- Let us illustrate index notation using operators that commonly appears in the governing equations of fluid dynamics.
- In our notation, the indices $i,\ j,\ k$ can take the following values,

$$\left.\begin{matrix} i \\ j \\ k \end{matrix}\right\} 1, 2, 3$$

- These two sets of equations are the same, we simply wrote them using different notations.

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Vector notation

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

Index notation

- One free index results in a vector.
- It represents a gradient.

$$\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial x_1} + \frac{\partial \phi}{\partial x_2} + \frac{\partial \phi}{\partial x_3}$$

- In vector notation, is equivalent to,

$$\frac{\partial \phi}{\partial x_i} = \operatorname{grad} \phi = \nabla \phi$$

- The gradient will increase the rank of a tensor. That is, a zero-rank tensor (scalar), will become a first-rank tensor (vector), and a first-rank tensor will become a second-rank tensor (tensor).

- One repeated index results in a scalar.
- It is the sum over the index.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

- In vector notation, is equivalent to,

$$\frac{\partial u_i}{\partial x_i} = \operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u}$$

- The divergence will decrease the rank of a tensor. That is, a second-rank tensor (tensor), will become a first-rank tensor (vector), and a first-rank tensor will become a zero-rank (tensor).

- Two free indices results in a tensor.

$$u_i u_j = \begin{pmatrix} u_1 u_1 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2 u_2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3 u_3 \end{pmatrix} \quad \rightarrow \quad \mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^T = \mathbf{a}\mathbf{b}$$

$$\frac{\partial \phi_i}{\partial x_j} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_1}{\partial x_3} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_2}{\partial x_3} \\ \frac{\partial \phi_3}{\partial x_1} & \frac{\partial \phi_3}{\partial x_2} & \frac{\partial \phi_3}{\partial x_3} \end{pmatrix} \quad \rightarrow \quad \nabla \phi$$

$$\frac{\partial \phi_j}{\partial x_i} = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_3}{\partial x_1} \\ \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_2}{\partial x_2} & \frac{\partial \phi_3}{\partial x_2} \\ \frac{\partial \phi_1}{\partial x_3} & \frac{\partial \phi_2}{\partial x_3} & \frac{\partial \phi_3}{\partial x_3} \end{pmatrix} \quad \rightarrow \quad \nabla \phi^T$$

- Two free indices results in a tensor.

$$-\rho\overline{u_i'u_j'} = 2\mu_t S_{ij} - \frac{2}{3}k\delta_{ij}$$

$$\delta_{ij}\begin{cases} = 1 & \text{if} \quad i = j \\ = 0 & \text{otherwise} \end{cases}$$

Kronecker delta

$$-\rho\overline{u_i u_j} = \begin{pmatrix} 2\mu S_{11} - \frac{2}{3}k & 2\mu S_{12} & 2\mu S_{13} \\ 2\mu S_{21} & 2\mu S_{22} - \frac{2}{3}k & 2\mu S_{23} \\ 2\mu S_{31} & 2\mu S_{32} & 2\mu S_{33} - \frac{2}{3}k \end{pmatrix}$$

- In vector notation, is equivalent to,

$$-\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T\bar{\mathbf{S}}^R - \frac{2}{3}\rho k\mathbf{I}$$

- Two repeated indices (*j*) and one free indices (*i*) results in a tensor.
- Summation in *j* and it will form a vector in *i*.

$$\frac{\partial(\overline{u_i u_j})}{\partial x_j} = \frac{\partial(\overline{u_i u_1})}{\partial x_1} + \frac{\partial(\overline{u_i u_2})}{\partial x_2} + \frac{\partial(\overline{u_i u_3})}{\partial x_3}$$

→ Summation in *j*

$$\frac{\partial(\overline{u_i u_1})}{\partial x_1} + \frac{\partial(\overline{u_i u_2})}{\partial x_2} + \frac{\partial(\overline{u_i u_3})}{\partial x_3} =$$

$$\left(\frac{\partial(\overline{u_1 u_1})}{\partial x_1} + \frac{\partial(\overline{u_1 u_2})}{\partial x_2} + \frac{\partial(\overline{u_1 u_3})}{\partial x_3}\right)\vec{i} +$$

$$\left(\frac{\partial(\overline{u_2 u_1})}{\partial x_1} + \frac{\partial(\overline{u_2 u_2})}{\partial x_2} + \frac{\partial(\overline{u_2 u_3})}{\partial x_3}\right)\vec{j} +$$

$$\left(\frac{\partial(\overline{u_3 u_1})}{\partial x_1} + \frac{\partial(\overline{u_3 u_2})}{\partial x_2} + \frac{\partial(\overline{u_3 u_3})}{\partial x_3}\right)\vec{k}$$

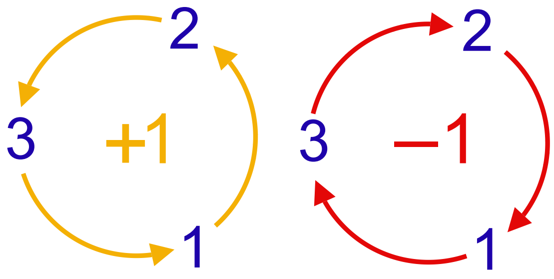
→ Vector in *i*

- Permutation or Levi-Civita operator.

$$\varepsilon_{ijk} \begin{cases} = 0 & \text{if any two of } i, j, k \text{ are the same} \\ = 1 & \text{for even permutation} \\ = -1 & \text{for odd permutation} \end{cases}$$

$$\text{for even permutation} = 123, 312, 231$$

$$\text{for odd permutation} = 321, 132, 213$$



- Using the Levi-Civita operator in the following way,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}$$

- Results in the following vector,

$$\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \underbrace{\varepsilon_{i11}\frac{\partial u_1}{\partial x_1}}_{\substack{j=1\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i12}\frac{\partial u_2}{\partial x_1}}_{\substack{j=1\\k=2\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=+1}} + \underbrace{\varepsilon_{i13}\frac{\partial u_3}{\partial x_1}}_{\substack{j=1\\k=3\\i=1\to\varepsilon=0\\i=2\to\varepsilon=-1\\i=3\to\varepsilon=0}} + \ldots$$

$$\underbrace{\varepsilon_{i21}\frac{\partial u_1}{\partial x_2}}_{\substack{j=2\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=-1}} + \underbrace{\varepsilon_{i22}\frac{\partial u_2}{\partial x_2}}_{\substack{j=2\\k=2\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i23}\frac{\partial u_3}{\partial x_2}}_{\substack{j=2\\k=3\\i=1\to\varepsilon=+1\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \ldots$$

$$\underbrace{\varepsilon_{i31}\frac{\partial u_1}{\partial x_3}}_{\substack{j=3\\k=1\\i=1\to\varepsilon=0\\i=2\to\varepsilon=+1\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i32}\frac{\partial u_2}{\partial x_3}}_{\substack{j=3\\k=2\\i=1\to\varepsilon=-1\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}} + \underbrace{\varepsilon_{i33}\frac{\partial u_3}{\partial x_3}}_{\substack{j=3\\k=3\\i=1\to\varepsilon=0\\i=2\to\varepsilon=0\\i=3\to\varepsilon=0}}$$

- Using the Levi-Civita operator in the following way,

$$\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

- Results in a vector.
- After some algebra, we obtain the following vector,

$$\varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \vec{i} + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \vec{j} + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \vec{k}$$

- In vector notation, is equivalent to,

$$\operatorname{curl} \mathbf{u} = \nabla \times \mathbf{u}$$

- A few additional operators in index notation that you will find in the governing equations of fluid dynamics.
    - Strain rate tensor,

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

    - In vector notation, is equivalent to,

$$\mathbf{S} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$$

- A few additional operators in index notation that you will find in the governing equations of fluid dynamics.
  - Laplacian,

$$\frac{\partial^2 u_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{bmatrix} \frac{\partial^2 u_1}{\partial x_1^2} & \frac{\partial^2 u_1}{\partial x_2^2} & \frac{\partial^2 u_1}{\partial x_3^2} \\ \frac{\partial^2 u_2}{\partial x_1^2} & \frac{\partial^2 u_2}{\partial x_2^2} & \frac{\partial^2 u_2}{\partial x_3^2} \\ \frac{\partial^2 u_3}{\partial x_1^2} & \frac{\partial^2 u_3}{\partial x_2^2} & \frac{\partial^2 u_3}{\partial x_3^2} \end{bmatrix}$$

  - In vector notation, is equivalent to,

$$\nabla \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u} = \Delta \mathbf{u}$$

- The Laplacian operator will not change the rank of a tensor.

- A few additional operators in index notation that you will find in the governing equations of fluid dynamics.
    - Every second-rank tensor, *e.g.*, the gradient of a vector, can be decomposed into a symmetric part and an anti-symmetric part.

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{Anti-symmetric part}}$$

    - In vector notation, is equivalent to,

$$\nabla \mathbf{u} = \underbrace{\frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}\left(\nabla \mathbf{u} - \nabla \mathbf{u}^T\right)}_{\text{Anti-symmetric part}}$$

    - The symmetric part is equivalent to the strain rate tensor and the anti-symmetric part is equivalent to the spin tensor (vorticity).