- What is the circled term?

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nabla \cdot \left(\nu \nabla \bar{\mathbf{u}} + \frac{1}{\rho}\boldsymbol{\tau}^{R} \right)$$

- The circled term represents the total stresses,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nabla \cdot \left(\nu \nabla \bar{\mathbf{u}} + \frac{1}{\rho}\boldsymbol{\tau}^R \right)$$

- The total stresses can be decomposed in a laminar contribution and a turbulent contribution.

$$\boldsymbol{\tau}_{Total} = \boldsymbol{\tau}_{Laminar} + \boldsymbol{\tau}_{Turbulent}$$

- Where each contribution is given by,

$$\boldsymbol{\tau}_{Laminar} = \nu \nabla \bar{\mathbf{u}} \qquad \boldsymbol{\tau}_{Turbulent} = \frac{1}{\rho}\boldsymbol{\tau}^R$$

- The laminar (or viscous) stresses can be computed from the molecular viscosity and the mean velocity gradient,

$$\boldsymbol{\tau}_{Laminar} = \nu \nabla \bar{\mathbf{u}}$$

- The turbulent stresses (Reynolds stresses), can be computed as follows,

$$\boldsymbol{\tau}_{Turbulent} = \frac{1}{\rho} \boldsymbol{\tau}^{R}$$

- As we have seen, this term is a little bit trickier to compute.
- This term can be approximated or modeled (*e.g.*, by using the Boussinesq approximation).
- Or it can be resolved (DNS, LES, DES, experiments).

- We can approximate the Reynolds stresses by using the Boussinesq approximation,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T\overline{\mathbf{D}}^R - \frac{2}{3}\rho k\mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho k\mathbf{I}$$

- Or we can directly resolve it,

$$\tau^R = \tau_{ij}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

- We can derive a transport equation for each term of the Reynolds stress tensor $\boldsymbol{\tau}^R$, or we can resolve the velocity fluctuations using SRS simulations (DES, LES, DNS).

- If we make the assumptions that we are working with a 2D boundary layer, the total stress can be approximated as follows,

$$\boldsymbol{\tau}_{Total} = \nu \frac{\partial \bar{u}}{\partial y} + \overline{u'v'}$$

- With no simplifications (3D case), the total shear stress is computed as follows,

$$\boldsymbol{\tau}_{Total} = \nu \nabla \bar{\mathbf{u}} + \frac{1}{\rho} \boldsymbol{\tau}^{R}$$

- Statistical moments.

$$\mu = E(X)$$ ← Mean

$$\sigma^2 = E[(X-\mu)^2]$$ ← Variance

$$S = \frac{\mu_3}{\sigma^3} = \frac{E[(X-\mu)^3]}{\sigma^3}$$ ← Skewness

$$T = \frac{\mu_4}{\sigma^4} = \frac{E[(X-\mu)^4]}{\sigma^4}$$ ← Kurtosis

$$\frac{\mu_n}{\sigma^n} = \frac{E[(X-\mu)^n]}{\sigma^n}$$

- In lecture 5, we talked about first order and second order closure methods.
- Let us recall these methods.
- In analogy to the statistical moments,
  - First order closure methods (or first moment closure), approximate the solution using the mean velocity field.
  - They are based on the Boussinesq approximation,

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

  - Second order closure methods (or second moment closure), compute the solution using the mean turbulent field,

$$\overline{u_i' \mathcal{N}(u_j) + u_j' \mathcal{N}(u_i)} = 0$$

  - Where

$$\mathcal{N}(u_i) = \rho \frac{\partial u_i}{\partial t} + \rho u_k \frac{\partial u_i}{\partial x_k} + \frac{\partial p}{\partial x_i} - \mu \frac{\partial^2 u_i}{\partial x_k x_k} = 0$$

- Second order closure methods are based on the Reynolds stress equations,

$$\frac{\partial \tau_{ij}^R}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}^R}{x_k} = -\left(\tau_{ik}^R \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}^R \frac{\partial \bar{u}_i}{\partial x_k}\right) 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} + \ldots$$

$$\ldots + \frac{1}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}}\right) + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ij}^R}{\partial x_k}\right) + \frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)$$

- The previous equations can be rewritten as,

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \epsilon_{ij} - \Pi_{ij} + \frac{\partial}{\partial x_k}\left[\nu \frac{\partial \tau_{ij}}{\partial x_k} + C_{ijk}\right]$$

$$\epsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \qquad \rho C_{ijk} = \rho \overline{u_i' u_j' u_k'} + \overline{p' u_i'} \delta_{jk} + \overline{p' u_j'} \delta_{ik} \qquad \Pi_{ij} = \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}$$

- There are higher order closure methods.
- For example, the equations for the third order moments, read as,

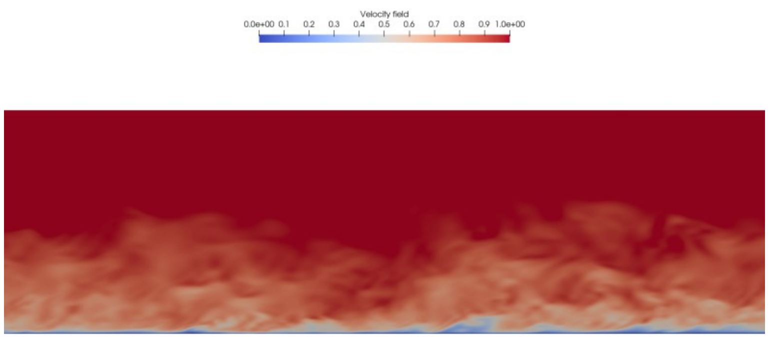
$$
\begin{aligned}
\frac{\partial \overline{u_i u_j u_l}}{\partial t} + \overline{U}_k \frac{\partial \overline{u_i u_j u_l}}{\partial x_k} = {} & \overline{u_i u_j} \frac{\partial \overline{u_l u_k}}{\partial x_k} + \overline{u_j u_l} \frac{\partial \overline{u_i u_k}}{\partial x_k} + \overline{u_l u_i} \frac{\partial \overline{u_j u_k}}{\partial x_k} \\
& - \overline{u_j u_l u_k} \frac{\partial \overline{U}_i}{\partial x_k} - \overline{u_i u_j u_k} \frac{\partial \overline{U}_l}{\partial x_k} - \overline{u_l u_i u_k} \frac{\partial \overline{U}_j}{\partial x_k} \\
& \underbrace{- \frac{\partial \overline{u_i u_j u_l u_k}}{\partial x_k}}_{T_{ijl}} \underbrace{- \frac{1}{\varrho} \left(\overline{u_l u_j \frac{\partial p}{\partial x_i}} + \overline{u_i u_j \frac{\partial p}{\partial x_l}} + \overline{u_l u_i \frac{\partial p}{\partial x_j}} \right)}_{\Pi_{ijl}} \\
& \underbrace{- 2\nu \left(\overline{u_i \frac{\partial u_j}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_j \frac{\partial u_i}{\partial x_k} \frac{\partial u_l}{\partial x_k}} + \overline{u_l \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}} \right)}_{\epsilon_{ijl}} + \nu \frac{\partial^2 \overline{u_i u_j u_l}}{\partial x_k \partial x_k}.
\end{aligned}
$$

- Notice that we keep multiplying by the mean fluctuating velocity (in analogy to the statistical moments).

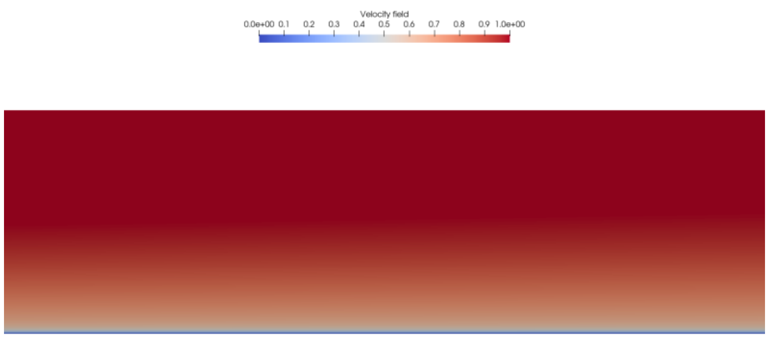
- Recall that the turbulent kinetic energy can be computed as follows,

$$k = \frac{1}{2}\overline{u_i^{'}u_i^{'}} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \frac{1}{2}\left(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}\right)$$

- If you are running SRS simulations, you need to compute the average of the product of the fluctuations.
- The product of the fluctuations are derived from the primitive variables.
- Therefore, before running, remember to set the unsteady statistics and define the products of the fluctuating quantities.



Instantaneous field

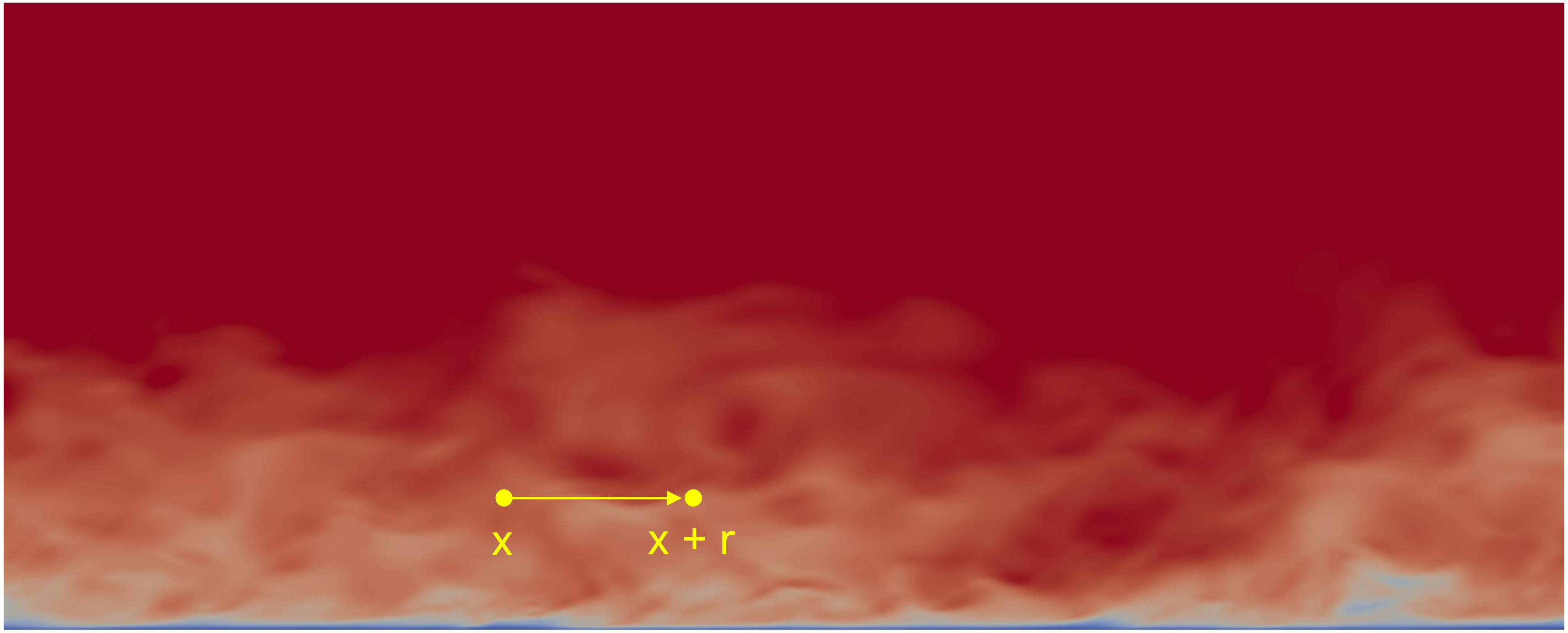


Mean field

- To stress this point.
  - All the correlations (or products) appearing in the Reynolds stress tensor, are computed from the instantaneous field.
  - Therefore, you need to compute the unsteady statistics and define these variables, that is, the products of the fluctuations.
  - Recall that the Reynolds stress tensor read as,

$$\tau^R = \tau_{ij}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

- Two-point correlation.



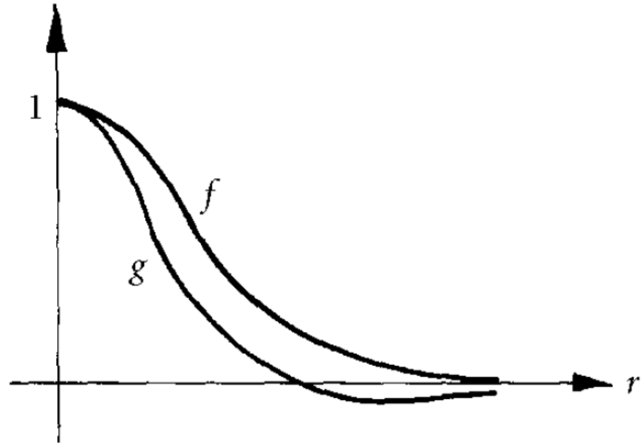
$$R_{ij}\left(\mathbf{x},t;\mathbf{r}\right)=\overline{u_i^{'}\left(\mathbf{x},t\right)u_j^{'}\left(\mathbf{x}+\mathbf{r},t\right)}$$

← Two-point velocity correlation tensor

$$R_{ij}\left(\mathbf{x},t;t'\right)=\overline{u_i^{'}\left(\mathbf{x},t\right)u_j^{'}\left(\mathbf{x},t+t'\right)}$$

← Autocorrelation tensor

- Two-point correlation.



$$f\left(x;r\right)=\frac{R_{11}\left(x;r\right)}{\overline{u_1'^2}}$$

Longitudinal velocity correlation coefficient

$$g\left(x;r\right)=\frac{R_{22}\left(x;r\right)}{\overline{u_2'^2}}$$

Lateral velocity correlation coefficient

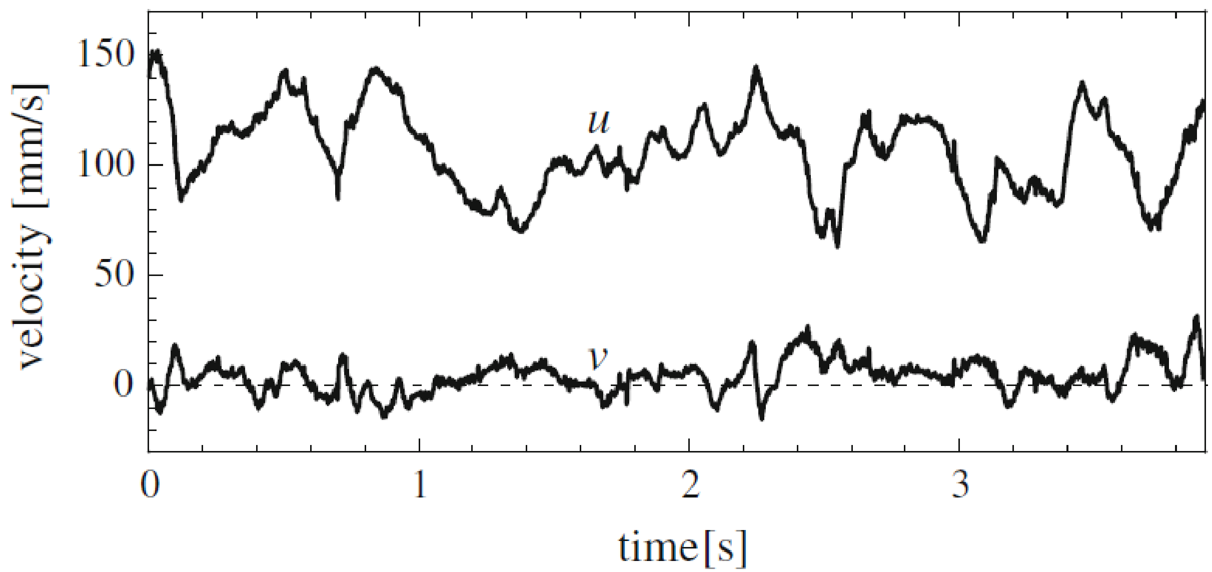
$$L_{11}=\int_0^\infty f\left(x;r\right)dr_1$$

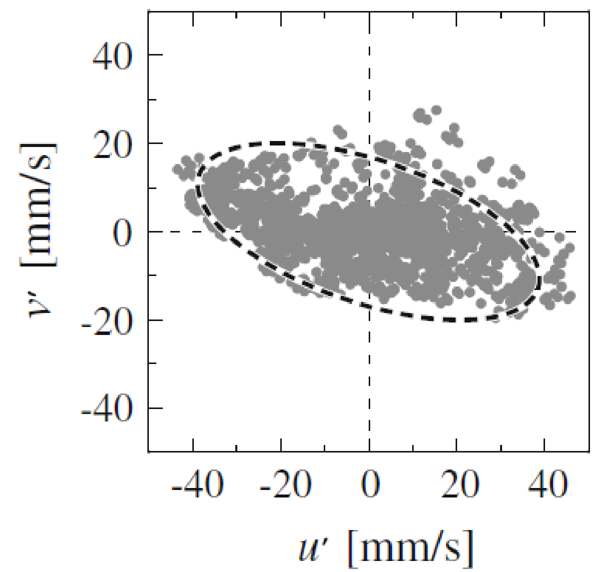
Longitudinal length scale

$$L_{22}=\int_0^\infty g\left(x;r\right)dr_2$$

Lateral length scale

- Time series of the measured stream wise and wall-normal velocity components measured by means of laser-Doppler velocimetry.





- **Left image.** Time series of the measured stream wise (u component) and wall-normal (v component) velocity measured by means of laser-Doppler velocimetry in a turbulent boundary layer over a flat plate
- **Right image.** The correlation of the fluctuations u' = u − u and v' = v − v. The ellipse represents the correlation between the two velocity components, i.e. u'v', or Reynolds stress. The main axes of the ellipse are proportional to σ_u and σ_v, respectively. The measurement was taken at a distance of $y^+ = 20$ viscous wall-units from the plate.
- Images adapted from reference [1].

[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

- The turbulence intensity is defined as follows,

$$I = \frac{\mathbf{u}_{RMS}}{U_\infty}$$

- Where RMS stands for root mean square.
- The RMS of the velocity is equivalent to the velocity fluctuations,

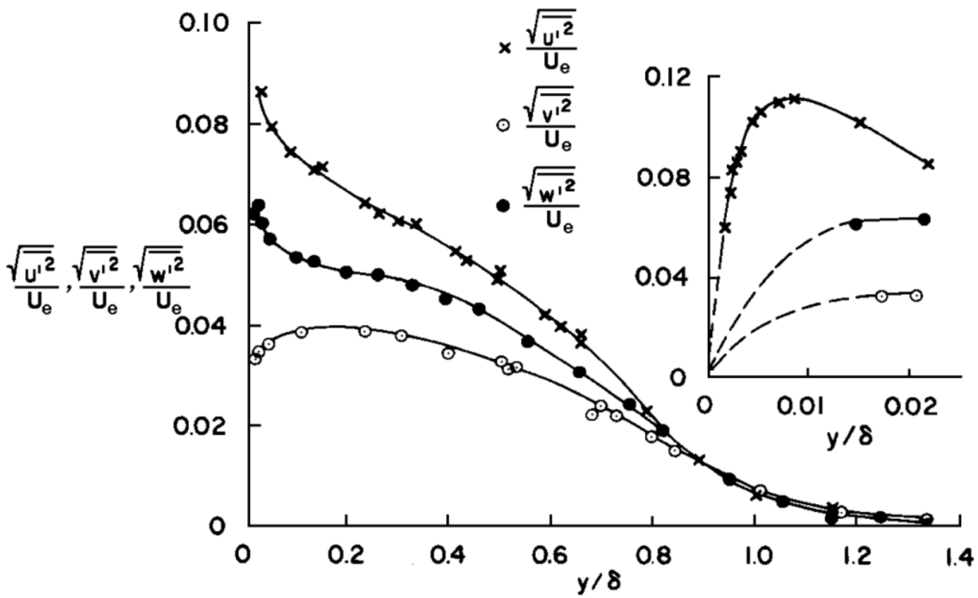
$$\mathbf{u}' = \mathbf{u}_{RMS} = \sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}$$

- The standard deviation is the same as the root mean squared, if it is defined about zero.
- We use the RMS to compute the turbulent intensity because otherwise you will get turbulent intensity levels that are too high.
- When you see the quantity RMS in solvers, it means that the fluctuations are computed about zero.

- The normal Reynolds stresses can be normalized relative to the mean flow velocity, as follows,

$$\widehat{u} \equiv \frac{\sqrt{\overline{u'^2}}}{\overline{u}} \qquad \widehat{v} \equiv \frac{\sqrt{\overline{v'^2}}}{\overline{u}} \qquad \widehat{w} \equiv \frac{\sqrt{\overline{w'^2}}}{\overline{u}}$$

- These three quantities are known as relative intensities.



Turbulence intensities for a flat-plate boundary layer of thickness [1].

$$\overline{u'^2} : \overline{v'^2} : \overline{w'^2} \approx 4 : 2 : 3$$

[1] P. Klebanoff. “Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient”. NACA TN 1247, 1955.