ m$). (f) The Namib desert (h $\approx 10-500 \ m$). Adapted from reference [1].

[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer. 2016.

- In CFD, the roughness must be modeled.
- In theory, it can be resolved with very fine meshes but the computational requirements are too high.
- Wall roughness increases the wall shear stress and heat transfer rate.
- It also breaks up the viscous sublayer.
- By looking at the nondimensional velocity plot, the wall roughness shifts the nondimensional velocity downwards by a factor of ΔB .



- There are many ways to add roughness to the solution.
- Let us study maybe the most common wall function for roughness.
- This implementation is based on the standard wall functions.

STEP 1.
$$U^+ = \frac{1}{\kappa} \ln (Ey^+) - \Delta B$$
 Roughness correction coefficient
STEP 2. $U^+ = \frac{1}{\kappa} \ln (Ey^+) - \ln (e^{\Delta B})$

Step 3.
$$U^+ = rac{1}{\kappa} \ln\left(rac{Ey^+}{e^{\Delta B}}
ight)$$

Where we used the following logarithm rules to derive the previous relations,

$$\ln(e^x) = x \qquad \ln(x) - \ln(y) = \ln\left(\frac{x}{y}\right)$$

• If we introduce the following relation,

$$E' = \frac{E}{e^{\Delta B}}$$

• Into the relation corrected for wall roughness,

$$U^{+} = \frac{1}{\kappa} \ln \left(\frac{Ey^{+}}{e^{\Delta B}} \right)$$

• We obtain the following equation,

$$U^{+} = \frac{1}{\kappa} ln \left(E' y^{+} \right)$$

- Which is identical to the standard wall function formulation (except for the variable E').
- We now have a way to work with smooth and rough walls using the same log-law relation.

- Let us now address the roughness correction factor ΔB_{+}
- This factor can be computed as follows [1, 2],

Transitional

$$\kappa \left(87.75 \right) = 2.25 < K_s^+ < 90$$

$$\Delta B = \frac{1}{\kappa} \ln \left(1 + C_s K_s^+ \right) \qquad \qquad K_s^+ > 90 \qquad \qquad \text{Fully rough}$$

• Notice that ΔB is a function of the nondimensional roughness height K_s^+ and the roughness constant C_s .

[2] P. Ligrani, R. Moffat. Structure of transitionally rough and fully rough turbulent boundary layers. J. of Fluid Mechanics, 162, 69-98. 1986.

^[1] T. Cebeci, P. Bradshaw. Momentum Transfer in Boundary Layers. Hemisphere Publishing Corporation. 1977.

- The hydraulically smooth condition exists when roughness heights are so small that the roughness is buried in the viscous sublayer.
- The fully rough flow condition exists when the roughness elements are so large that the sublayer is completely eliminated, and the flow can be considered as independent of molecular viscosity; that is, the velocity shift is proportional to $\ln(K_s^+)$.
- The transitional region is characterized by reduced sublayer thickness, which is caused by diminishing effectiveness of wall damping.
- Because molecular viscosity still has some role in the transitional region, the geometry of roughness elements has a relatively large effect on the velocity shift.

• The nondimensional roughness height can be computed as follows,

- Where K_s is the typical roughness height (sand grain diameter).
- The roughness constant C_s is often equal to 0.5. This constant represents the shape and distribution of the roughness elements (sand grains).
- It is recommended to fix C_s and adjust K_s^+ .
- Remember, in our discussion, the roughness regime is subdivided into the three regimes.
 - Hydraulically smooth.
 - Transitional.
 - Fully rough.

- The subdivision of the roughness regime and the dependence of the constant C (law of the wall) and the roughness correction factor \Delta B are supported on experimental data (Nikuradse's data [1]).
- The constant C and the roughness correction factor depend on the roughness parameters K_s^+ and C_s .



Constant in the law of the wall as a function of surface roughness. Based on measurements of Nikuradse [1]. Adapted from [2].



Effect of wall roughness on the roughness correction factor and universal velocity profiles [3].

[2] D. Wilcox. Turbulence modeling for CFD. DCW Industries, Inc. 2006.

^[1] J. Nikuradse. Law of Flow in Rough Pipes. Technical Memorandum 1292, National Advisory Committee for Aeronautics. 1950.

^[3] F. Clauser. The turbulent boundary layer. Advan. Appl. Mech. 4, 1. 1956.

• Plots of mean velocity distribution for uniform roughness at several K_s^+ values.



Plots of mean velocity distribution for uniform roughness at several values of nondimensional roughness height [1].

Plot of roughness mean velocity profiles [2].

[1] T. Cebeci, A. M. O. Smith. Analysis of turbulent boundary layers. Academic Press. 1974.

[2] U. Oriji, X. Yang, P. Tucker. Hybrid RANS/ILES for Aero Engine Intake. Proceedings of the ASME Turbo Expo 2014. Paper No: GT2014-26472. 2014.

- There is no universal roughness function valid for all types of roughness.
- Many methods to take into account the surface roughness are available in the literature, just to name a few,
 - [1] T. Cebeci, P. Bradshaw. Momentum transfer in boundary layers. Hemisphere Publishing Corporation. 1977.
 - [2] P. Ligrani, R. Moffat. Structure of transitionally rough and fully rough turbulent boundary layers. J. of Fluid Mechanics, 162, 69-98. 1986.
 - [3] B. Aupoix, R. Spalart. Extensions of the Spalart-Allmaras turbulence model to account for wall roughness. Int. J. of Heat and Fluid Flow. 24. 2003.
 - [4] D. Wilcox. Turbulence modeling for CFD. DCW Industries, Inc. 2006.
 - [5] U. Oriji, X. Yang, P. Tucker. Hybrid RANS/ILES for aero engine intake. Proceedings of the ASME Turbo Expo 2014. Paper No: *GT2014-26472*. 2014.
 - [6] B. Krishnappan. Laboratory verification of a turbulent flow model. J. of Hydraulic Eng. 110. 1984.
 - [7] J. Yoon, V. Patel. A numerical model of flow in channels with sand-dune beds and ice covers. IIHR Report no. 362. 1993
 - [8] J. Rotta. Turbulent boundary layers in incompressible flows. Progress in Aerospace Science. 1982.

- Finally, as we have seen, a rough wall increases friction.
- But it happens that special rough surfaces can increase drag reduction.
- Well designed riblets can shift the nondimensional velocity profile upwards, and closer to the viscous law nondimensional profile.
- It appears that small riblets that are aligned with the flow direction can achieve a significant drag reduction up to 10%.
- The maximum drag reduction occurs for a rib spacing s between 14 and 20 viscous wall units, i.e., 14 < s⁺ < 20, as illustrated in the figure.
- This is an example of biomimetics, since the development and application of ribbed surfaces has been inspired by the presence of ribbed scales on sharks



- Besides using riblets for drag reduction, a similar effect can be achieved by polymer additives.
- Small amounts of certain polymer additives to fluids can achieve a significant reduction of friction drag, which is know as the Toms effect [1, 2, 3].
- The figure below shows experimental data of the measured velocity profiles for various flows with polymer additives.
 - The profiles for drag-reducing flows have logarithmic profiles with the same slope as for a Newtonian fluid, but with an offset, or effective slip.



[1] B. Toms. Some observations on the flow of linear polymer solutions through straight tubes at large Reynolds numbers. In Proc. 1st. Int. Congr. Rheol., vol. 2, pp. 135–141. 1948.

[2] P. Virk. Drag reduction fundamentals. AIChE J. 21, 625-656. 1975.

[3] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer. 2016.

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 One approach to achieving a more appropriate description of the Reynolds-stress tensor without introducing any additional transport equations (as in the RSM models) is to add extra high order terms to the Boussinesq approximation.

$$\rho \overline{u_i' u_j'} = \frac{2}{3} \rho k \delta_{ij} - 2\mu_t S_{ij} + f(S_{ij}, \Omega_{ij})$$

- Where $f(S_{ij}, \Omega_{ij})$ is a nonlinear function dependent on the mean strain rate S_{ij} tensor and spin tensor (rotation) Ω_{ij} .
- Recall that the mean strain rate tensor and spin tensor are defined as follows,

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \qquad \Omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

- These models are known as nonlinear eddy viscosity models (NLEVM).
- This idea was originally proposed by Lumley [1,2], and many NLEVM has been proposed since then.

^[1] J. Lumley. Toward a turbulent constitutive equation. Journal of Fluid Mechanics. 1970.

^[2] J. Lumley. Computational modeling of turbulent flows. Advances in Applied Mechanics. 1978.

- The NLEVM approach can be seen as a remedy to the deficiencies of the EVM.
- Where the main deficiencies of the EVM are:
 - Inability to proper describe the anisotropic behavior in shear layers. In the EVM the normal stresses are isotropic, $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = (2/3)k$.
 - Flow in ducts with secondary motions.
 - Overpredicting production of turbulent kinetic energy in stagnation points.
 - Failure to reproduce the asymmetric behavior of the velocity profiles in the presence of streamlined geometries (strong curvature).
 - Underpredicting turbulent viscosity in the presence of system rotation (strong vortices).
- In comparison to the EVM, the NLEVM models are more computational expensive (as they need to solve more terms and are wall resolving).
- They are also harder to convergence.
- However, they do offer improved prediction capabilities for certain complex turbulent flows.
- Despite the many apparent advantages of NLEVM, they are not widely used.
- EVM still are the workhorse of turbulence modeling.

- The NLEVM are usually quadratic or cubic.
- Let us briefly discussed the NLEVM by Shih et al [1], which is cubic.
- In this model, the Reynolds stresses are computed as follows,

$$-\rho \overline{u_i' u_j'} = -\frac{2}{3}\rho k \delta_{ij} + \mu_t 2S_{ij}^* + A_3 \frac{\rho k^3}{\epsilon^2} \left[\bar{S}_{ik} \bar{\Omega}_{kj} - \bar{\Omega}_{ik} \bar{S}_{kj} \right]$$
$$-2A_5 \frac{\rho k^4}{\epsilon^3} \left[\bar{\Omega}_{ik} \bar{S}_{kj}^2 - \bar{S}_{ik}^2 \bar{\Omega}_{kj} + \bar{\Omega}_{ik} \bar{S}_{km} \bar{\Omega}_{mj} - \frac{1}{3} \bar{\Omega}_{kl} \bar{S}_{lm} \bar{\Omega}_{mk} \delta_{ij} + I_s S_{ij}^* \right]$$

• Where I_s , S^{st}_{ij} , and S^2_{ij} are given by the following relationships,

$$I_{s} = \frac{1}{2} \left[\bar{S}_{kk} \bar{S}_{mm} - \bar{S}_{kk}^{2} \right] \qquad \qquad S_{ij}^{*} = \bar{S}_{ij} - \frac{1}{3} \bar{S}_{kk} \delta_{ij} \qquad \qquad S_{ij}^{2} = S_{ik} S_{kj}$$

- This model is particularly suited for swirling flows.
- It was developed to deal with aircraft engine combustors that generally involve turbulent swirling flows in order to enhance fuel-air mixing and flame stabilization.
- The model includes third order terms, so it offers extra accuracy.
- The method also satisfy the constraints of rapid distortion theory (RDT) and realizability.
- All the coefficients appearing in the nonlinear constitutive equation are calibrated using DNS and experimental data.
- The coefficient C_{μ} is not constant, it depends on the strain rate tensor.
- The value of the turbulent kinetic energy k and the dissipation rate ϵ are obtained from low-RE $k \epsilon$ turbulence models (wall resolving).
- Also, the value of the turbulent eddy viscosity is computed using the relations from low-RE $k-\epsilon$ turbulence models.