Turbulence and CFD models: Theory and applications

Part 7

1. Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation
2. The $k-\omega$ SST turbulence model
3. Vorticity based models
4. Third-order and higher order moment closure methods

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Generalization of the $k - \epsilon$ turbulence model and low-Reynolds formulation

- The equations of the $k - \epsilon$ turbulence model can be generalized as follows,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right] + L_k$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} f_1 \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} f_2 \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right] + L_\epsilon$$

$$\nu_t = f_\mu C_\mu \frac{k^2}{\epsilon}$$

- At this point, by changing the values of the coefficients f_1, f_2, f_μ, L_k, and L_ϵ we can obtain different formulations of the $k - \epsilon$ turbulence model.
- For example, by setting the coefficients f_1, f_2, f_μ, to one and the coefficients L_k and L_ϵ to zero, we recast the standard $k - \epsilon$ turbulence model.

Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation

- One of the drawbacks of the standard $k-\epsilon$ model is that it can only be used with wall functions.
    - Actually, and without any modification to the standard $k-\epsilon$ turbulence model, the equations can be integrated in the viscous sublayer all the way down to wall, but the results will deteriorate.
- The standard $k-\epsilon$ turbulence model is a wall modeling model.
- This also applies to the RNG $k-\epsilon$ [1, 2] and realizable $k-\epsilon$ [3] turbulence models, as both models are variants of the standard $k-\epsilon$ turbulence model.
- To avoid this limitation, we can integrate the governing equations in the viscous sublayer all the way to the wall by adding a few modifications to the original formulation.
- The resulting formulations are known as low-Reynolds number $k-\epsilon$ turbulence models.
- The terminology low-Reynolds number refers to the Reynolds number measured normal to the wall (something similar to the y^+ normal to the wall) and not the system Reynolds number.

[1] V. Yakhot, S. Orszag. Renormalization group analysis of turbulence: 1. Basic theory. Journal of Scientific Computing. Vol. 1, pp. 3-51. 1986.
[2] V. Yakhot, S.A. Orszag, S. Thangam, and C.G. Speziale. Development of Turbulence Models for Shear Flows by a Double Expansion technique. Physics of Fluids A Fluid Dynamics, 4(7), 1992.
[3] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New k-epsilon Eddy-Viscosity Model for High Reynolds Number Turbulent Flows - Model Development and Validation. Computers Fluids, 24(3):227-238, 1995.

Generalization of the $k - \epsilon$ turbulence model and low-Reynolds formulation

- The first low-Reynolds number $k - \epsilon$ turbulence model was developed by Jones and Launder [1, 2], and subsequently it has been modified by several authors.
- The primary modifications introduced by Jones and Launder [1, 2] were to include turbulence Reynolds number dependency functions f_1, f_2, and f_μ.
- The purpose of these functions is to correct or damp the behavior of the turbulent viscosity as we approach to the walls.
- The main idea is getting asymptotically consistent near wall behavior.
- Furthermore, additional terms L_k and L_ϵ were added to the equations to account for the dissipation which may not be isotropic.
- Recall that the turbulence Reynolds number is related to the Reynolds number of the integral scales and can be computed as follows,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu} \approx \frac{\nu_t}{\nu}$$

[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.

Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation

- Closure functions and coefficients for the $k-\epsilon$ turbulence models.

Model	f_1	f_2	f_μ	L_k	L_ϵ
Standard	1.0	1.0	1.0	0	0
Jones-Launder [1,2]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-2.5}{1+0.02Re_T}\right]$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial x_j}\right)^2$	$2\nu\nu_t\left(\frac{\partial^2 u_i}{\partial x_j x_k}\right)^2$
Launder-Sharma [3]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-3.4}{(1+0.02Re_T)^2}\right]$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial x_j}\right)^2$	$2\nu\nu_t\left(\frac{\partial^2 u_i}{\partial x_j x_k}\right)^2$
Hoffman [4]	1.0	$1-0.3\exp(-Re_T^2)$	$\exp\left[\frac{-1.75}{1+0.02Re_T}\right]$	$-\frac{\nu}{y}\frac{\partial k}{\partial y}$	0
Nagano-Hishida [5]	1.0	$1-0.3\exp(-Re_T^2)$	$[1-\exp(-Re_T/26.5)]^2$	$-2\nu\left(\frac{\partial k^{1/2}}{\partial y}\right)^2$	$\nu\nu_t(1-f_\mu)\left(\frac{\partial^2 u}{\partial y^2}\right)^2$
Chien [6]	1.0	$1-\frac{0.4}{1.8}\exp(-Re_T^2/36)$	$1-\exp(-0.0115d^+)$ $d^+=d\rho u_\tau/\mu$	$-2\frac{\mu k}{\rho d^2}$	$-2\frac{\mu\epsilon}{\rho d^2}\exp(-d^+/2)$

References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] G. Hoffman. Improved form of the low Reynolds number k-epsilon turbulence model. Physics of Fluids, vol. 18(3), pp. 309-312,1975.
[5] Y. Nagado, M. Hishida. Improved form of the k-epsilon model for wall turbulent shear flows. Journal of Fluids Engineering, vol. 109, pp. 156-160, 1987.
[6] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation

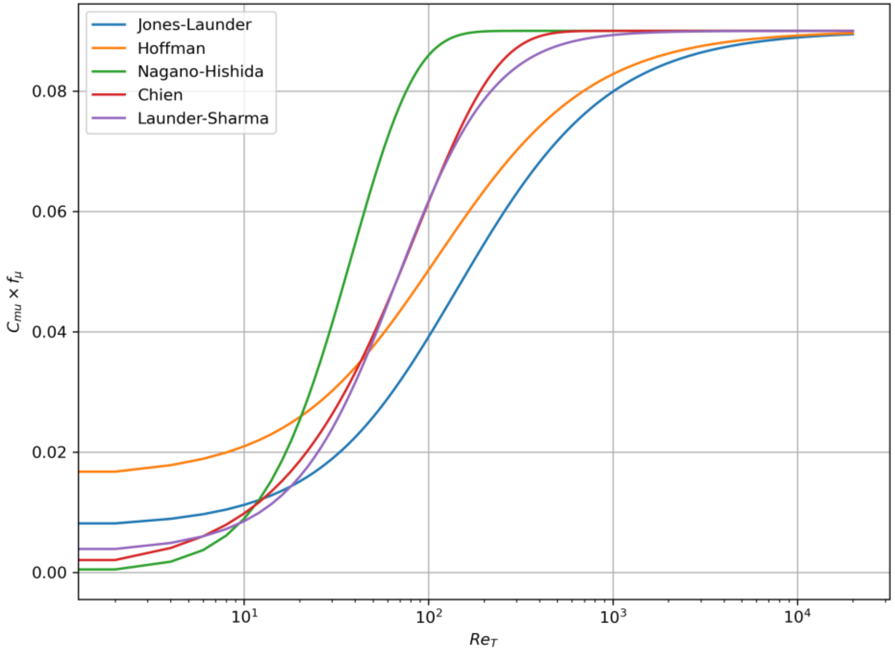
- Closure functions and coefficients for the $k-\epsilon$ turbulence models.

Model	C_μ	$C_{\epsilon 1}$	$C_{\epsilon 2}$	σ_k	σ_ϵ
Standard	0.09	1.44	1.92	1.0	1.3
Jones-Launder [1,2]	0.09	1.44	1.92	1.0	1.3
Launder-Sharma [3]	0.09	1.44	1.92	1.0	1.3
Hoffman [4]	0.09	1.81	2.0	2.0	3.0
Nagano-Hishida [5]	0.09	1.45	1.9	1.0	1.3
Chien [6]	0.09	1.35	1.8	1.0	1.3

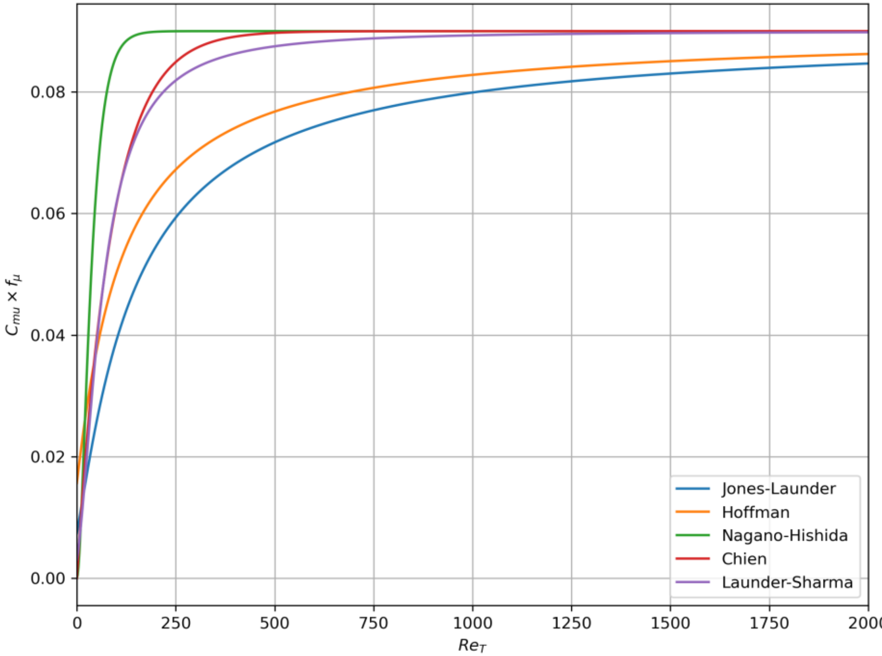
References:
[1] W. Jones, B. Launder. The prediction of laminarization with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 15, pp. 301–314, 1972.
[2] W. Jones, B. Launder. The calculation of low-Reynolds number phenomena with a two-equation model of turbulence. Int. J. Heat Mass Transfer, vol. 16, pp. 1119–1130, 1973.
[3] B. Launder, B. Sharma. Application of the energy dissipation model of turbulence to the calculation of flow near a spinning disc. Letters in Heat and Mass Transfer, Vol. 1(2), pp. 131-138. 1974.
[4] G. Hoffman. Improved form of the low Reynolds number k-epsilon turbulence model. Physics of Fluids, vol. 18(3), pp. 309-312,1975.
[5] Y. Nagado, M. Hishida. Improved form of the k-epsilon model for wall turbulent shear flows. Journal of Fluids Engineering, vol. 109, pp. 156-160, 1987.
[6] K. Chien. Predictions of Channel and Boundary-Layer Flows with a Low-Reynolds-Number Turbulence Model. AIAA Journal, vol. 20(1), pp. 33-38, 1982.

Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation

- Closure coefficient C_μ as a function of Re_T.
- These plots illustrate the damping effect towards the walls of the function f_μ.



Semi-logarithmic scale



Linear scale

Roadmap to Lecture 6

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1. ~~Generalization of the $k-\epsilon$ turbulence model and low-Reynolds formulation~~
2. **The $k-\omega$ SST turbulence model**
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The $k-\omega$ SST turbulence model

- The equations of the $k-\omega$ SST turbulence model [1] are the following ones,

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\mathbf{u} \omega) = \frac{\alpha}{\nu_t} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\omega} \right) \nabla \omega \right] + 2 \left(1 - F_1 \right) \frac{\sigma_{\omega 2}}{\omega} \nabla k \cdot \nabla \omega$$

$$\mu_t = \rho \frac{a_1 k}{\max \left(a_1 \omega, F_2 \Omega \right)}$$

$$\Omega = \sqrt{2 W_{ij} W_{ij}}$$

Magnitude of the vorticity tensor

$$W_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i} \right)$$

Anti-symmetric part of the velocity gradient (vorticity tensor)

- This model features several closure coefficients, blending functions, and auxiliary relations.
- Many of the coefficients used in this model are computed by a blend function between the respective constants of the $k-\epsilon$ and $k-\omega$ models.

[1] F. Menter. Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications. **AIAA Journal**, Vol. 32, No. 8, 1994, 1598-1605, 1994.

The $k-\omega$ SST turbulence model

- Letting ϕ denote any one of the parameters $\alpha, \sigma_k, \sigma_\omega, \beta, \beta^*, \mu_t$, then each of these parameters varies between a near-wall state ϕ_1 and a far-wall state ϕ_2 according to,

$$\phi = F_1\phi_1 + (1 - F_1)\phi_2$$

- Where F_1 and arg_1 are defined as follows,

$$F_1 = \tanh(\mathrm{arg}_1^4) \qquad \mathrm{arg}_1 = \min\left[\max\left(\frac{\sqrt{k}}{\beta^*\omega d}, \frac{500\nu}{d^2\omega}\right), \frac{4\rho\sigma_{\omega 2}k}{CD_{k\omega}d^2}\right]$$

- Notice that $0 \leq F_1 \leq 1$, and $arg_1 \geq 0$.
- In the equations, d is the distance from the wall. So, as d increases, the two expressions in the maximum of arg_1 become smaller as well the term that the maximum is being compared to.
- Thus, arg_1 diminishes with d, causing F_1 to approach zero and ϕ to approach the far-field value ϕ_2 (the coefficients of the $k-\epsilon$ turbulence model).
- The opposite behavior occurs as the wall is approached with,

$$\mathrm{arg}_1 \to \infty, \quad F_1 \to 1, \quad \phi \to \phi_1$$

- That is, we obtain the coefficients of the $k-\omega$ turbulence model.

The $k-\omega$ SST turbulence model

- The additional expressions and coefficients used in the formulation are,

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j}, 10^{-20}\right)$$

$$F_2 = \tanh(\arg_2^2) \qquad \arg_2 = \max\left(\frac{2\sqrt{k}}{\beta^*\omega d}, \frac{500\nu}{d^2\omega}\right)$$

$$\alpha_1 = \frac{\beta_1}{\beta^*} - \frac{\sigma_{\omega 1}\kappa^2}{\sqrt{\beta^*}} \qquad \alpha_2 = \frac{\beta_2}{\beta^*} - \frac{\sigma_{\omega 2}\kappa^2}{\sqrt{\beta^*}}$$

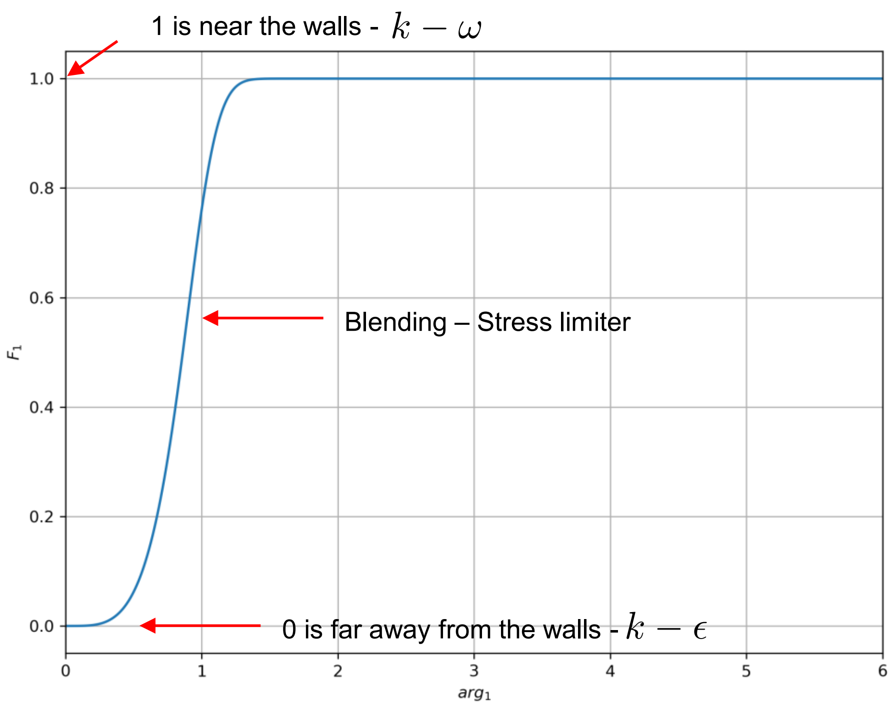
$$\sigma_{k1} = 0.85, \quad \sigma_{k2} = 1.0, \quad \sigma_{\omega 1} = 0.5, \quad \sigma_{\omega 2} = 0.856,$$

$$\beta_1 = 0.075, \quad \beta_2 = 0.0828, \quad \beta^* = 0.09, \quad \kappa = 0.41, \quad a_1 = 0.31$$

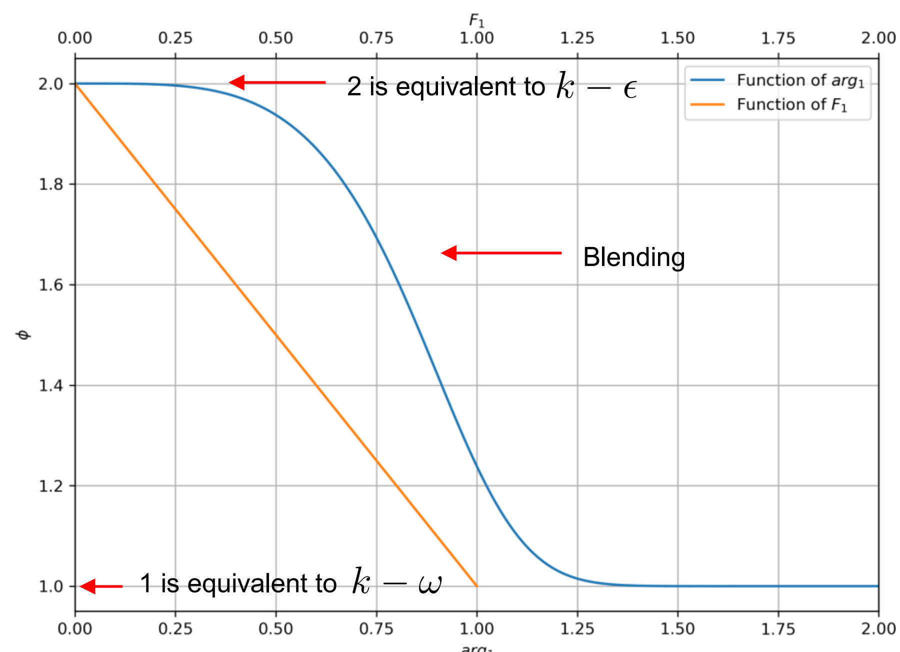
$\approx \sqrt{C_\mu}$ or $\approx \sqrt{\beta^*}$

The $k-\omega$ SST turbulence model

- Plot of the function F_1 as a function of arg_1.
- When $F_1 = 0$ the formulation is in the far-field conditions. That is, the formulation uses the $k-\epsilon$ model.
- And when $F_1 = 1$ the formulation is in the near wall region. That is, the formulation uses the $k-\omega$ model.
- The function F_1 represents a blending between the two turbulence models.



- Plot of any of the parameters ϕ of the turbulence model as a function of F_1.
- When F1 = 0 we use the parameters of the $k-\epsilon$ model.
- This function indicates how the parameters vary between a near-wall state and a far-wall state.
- The function ϕ blends the parameters of the two turbulence models.



The $k-\omega$ SST turbulence model

- It is worth noting that the SST model conforms to the $k-\epsilon$ model away from walls.
- And close to the walls it uses a y^+ insensitive formulation.
- The SST model and the Wilcox $k-\omega$ model share many common parameters.
- By using the following relationship,

$$\epsilon = \beta^* \omega k$$

- And by taking its substantial derivative, and after substituting for the k and ω derivatives using the solvable equations of the SST models, gives a differential equation for ϵ of essentially the same form as the solvable equation of the dissipation rate ϵ of the $k-\epsilon$ turbulence model.
- Substituting the far-field form of the constants into the expression yields an equation for the turbulent dissipation rate ϵ that conforms to that in the $k-\epsilon$ closure with only small differences.
- The $k-\omega$ SST model (and its variants), is considered the most efficient and general RANS/URANS turbulence model.
- Therefore, it is strongly recommended to use this model.

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Vorticity based models

- So far, we have used the Boussinesq hypothesis to model the Reynolds stress tensor.
- However, we must be aware that different approaches do exist.
- For example, an entirely different approach toward handling RANS was originally considered by Taylor [1] and subsequent authors [2,3,4].
- To avoid the appearance of the Reynolds stress tensor, they proposed the use of the following identity,

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial k}{\partial x_i} - \epsilon_{ijk} \overline{u_j \omega_k}$$

- Using this identity, we can write the momentum equation of the RANS equations in the vorticity transport form,

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_i \frac{\partial \bar{u}_i}{\partial x_j} = - \frac{\partial (\bar{p}/\rho + k)}{\partial x_i} + \nu \nabla^2 \bar{u}_i + \epsilon_{ijk} \overline{u_j \omega_k}$$

- In this approach, a model must be sought for the vorticity flux term $\overline{u_j \omega_k}$.
- Closures schemes based on this approach remain largely undeveloped.

References:
[1] G. Taylor. The transport of vorticity ad heat through fluids in turbulent motion. Proc. Roy. Soc., 135A, 1932.
[2] J. Hinze. Turbulence. McGraw-Hill. 1975.
[3] B. Perot, P. Moin. A new approach to turbulence modeling. Center for Turbulence Research. Proc. Summer Program. 1996.
[4] S. Goldstein. A Note on the Vorticity-Transport Theory of Turbulent Motion. Mathematical Proceedings of the Cambridge Philosophical Society, 31(3). 1935.

Roadmap to Lecture 6

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Third-order and higher order moment closure methods.

- We have seen that in order to derive the Reynolds stress transport equations, we need to multiply the Navier-Stokes operator $\mathcal{N}(u_i)$, by the velocity fluctuations, as follows,

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k} = 0$$

$$\overline{u_i' \mathcal{N}(u_j) + u_j' \mathcal{N}(u_i)} = 0$$

- Basically, we are multiplying the exact momentum equations by the velocity fluctuations in order to obtain governing equations for $\tau_{ij} = -\overline{u_i' u_j'}$.
- In doing so, we are increasing the order of closure of the equations, from first-order moment closure to second-order moment closure (in analogy to statistical moments).
- In theory, we can continue increasing the order of the moment closure up to infinite.
- So, we can derive third-order moment closure equations and so on.

Third-order and higher order moment closure methods.

- However, as we keep increasing the moment, higher order correlations will keep appearing in the equations.
- For example, in the **exact** Reynolds stress transport equations, which are second-order moment closure equations, a triple correlation appears, namely,

$$\overline{u_i' u_j' u_k'}$$

- We could derive a set of governing equations for this triple correlation, but the resulting equations will contain quadruple correlations.
- Therefore, it is easier to model this term.
- In the third-order moment closure equations, the quadruple correlation is expressed as follows,

$$\overline{u_i' u_j' u_k' u_l'}$$

- It is worth noting that third-order moment closure models do exist, but they are not widely diffused, and they do not guarantee better results.

Third-order and higher order moment closure methods.

- For example, the equations for the third order moments, read as,

$$
\begin{aligned}
\frac{\partial \overline{u_i u_j u_l}}{\partial t} + \overline{U}_k \frac{\partial \overline{u_i u_j u_l}}{\partial x_k} &= \overline{u_i u_j} \frac{\partial \overline{u_l u_k}}{\partial x_k} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_k} + \overline{u_l u_i} \frac{\partial \overline{u_j u_k}}{\partial x_k} \\
&\quad - \overline{u_j u_l u_k} \frac{\partial \overline{U}_i}{\partial x_k} - \overline{u_i u_j u_k} \frac{\partial \overline{U}_l}{\partial x_k} - \overline{u_l u_i u_k} \frac{\partial \overline{U}_j}{\partial x_k} \\
&\quad \underbrace{- \frac{\partial \overline{u_i u_j u_l u_k}}{\partial x_k}}_{T_{ijl}} \underbrace{- \frac{1}{\varrho} \left(\overline{u_l u_j \frac{\partial p}{\partial x_i}} + \overline{u_i u_j \frac{\partial p}{\partial x_l}} + \overline{u_l u_i \frac{\partial p}{\partial x_j}} \right)}_{\Pi_{ijl}} \\
&\quad \underbrace{-2\nu \left(\overline{u_i \frac{\partial u_j}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_j \frac{\partial u_i}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_l \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \right)}_{\epsilon_{ijl}} + \nu \frac{\partial^2 \overline{u_i u_j u_l}}{\partial x_k \partial x_k} .
\end{aligned}
$$

- For the interested reader, works related to third-order moment closure turbulence models can be found in references [1,2,3,4].

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[1] R. Amano, J. Chai. Closure models of turbulent third-order momentum and temperature fluctuations. NASA-CR-180421. 1987
[2] R. Amano, P. Goel. A study of Reynolds-Stress closure model. NASA-CR-174342. 1985.
[3] R. Amano, P. Goel. Improvement of the second- and third-moment modeling of turbulence: A study of Reynolds-stress closure model. NASA-CR-176478. 1986.
[4] R. Amano, J. Chai, J. Chen. Higher order turbulence closure models. NASA-CR-183236. 1988.