- Remember, before starting to use any turbulence model, it is strongly recommended to know its range of applicability and limitations.
- It is also important to know the recommend values for the boundary conditions and initial conditions.
- And any other information that may be useful when setting the simulation.
- Therefore, it is extremely recommended to read the original source of documentation of the model.
- This can be a paper or the help system of the CFD solver you are using.
- From now on, we will always mention the specific version of the turbulence model that we are going to use, and we will give some useful references.
- You do not do CFD and turbulence modeling without understanding the theoretical background.
    - So, do not say that you have not been warned.

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
2. ~~Revisiting the closure problem~~
3. **Two equations models – The $k - \epsilon$ model**
4. Two equations models – The $k - \omega$ model
5. One equation model – The Spalart-Allmaras model

Two equations models – The $k-\epsilon$ model

- This is one of the most popular family of two-equation turbulence models.
- It is based on the Boussinesq hypothesis (EVM).
- The initial development of this model can be attributed to Chou [1], circa 1945.
- Launder and Spalding [2] and Launder and Sharma [3] further developed and calibrated the model and created what is generally referred to as the Standard $k-\epsilon$ model.
- This is the model that we are going to address hereafter.
- There are many variations of this model.
- Each one designed to add new capabilities and overcome the limitations of the standard $k-\epsilon$ model.
- The most notable limitation of the standard $k-\epsilon$ model is that it requires the use of wall functions.
- Variants of this model include the RNG $k-\epsilon$ model [3] and the Realizable $k-\epsilon$ model [4], just to name a few.

References:
[1] P. Y. Chou. On Velocity Correlations and the Solutions of the Equations of Turbulent Fluctuation. Quarterly of Applied Mathematics. 1945.
[2] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.
[3] B. E. Launder, B. I. Sharma. Application of the Energy-Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc. Letters in Heat and Mass Transfer. 1974.
[4] V. Yakhot, S. A. Orszag. Renormalization Group Analysis of Turbulence I Basic Theory. Journal of Scientific Computing. 1986.
[5] T. Shih, W. Liou, A. Shabbir, Z. Yang, J. Zhu. A New - Eddy-Viscosity Model for High Reynolds Number Turbulent Flows - Model Development and Validation. Computers Fluids. 1995.

Two equations models – The $k-\epsilon$ model

- It is called $k-\epsilon$ because it solves two additional equations for modeling the turbulent viscosity, namely, the turbulent kinetic energy k and the turbulence dissipation rate ϵ.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{C_\mu k^2}{\epsilon}$$

- With the following closure coefficients,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- And auxiliary relationships,

$$\omega = \frac{\epsilon}{C_\mu k} \qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

Two equations models – The $k - \epsilon$ model

- The **solvable** closure equations of the standard $k - \epsilon$ turbulence model have been manipulated so there are no terms including fluctuating quantities (*i.e.*, velocity and pressure), and doble or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The turbulence dissipation rate is modeled using a second transport equation.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \overbrace{\tau^R : \nabla \bar{\mathbf{u}}}^{\text{Production}} - \overbrace{\epsilon}^{\text{Dissipation}} + \overbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]}^{\text{Diffusion}}$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = \underbrace{C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underbrace{C_{\epsilon_2} \frac{\epsilon^2}{k}}_{\text{Dissipation}} + \underbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]}_{\text{Diffusion}}$$

Two equations models – The $k - \epsilon$ model

- The transport equation of the turbulence dissipation rate ϵ used in this model can be derived by taking the following moment of the NSE equations,

$$2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} \left[\mathcal{N}\left(u_i\right)\right]} = 0$$

- Where the operator $\mathcal{N}(u_i)$ is equal to,

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k}$$

- The **exact** turbulence dissipation rate transport equation is far more complicated than the turbulent kinetic energy equation.
- This equation contains several new unknown double and triple correlations of fluctuating velocity, pressure, and velocity gradients.

Two equations models – The $k - \epsilon$ model

- There is a lot of algebra involved in the derivation of the **exact** turbulence dissipation rate transport equation.
- The final equation looks like this,

$$
\begin{aligned}
\underbrace{\frac{\partial \epsilon}{\partial t}}_{1} + \underbrace{\overline{u}_j \frac{\partial \epsilon}{\partial x_j}}_{2} = & \underbrace{-2\nu \frac{\partial \overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)}_{3} \underbrace{-2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j}\overline{u_k' \frac{\partial u_i'}{\partial x_j}}}_{4} \\
& \underbrace{-2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}}_{5} \underbrace{-2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m}\frac{\partial^2 u_i'}{\partial x_k \partial x_m}}}_{6} \\
& \underbrace{+\nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j}\right)}{\partial x_j}}_{7} \underbrace{-\nu \frac{\partial \left(\overline{u_j' \frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}}_{8} \underbrace{-2\frac{\nu}{\rho}\frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}}_{9}
\end{aligned}
$$

1. Transient rate of change term.
2. Convective term.
3. Production term that arises from the product of the gradients of the fluctuating and mean velocities.
4. Production term that generates additional dissipation based on the fluctuating and mean velocities.
5. Dissipation (destruction) associated with eddy velocity fluctuating gradients.
6. Dissipation (destruction) arising from eddy velocity fluctuating diffusion.
7. Viscous diffusion.
8. Diffusive turbulent transport resulting from the eddy velocity fluctuations.
9. Dissipation of turbulent transport arising from eddy pressure and fluctuating velocity gradients.

Two equations models – The $k-\epsilon$ model

- To derive the **solvable** transport equation of the turbulence dissipation rate, we need to use approximations in place of the terms that contain fluctuating quantities (velocity, pressure, and so on).
- The following approximations can be added to the **exact** turbulence dissipation rate transport equation in order to obtain the **solvable** transport equation.
- Notice that the gradient diffusion hypothesis and the product rule is consistently used when deriving the turbulence dissipation rate.

$$\underbrace{-2\nu\frac{\partial\overline{u}_i}{\partial x_j}\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}+\overline{\frac{\partial u_k'}{\partial x_i}\frac{\partial u_k'}{\partial x_j}}\right)}_{3}\underbrace{-2\nu\frac{\partial^2\overline{u}_i}{\partial x_k\partial x_j}\overline{u_k'\frac{\partial u_i'}{\partial x_j}}}_{4}\longrightarrow C_{\epsilon 1}\frac{\epsilon}{k}\tau_{ij}\frac{\partial\overline{u}_i}{\partial x_j}$$

Production

$$\underbrace{-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_m}\frac{\partial u_k'}{\partial x_m}}}_{5}\underbrace{-2\nu^2\overline{\frac{\partial^2 u_i'}{\partial x_k\partial x_m}\frac{\partial^2 u_i'}{\partial x_k\partial x_m}}}_{6}\longrightarrow -C_{\epsilon 2}\frac{\epsilon^2}{k}$$

Dissipation

$$\underbrace{\nu\frac{\partial\left(\frac{\partial\epsilon}{\partial x_j}\right)}{\partial x_j}}_{7}\underbrace{-\nu\frac{\partial\left(\overline{u_j'\frac{\partial u_i'}{\partial x_m}\frac{\partial u_i'}{\partial x_m}}\right)}{\partial x_j}}_{8}\underbrace{-2\frac{\nu}{\rho}\frac{\partial\left(\overline{\frac{\partial p'}{\partial x_m}\frac{\partial u_j'}{\partial x_m}}\right)}{\partial x_j}}_{9}\longrightarrow\frac{\partial}{\partial x_j}\left[\left(\nu+\frac{\nu_t}{\sigma_\epsilon}\right)\frac{\partial\epsilon}{\partial x_j}\right]$$

Diffusion

Two equations models – The $k - \epsilon$ model

- By substituting the previous approximations in the **exact** turbulence dissipation rate transport equation, we derive the **solvable** equation.
- It is not easy to elucidate the behavior of each term appearing in the **exact** turbulence dissipation rate transport equation.
- All the approximations added are based on DNS simulations, experimental data, analytical solutions, or engineering intuition.
- The **solvable** turbulence dissipation rate transport equation takes the following form,

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} = \underbrace{C_{\epsilon_1} \frac{\epsilon}{k} \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{Production}} - \underbrace{C_{\epsilon_2} \frac{\epsilon^2}{k}}_{\text{Dissipation}} + \underbrace{\frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right]}_{\text{Diffusion}}$$

Two equations models – The $k - \epsilon$ model

- The standard $k - \epsilon$ turbulence model use wall functions.
- The wall boundary conditions for the solution variables are all taken care of by the wall functions implementation.
- Therefore, when using commercial solvers (Fluent in our case) you do not need to be concerned about the boundary conditions at the walls.
- Using the standard walls functions approach developed by Launder and Spalding [1], the numerical values of the boundary conditions at the walls are computed as follows,

$$\frac{\partial k}{\partial n} = 0 \qquad \epsilon_P = \frac{C_\mu^{3/4} k_P^{3/2}}{\kappa y_P}$$

Where the subscript P means cell center

- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the turbulence dissipation rate).

[1] B. E. Launder, D. B. Spalding. The Numerical Computation of Turbulent Flows. Computer Methods in Applied Mechanics and Engineering. 1974.

Roadmap to Lecture 6

Part 2

1. ~~Revisiting the Reynolds stress transport equation and the turbulent kinetic energy equation~~
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4. **Two equations models – The $k - \omega$ model**
5. One equation model – The Spalart-Allmaras model

Two equations models – The $k-\omega$ model

- This is probably the most widely used family of two-equation turbulence models.
- It is based on the Boussinesq hypothesis (EVM).
- The initial development of this model can be attributed to Kolmogorov [1], circa 1942. This was the first two-equation model of turbulence.
- The method was further developed and improved by Saffman [2], Launder and Spalding [3], Wilcox [4,5], Menter [6] and many more.
- There are many variations of this model. Hereafter, we will address the $k-\omega$ Wilcox 1988 model, which probably is the first formulation of the modern $k-\omega$ family of turbulence models.
- Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawbacks of the $k-\omega$ Wilcox 1988 model are its limitation to resolve streamline curvature and its overly sensitivity to initial conditions.
- This family of models is y^+ insensitive.
- Variants of this model include the Wilcox 1998 $k-\omega$, Wilcox 2006 $k-\omega$, and Menter 2003 $k-\omega$ SST.

References:
[1] A. N. Kolmogorov. Equations of Turbulent Motion in an Incompressible Fluid. Physics. 1941.
[2] P. Saffman. A Model for Inhomogeneous Turbulent Flow. Proceedings of the Royal Society of London. 1970.
[3] B. E. Launder, D. B. Spalding. Mathematical Models of Turbulence. Academic Press. 1972.
[4] D. C. Wilcox. Reassessment of the Scale-Determining Equation for Advanced Turbulence Models. AIAA Journal, 1988.
[5] D. C. Wilcox. Turbulence Modeling for CFD. DCW Industries, 2010.
[6] F. Menter, M. Kuntz, R. Langtry. Ten Years of Industrial Experience with the SST Turbulence Model. Turbulence, Heat and Mass Transfer. 2003.

Two equations models – The $k-\omega$ model

- It is called $k-\omega$ because it solves two additional equations for modeling the turbulence, namely, the turbulent kinetic energy k and the specific turbulence dissipation rate ω.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[(\nu + \sigma^* \nu_t) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[(\nu + \sigma \nu_t) \nabla \omega \right]$$

- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_t = \frac{k}{\omega}$$

- With the following closure coefficients,

$$\alpha = 5/9 \qquad \beta = 3/40 \qquad \beta^* = 9/100 \qquad \sigma = 1/2 \qquad \sigma^* = 1/2$$

- And auxiliary relationships,

$$\epsilon = \beta^* \omega k \qquad\qquad l = \frac{k^{1/2}}{\omega}$$

Two equations models – The $k-\omega$ model

- The closure equations of the Wilcox (1988) $k-\omega$ model have been manipulated so there are no terms including fluctuating quantities (*i.e.*, velocity and pressure), and doble or triple correlations of the fluctuating quantities.
- Remember, the Reynolds stress tensor is modeled using the Boussinesq approximation.
- The specific turbulence dissipation rate is modeled using a second transport equation.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \overbrace{\tau^R : \nabla \bar{\mathbf{u}}}^{\text{Production}} - \overset{\text{Dissipation}}{\beta^* k \omega} + \overbrace{\nabla \cdot \left[\left(\nu + \sigma^* \nu_t \right) \nabla k \right]}^{\text{Diffusion}}$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \underbrace{\alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underset{\text{Dissipation}}{\beta \omega^2} + \underbrace{\nabla \cdot \left[\left(\nu + \sigma \nu_t \right) \nabla \omega \right]}_{\text{Diffusion}}$$

Two equations models – The $k-\omega$ model

- In the Wilcox (1988) $k-\omega$ turbulence model, the production, dissipation, and diffusion terms or the specific turbulence dissipation rate ω are given by,

$$P^{\omega} = \alpha \frac{\omega}{k} \tau_{ij} \frac{\partial \overline{u}_i}{\partial x_j} = \alpha \frac{\omega}{k} P^k$$

$$\epsilon^{\omega} = \beta \omega^2$$

$$D^{\omega} = \frac{\partial}{\partial x_j} \left[(\nu + \sigma \nu_T) \frac{\partial \omega}{\partial x_j} \right]$$

Two equations models – The $k - \omega$ model

- The transport equation for the specific turbulence dissipation rate ω can be derived from the transport equation of the turbulence dissipation rate ϵ.
- The model can be thought as the ratio of ϵ to k.
- To derive the transport equation of the turbulence dissipation rate ω, we can start by using the following relation,

$$\epsilon = \beta^* \omega k$$

- Then, by using the product rule we can obtain the material derivative of the specific turbulence dissipation rate ω.
- At this point, we can substitute the material derivative of the variables of the $k - \epsilon$ turbulence model into the material derivative of the specific turbulence dissipation rate ω (the material derivative we just obtained).
- By proceeding in this way, we can obtain the **exact** equations of ω.
- To derive the **solvable** equations of the Wilcox (1988) $k - \omega$ turbulence model, we just need to insert the approximations into the exact equations (Boussinesq hypothesis, gradient diffusion hypothesis, and so on).

Two equations models – The $k-\omega$ model

- By using the following equations, it is possible to derive an **exact** transport equation for the specific turbulence dissipation rate ω.

$$\frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j} = \tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon + \frac{\partial}{\partial x_j} \left[\nu \frac{\partial k}{\partial x_j} - \frac{1}{2} \overline{u_i' u_i' u_j'} - \frac{1}{\rho} \overline{p' u_j'} \right]$$

$$\frac{\partial \epsilon}{\partial t} + \overline{u}_j \frac{\partial \epsilon}{\partial x_j} = -2\nu \frac{\partial \overline{u}_i}{\partial x_j} \left(\overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} + \overline{\frac{\partial u_k'}{\partial x_i} \frac{\partial u_k'}{\partial x_j}} \right) - 2\nu \frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} \overline{u_k' \frac{\partial u_i'}{\partial x_j}} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}}$$

$$-2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_m}} + \nu \frac{\partial \left(\frac{\partial \epsilon}{\partial x_j} \right)}{\partial x_j} - \nu \frac{\partial \left(\overline{u_j' \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m}} \right)}{\partial x_j} - 2 \frac{\nu}{\rho} \frac{\partial \left(\overline{\frac{\partial p'}{\partial x_m} \frac{\partial u_j'}{\partial x_m}} \right)}{\partial x_j}$$

$$\epsilon = \beta^* \omega k$$

- The new **exact** transport equation for the specific turbulence dissipation rate ω can be derived from the turbulence dissipation rate equation ϵ; therefore, they share many similarities.

Two equations models – The $k-\omega$ model

- As for the turbulence dissipation rate equation ϵ, there is a lot of algebra involved.
- Hereafter, we will show the most important steps.
- By using the product rule, we can write $\epsilon = \beta^* \omega k$ as follows,

$$\frac{d\epsilon}{dt} = \beta^* k \frac{d\omega}{dt} + \beta^* \omega \frac{dk}{dt} \quad \Longrightarrow \quad \frac{d\omega}{dt} = \frac{1}{\beta^* k}\frac{d\epsilon}{dt} - \frac{\omega}{k}\frac{dk}{dt}$$

- Where d/dt is the material derivative (dependent of the mean velocity),

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{u}_j \frac{\partial}{\partial x_j}$$

- By substituting the following relations into $d\omega/dt$,

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \overline{u}_j \frac{\partial \epsilon}{\partial x_j}$$

Exact transport equation of turbulence dissipation rate

$$\frac{dk}{dt} = \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j}$$

Exact transport equation of turbulence kinetic energy

- And doing a lot algebra, we obtain the **exact** equations of ω.

Two equations models – The $k-\omega$ model

- The final **exact** equation of the specific turbulence dissipation rate ω, can be written as follows,

$$
\begin{aligned}
\frac{d\omega}{dt} = \frac{\omega}{k}\left[-\tau_{ij} - 2\nu\frac{\overline{u'_{i,k}u'_{j,k}} + \overline{u'_{k,i}u'_{k,j}}}{\beta^*\omega}\right]\frac{\partial \overline{u}_i}{\partial x_j} \\
-2\nu\frac{u'_k u'_{i,j}}{\beta^* k}\frac{\partial^2 \overline{u}_i}{\partial x_k \partial x_j} \\
-\left[2\nu\frac{\overline{u'_{i,k}u'_{i,m}u'_{k,m}} + \nu\overline{u'_{i,km}u'_{i,km}}}{\beta^*\omega} - \beta^*\omega^2\right] \\
+\frac{\partial}{\partial x_j}\left[\nu\frac{\partial \omega}{\partial x_j} - \nu\frac{\overline{u'_j u'_{i,m}u'_{i,m}}}{\beta^* k}\right] + \frac{1}{2}\omega\frac{\overline{u'_j u'_i u'_i}}{k} - 2\nu\frac{\overline{p'_{,m}u'_{j,m}}}{\beta^*\rho k} + \omega\frac{\overline{p'u'_j}}{\rho k} \\
+\frac{1}{k}\left[2\nu\frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\rho}\overline{p'u'_j}\right]\frac{\partial \omega}{\partial x_j} \\
+\frac{1}{k^2}\left[\frac{\omega}{2}\overline{u'_j u'_i u'_i} - \frac{1}{\beta^*}\nu\overline{u'_j u'_{i,m}u'_{i,m}} + \frac{\omega}{\rho}\overline{p'u'_j} - \frac{2}{\beta^*}\frac{\nu}{\rho}\overline{p'_{,m}u'_{j,m}}\right]\frac{\partial k}{\partial x_j}
\end{aligned}
$$

- As for the **exact** turbulence dissipation rate transport equation, it is not easy to elucidate the behavior of each term appearing in this equation.
- As this equation was derived from **exact** turbulence dissipation rate transport equation, we can use similar approximations.

Two equations models – The $k-\omega$ model

- The $k-\omega$ family of turbulence models are y^+ insensitive.
- These models work by blending the viscous sublayer formulation and the logarithmic layer formulation based on the y^+.
- Unlike the standard $k-\epsilon$ model and some other models, the $k-\omega$ models can be integrated through the viscous sublayer without the need for wall functions.
- The wall boundary conditions for the turbulent variables can be computed as follows,

$$k = 0 \qquad \omega = \frac{6\nu}{\beta_0 d^2} \qquad \beta_0 \approx 0.075$$

d is the distance to the first cell center normal to the wall

- The free-stream values can be computed using the method introduced in Lecture 4.
- It is strongly recommended to not initialize these quantities with the same value or with values close to zero (in particular the specific turbulence dissipation rate).

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5. **One equation model – The Spalart-Allmaras model**

One equation model – The Spalart-Allmaras model

- The Spalart-Allmaras model [1,2] is a one-equation model that solves a model transport equation for the modified turbulent kinematic viscosity.
- By far this is the most popular and successful one-equation model.
- It also has been adopted as the foundation for DES models [3].
- The Spalart-Allmaras model was designed specifically for aerospace applications involving wall-bounded flows.
- In its original form, the Spalart-Allmaras model is a wall resolving method, requiring the use of fine meshes in order to resolve the viscous sublayer.
- Over the years this method has been improved. Each variation is designed to add new capabilities and overcome the limitations of the predecessor formulations.
- The most notable drawback is its limitation to deal with massive flow separation.
- Variants of this model include the addition of rotation/curvature corrections, trip terms, production limiters, strain adaptive formulations, wall roughness corrections, compressibility corrections, extension to y^+ insensitive treatment, and so on.
- Hereafter, we will address the model formulation described in reference [1] (which is probably the original formulation).

[1] P. Spalart, S. Allmaras, A One-Equation Turbulence Model for Aerodynamic Flows, Recherche Aerospatiale, No. 1, pp. 5-21, 1994.
[2] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. AIAA Conference Paper AIAA-92-0439, 1992.
[3] M. Shur, P. R. Spalart, M. Strelets, A. Travin. Detached-Eddy Simulation of an Airfoil at High Angle of Attack. 1999.

One equation model – The Spalart-Allmaras model

- The Spalart-Allmaras model is based on the Boussinesq hypothesis (EVM).

$$\tau_{ij}^{R} = -\rho\overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3}\rho k \delta_{ij}$$

- Where the circled term is generally ignored because information about the turbulent kinetic energy k is not readily available.
- Details regarding a nonlinear implementation that also includes an approximation for the term,

$$-\frac{2}{3}\rho k \delta_{ij}$$

can be found in references [1, 2, 3].

- Most of the one equation turbulence models (unless they are based on a transport equation for the turbulent variable k), do not provide information about the turbulent kinetic energy.

[1] M. Mani, D. Babcock, C. Winkler, P. Spalart, Predictions of a Supersonic Turbulent Flow in a Square Duct, AIAA Paper 2013-0860, January 2013.
[2] C. Rumsey, H. Lee, T. Pulliam, Reynolds-Averaged Navier-Stokes Computations of the NASA Juncture Flow Model Using FUN3D and OVERFLOW, AIAA Paper 2020-1304, 2020.
[3] C. Rumsey, J. Carlson, T. Pulliam, P. Spalart, Improvements to the Quadratic Constitutive Relation Based on NASA Juncture Flow Data, AIAA Journal, Vol. 58, No. 10, pp. 4374-4384, 2020.

One equation model – The Spalart-Allmaras model

- In the Spalart-Allmaras model (SA), a closed equation for the turbulent eddy viscosity is artificially created that fits well a range of experimental and empirical data.
- To accomplish this, the SA ν_T equation is built up term by term in a series of calibrations involving flows of increasing complexity.

$$\underbrace{\nabla_t \nu_T}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \nu_T}_{\text{Convection}} = \underbrace{P^{\nu_T}}_{\text{Production}} + \underbrace{\epsilon^{\nu_T}}_{\text{Dissipation}} + \underbrace{D^{\nu_T}}_{\text{Diffusion}} + \underbrace{S^{\nu_T}}_{\text{Source terms}}$$

- The resulting model has gone through a number of developmental iterations beyond its original form and has been widely tested for different external aerodynamics applications.
- Probably, this is the turbulence model that has undergone more modifications.
- It is beyond the scope of this discussion to delve the different calibration steps of each term and the choice of the closure coefficients.
- The interested reader should take a look at the following link:
    - https://turbmodels.larc.nasa.gov/spalart.html

One equation model – The Spalart-Allmaras model

- Before introducing the SA model, it is worth mentioning that this model does not actually solves a transport equation for the turbulent eddy viscosity ν_T.
- It solves a modified version of the turbulent eddy viscosity.
- Namely, modified eddy viscosity $\tilde{\nu}$.
- Therefore, we deal with the following general transport equation,

$$\underbrace{\nabla_t \tilde{\nu}}_{\text{Transient term}} + \underbrace{\nabla \cdot \bar{\mathbf{u}} \tilde{\nu}}_{\text{Convection}} = \underbrace{P^{\tilde{\nu}}}_{\text{Production}} + \underbrace{\epsilon^{\tilde{\nu}}}_{\text{Dissipation}} + \underbrace{D^{\tilde{\nu}}}_{\text{Diffusion}} + \underbrace{S^{\tilde{\nu}}}_{\text{Source terms}}$$

- Also, as it is an artificial method, the structure of the production and dissipation terms is slightly different from that of the two and more equations models.

One equation model – The Spalart-Allmaras model

- The closure equations of the standard SA model are as follows,

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = c_{b1} \left(1 - f_{t2}\right) \tilde{S} \tilde{\nu} - \left[c_{w1} f_w - \frac{c_{b1}}{\kappa^2} f_{t2} \right] \left(\frac{\tilde{\nu}}{d} \right)^2$$
$$+ \frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[\left(\nu + \tilde{\nu}\right) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}$$

- Where $\tilde{\nu}$ is the modified eddy viscosity.
- This model uses the following relation for the kinematic eddy viscosity,

$$\nu_T = \tilde{\nu} f_{v1}$$

- Where f_{v1} can be interpreted as a wall damping function [1].

[1] G. Mellor, H. Herring. Two methods of calculating turbulent boundary layer behavior based on numerical solutions of the equations of motion. 1968.

One equation model – The Spalart-Allmaras model

- With the following closure relationships,

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \qquad \chi = \frac{\tilde{\nu}}{\nu} \qquad f_{t2} = c_{t3} e^{-c_{t4}\chi^2}$$

$$\tilde{S} = \Omega + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}$$

d is the minimum distance to the nearest wall

$$f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}}$$

$$\Omega = \sqrt{2 W_{ij} W_{ij}}$$

Magnitude of the vorticity tensor

$$W_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} - \frac{\partial \overline{u}_j}{\partial x_i}\right)$$

Anti-symmetric part of the velocity gradient (vorticity tensor)

$$f_w = g\left[\frac{1 + c_{w3}^6}{g^6 + c_{w3}^6}\right]^{1/6} \qquad g = r + c_{w2}\left(r^6 - r\right) \qquad r = \min\left[\frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2}, 10\right]$$

One equation model – The Spalart-Allmaras model

- And with the following closure coefficients,

$$c_{b1} = 0.1355 \qquad c_{b2} = 0.622$$

$$c_{v1} = 7.1 \qquad \sigma = 2/3$$

$$c_{w1} = \frac{c_{b1}}{\kappa^2} + \frac{(1 + c_{b2})}{\sigma} \qquad c_{w2} = 0.3$$

$$c_{w3} = 2 \qquad \kappa = 0.41$$

$$c_{t3} = 1.2 \qquad c_{t4} = 0.5$$

One equation model – The Spalart-Allmaras model

- In the previous relationships, W_{ij} is the rotation tensor (anti-symmetric part of the velocity gradient) and d is the distance from the closest wall.
- Notice that the modified eddy viscosity equation depends on the distance from the closest wall, as well as on the gradient of the modified eddy viscosity gradient.
- Since $d \to \infty$ far from the walls, this model also predicts no decay of the eddy viscosity in a uniform stream.
- Inspection of the transport equation reveals that κd has been used as length scale.
- The length scale κd is also used in the term $\tilde{S}$, which is related to the vorticity.
- To avoid possible numerical problems, the vorticity parameter $\tilde{S}$ must never be allowed to reach zero or go negative. In references [1], a limiting method is reported.
- Many implementations of the SA model ignore the term f_{t2}, which was added to provide more stability when the trip term is used.
- Based on studies described in [2], the use of this form as opposed to the SA version with the trip term probably makes very little difference.
- The form of the Spalart-Allmaras model with the trip term included is given in reference [3].

[1] S. Allmaras, F. Johnson, P. Spalart. Modifications and Clarifications for the Implementation of the Spalart-Allmaras Turbulence Model. 2012.
[2] C. Rumsey. Apparent Transition Behavior of Widely-Used Turbulence Models. 2007.
[3] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.

One equation model – The Spalart-Allmaras model

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = \underbrace{c_{b1} \tilde{S} \tilde{\nu}}_{\text{Production}} - \underbrace{c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2}_{\text{Dissipation}} + \underbrace{\frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \overbrace{\frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}}^{\text{Extra diffusion source term - Wake profile spreading}}}_{\text{Diffusion}}$$

- The closure equations of the SA turbulence model have been derived using empirical relationships, dimensional analysis, and experimental and numerical data.
- Basically, this is an artificial (or synthetic) model with a theoretical background less rigorous than that of any of the turbulence models that we previously studied.
- Despite being a one equation model, the SA model performs very well for a specific group of application, namely, compressible high-speed external aerodynamics in aerospace applications.
- Note that this model has source terms (production and destruction) that are non-zero in the freestream, even when vorticity is zero.
- The source terms are, however, very small, proportional to $1/d^2$.

One equation model – The Spalart-Allmaras model

$$\frac{\partial \tilde{\nu}}{\partial t} + \overline{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = \underbrace{c_{b1} \tilde{S} \tilde{\nu}}_{\text{Production}} - \underbrace{c_{w1} f_w \left(\frac{\tilde{\nu}}{d} \right)^2}_{\text{Dissipation}} + \underbrace{\frac{1}{\sigma} \frac{\partial}{\partial x_j} \left[(\nu + \tilde{\nu}) \frac{\partial \tilde{\nu}}{\partial x_j} \right] + \overbrace{\frac{c_{b2}}{\sigma} \frac{\partial \tilde{\nu}}{\partial x_i} \frac{\partial \tilde{\nu}}{\partial x_i}}^{\text{Extra diffusion source term - Wake profile spreading}}}_{\text{Diffusion}}$$

- Notice that in this form of the **solvable** equations, we have dropped the term f_{t2}.
- Which implies that $c_{t3} = 0$.
- Many implementations of the SA model ignore the term f_{t2}, which was a numerical fix in the original model to slightly delay transition so that the trip term could be activated appropriately. So, it is argued that if the trip is not used, then f_{t2} is not necessary [1, 2].
- Based on studies [3], use of this form as opposed to the version that retains the term f_{t2}, probably makes very little difference, at least at reasonably high Reynolds numbers.

[1] L. Eca, M. Hoekstra, A. Hay, D. Pelletier, A Manufactured Solution for a Two-Dimensional Steady Wall-Bounded Incompressible Turbulent Flow, International Journal of Computational Fluid Dynamics, Vol. 21, Nos. 3-4, pp. 175-188, 2007.
[2] B. Aupoix, P. Spalart, Extensions of the Spalart-Allmaras Turbulence Model to Account for Wall Roughness, International Joumal of Heat and Fluid Flow, Vol. 24, pp. 454-462, 2003. [3] P. Spalart, S. Allmaras. A One-Equation Turbulence Model for Aerodynamic Flows. 1994.
[3] C. Rumsey, Apparent Transition Behavior of Widely-Used Turbulence Models, International Journal of Heat and Fluid Flow, Vol. 28, pp. 1460-1471, 2007.

One equation model – The Spalart-Allmaras model

- Remember, the standard SA model is wall resolving.
- The wall boundary conditions for the turbulent variables can be computed as follows,

$$\tilde{\nu} = 0 \qquad \nu_t = 0$$

- The freestream conditions can be computed using the following relationships,

$$\tilde{\nu}_{\mathrm{farfield}} = 3\nu_\infty \quad to \quad 5\nu_\infty \qquad\qquad \nu_{t_{\mathrm{farfield}}} = 0.210438\nu_\infty \quad to \quad 1.29423\nu_\infty$$

- If information related to the turbulent kinetic energy and turbulent dissipation is available, the freestream conditions can be computed as follows,

$$\tilde{\nu} = \nu_t = \frac{C_\mu k^2}{\epsilon} \qquad\qquad \tilde{\nu} = \nu_t = \frac{k}{\omega}$$