Turbulence and CFD models: Theory and applications

Roadmap to Lecture 5

1. **Governing equations of fluid dynamics**
2. **RANS equations – Reynolds averaging**
3. **The Boussinesq hypothesis**
4. **Sample turbulence models**

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“ That we have written an equation does not remove from the flow of fluids its charm or mystery or its surprise. ”

- Richard Feynman

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{S}_u$$

$$\frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) = \nabla \cdot q - \nabla \cdot (p\mathbf{u}) + \boldsymbol{\tau} \mathbf{:} \nabla \mathbf{u} + \mathbf{S}_e$$

Source terms

$$+$$

Additional equations to close the system

Relationships between two or more thermodynamics variables (p, ρ, T, e_t)

Additionally, relationships to relate the transport properties (μ, k)

- In the absent of models (turbulence, multiphase, mass transfer, combustion, particles interaction, chemical reactions, acoustics, and so on), this set of equations will resolve all scales in space and time.

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- I like to write the governing equations in matrix-vector form,

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{e}_i}{\partial x} + \frac{\partial \mathbf{f}_i}{\partial y} + \frac{\partial \mathbf{g}_i}{\partial z} = \frac{\partial \mathbf{e}_v}{\partial x} + \frac{\partial \mathbf{f}_v}{\partial y} + \frac{\partial \mathbf{g}_v}{\partial z}$$

- Where **q** is the vector of the conserved flow variables,

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix} \begin{array}{l} \leftarrow \text{CE} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \leftarrow \text{ME} \\ \leftarrow \text{EE} \end{array}$$

CE = Continuity equation   ME = Momentum equation (x,y,z)   EE = Energy equation

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- I like to write the governing equations in matrix-vector form.

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{e}_i}{\partial x} + \frac{\partial \mathbf{f}_i}{\partial y} + \frac{\partial \mathbf{g}_i}{\partial z} = \frac{\partial \mathbf{e}_v}{\partial x} + \frac{\partial \mathbf{f}_v}{\partial y} + \frac{\partial \mathbf{g}_v}{\partial z}$$

- The vectors $\mathbf{e}_i$, $\mathbf{f}_i$, and $\mathbf{g}_i$ contain the inviscid fluxes (or convective fluxes) in the x, y, and z directions,

$$\mathbf{e}_i = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ (\rho e_t + p)\, u \end{bmatrix}, \qquad \mathbf{f}_i = \begin{bmatrix} \rho v \\ \rho v u \\ \rho v^2 + p \\ \rho v w \\ (\rho e_t + p)\, v \end{bmatrix}, \qquad \mathbf{g}_i = \begin{bmatrix} \rho w \\ \rho w u \\ \rho w v \\ \rho w^2 + p \\ (\rho e_t + p)\, w \end{bmatrix} \begin{array}{l} \leftarrow \text{CE} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \leftarrow \text{ME} \\ \leftarrow \text{EE} \end{array}$$

CE = Continuity equation   ME = Momentum equation (x,y,z)   EE = Energy equation

Governing equations of fluid dynamics

- I like to write the governing equations in matrix-vector form.

$$\frac{\partial \mathbf{q}}{\partial t}+\frac{\partial \mathbf{e}_i}{\partial x}+\frac{\partial \mathbf{f}_i}{\partial y}+\frac{\partial \mathbf{g}_i}{\partial z}=\frac{\partial \mathbf{e}_v}{\partial x}+\frac{\partial \mathbf{f}_v}{\partial y}+\frac{\partial \mathbf{g}_v}{\partial z}$$

- The vectors **e**$_v$, **f**$_v$, and **g**$_v$ contain the viscous fluxes (or diffusive fluxes) in the x, y, and z directions,

$$\mathbf{e}_v=\begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} u\tau_{xx}+v\tau_{xy}+w\tau_{xz}-q_x \end{bmatrix}, \quad \mathbf{f}_v=\begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ u\tau_{yx}+v\tau_{yy}+w\tau_{yz}-q_y \end{bmatrix}, \quad \mathbf{g}_v=\begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \\ u\tau_{zx}+v\tau_{zy}+w\tau_{zz}-q_z \end{bmatrix}$$

CE

ME

EE

CE = Continuity equation ME = Momentum equation (x,y,z) EE = Energy equation

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- The heat fluxes in the vectors $\mathbf{e_v}$, $\mathbf{f_v}$, and $\mathbf{g_v}$ can be computed using Fourier's law of heat conduction as follows,

$$q_x = -k\,\frac{\partial T}{\partial x}, \qquad q_y = -k\,\frac{\partial T}{\partial y}, \qquad q_z = -k\,\frac{\partial T}{\partial z}$$

- If we assume that the fluid behaves as a Newtonian fluid (a fluid where the shear stresses are proportional to the velocity gradients), the viscous stresses can be computed as follows,

$$\tau_{xx} = \lambda\left(\nabla\cdot\mathbf{u}\right) + 2\mu\frac{\partial u}{\partial x} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)$$
$$\tau_{yy} = \lambda\left(\nabla\cdot\mathbf{u}\right) + 2\mu\frac{\partial v}{\partial y} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)$$
$$\tau_{zz} = \lambda\left(\nabla\cdot\mathbf{u}\right) + 2\mu\frac{\partial w}{\partial z} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$
$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$
$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- In virtually all practical aerodynamic problems, the working fluid can be assumed to be Newtonian
- In the normal viscous stresses $\tau_{xx}, \tau_{yy}, \tau_{zz}$, the variable λ is known as the second viscosity coefficient (or bulk viscosity).
- If we use Stokes hypothesis, the second viscosity coefficient can be approximated as follows,

$$\lambda = -\frac{2}{3}\mu$$

- Except for extremely high temperature or pressure, there is so far no experimental evidence that Stokes hypothesis does not hold.
- For gases and incompressible flows, Stokes hypothesis is a good approximation.

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- By using Stokes hypothesis and assuming a Newtonian flow, the viscous stresses can be expressed as follows,

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right), \qquad \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right), \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{zz} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

- So far we have five equations and seven variables.
- To close the system we need to find two more equations by determining the relationship that exist between the thermodynamics variables (p, ρ, T, e_t).

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- Choosing the internal energy e_i and the density ρ as the two independent thermodynamic variables, we can find equations of state of the form,

$$p = p(e_i, \rho) \qquad\qquad T = T(e_i, \rho)$$

- Assuming that the working fluid is a gas, that behaves as a perfect gas and is also a calorically perfect gas, the following relations for pressure *p* and temperature *T* can be used,

$$p = (\gamma - 1)\,\rho e_i, \qquad T = \frac{p}{\rho R_g} = \frac{(\gamma - 1)\,e_i}{R_g}$$

- Now our system of equations is closed. That is, seven equations and seven variables.

Governing equations of fluid dynamics

- To derive the thermodynamics relations for pressure *p* and temperature *T*, the following equations where used,

$$p = \rho R_g T$$

Equation of state

Recall that R_g is the specific gas constant

$$e_i = c_v T,$$

Internal energy

$$h = c_p T,$$

Enthalpy

$$\gamma = \frac{c_p}{c_v},$$

Ratio of specific heats

$$c_v = \frac{R_g}{\gamma - 1},$$

Specific heat at constant volume

$$c_p = \frac{\gamma R_g}{\gamma - 1}$$

Specific heat at constant pressure

$$e_t = e_i + \frac{1}{2}\left(u^2 + v^2 + w^2\right)$$

Total energy

Governing equations of fluid dynamics

- In our discussion, it is also necessary to relate the transported fluid properties (μ, k) to the thermodynamic variables.
- The molecular viscosity (or laminar viscosity) can be computed using Sutherland's formula,

$$\mu = \frac{C_1 T^{\frac{3}{2}}}{(T + C_2)}$$

- The thermal conductivity can be computed as follows,

$$k = \frac{c_p \mu}{Pr}$$

Prandtl number of the working fluid

Governing equations of fluid dynamics

- If you are working with high speed compressible flows, it is useful to introduce the Mach number.
- The Mach number is a non-dimensional parameter that measures the speed of the gas motion in relation to the speed of sound a,

$$a = \left[\left(\frac{\partial p}{\partial \rho} \right)_S \right]^{\frac{1}{2}} = \sqrt{\gamma \frac{p}{\rho}} = \sqrt{\gamma R_g T}$$

- Then, the Mach number can be computed as follows,

$$M_\infty = \frac{U_\infty}{a} = \frac{U_\infty}{\sqrt{\gamma (p/\rho)}} = \frac{U_\infty}{\sqrt{\gamma R_g T}}$$

- And never forget the definition of the Reynolds number,

$$Re_L = \frac{\rho U L}{\mu} = \frac{U L}{\nu}$$

Governing equations – Reynolds averaging

Governing equations of fluid dynamics

- The previous equations, together with appropriate equations of state, boundary conditions, and initial conditions, govern the unsteady three-dimensional motion of a viscous Newtonian compressible fluid.
- These equations solve all the scales in space and time. Therefore, we need to use very fine meshes and very small time-steps.
- Notice that besides the thermodynamics models (or constitutive equations) and a few assumptions (Newtonian fluid and Stokes hypothesis), we did not used any other model.
- Our goal now is to add turbulence models to these equations in order to avoid solving all scales.
- This will allow us use coarse meshes and larger time-steps.
- We can also use steady solvers.
- Before deriving the RANS equations, let us add a few simplifications to this beautiful set of equations.

Simplifications of the governing equations of fluid dynamics

- In many applications the fluid density can be assumed to be constant.
- If the flow is also isothermal, the viscosity is also constant.
- This is true not only for liquids, but also for gases if the Mach number is below 0.3.
- Such flows are known as incompressible flows.
- If the fluid is also Newtonian, the governing equations written in compact conservation differential form and in primitive variable formulation (u, v, w, p) reduce to the following set of equations,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Governing equations – Reynolds averaging

Simplifications of the governing equations of fluid dynamics

- In expanded three-dimensional Cartesian coordinates, the simplified governing equations can be written as follows,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial ww}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

- It is worth noting that the simplifications added do not make the equations easier to solve.
- The mathematical complexity is the same. We just eliminated a few variables, so from the computational point of few, it means less storage.
- Also, the convergence rate is not necessarily faster.

Governing equations – Reynolds averaging

Simplifications of the governing equations of fluid dynamics

- We can also write the simplified governing equations in matrix-vector form.

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{e}_i}{\partial x} + \frac{\partial \mathbf{f}_i}{\partial y} + \frac{\partial \mathbf{g}_i}{\partial z} = \frac{\partial \mathbf{e}_v}{\partial x} + \frac{\partial \mathbf{f}_v}{\partial y} + \frac{\partial \mathbf{g}_v}{\partial z}$$

$$\mathbf{q} = \begin{bmatrix} 0 \\ u \\ v \\ w \end{bmatrix} \qquad \mathbf{e}_i = \begin{bmatrix} u \\ u^2 + p \\ uv \\ uw \end{bmatrix}, \qquad \mathbf{f}_i = \begin{bmatrix} v \\ vu \\ v^2 + p \\ vw \end{bmatrix}, \qquad \mathbf{g}_i = \begin{bmatrix} w \\ wu \\ wv \\ w^2 + p \end{bmatrix}$$

$$\mathbf{e}_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \end{bmatrix}, \qquad \mathbf{f}_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \end{bmatrix}, \qquad \mathbf{g}_v = \begin{bmatrix} 0 \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{bmatrix}$$

Simplifications of the governing equations of fluid dynamics

- Recall that the viscous stresses tensor $\boldsymbol{\tau}$ can be written as follows,

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

- By using Stokes hypothesis and assuming a Newtonian flow, the viscous stresses can be expressed as follows,

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

$$\tau_{zz} = 2\mu \frac{\partial w}{\partial z} \qquad \tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

Governing equations – Reynolds averaging

Simplifications of the governing equations of fluid dynamics

- The viscous stress tensor can be written in compact vector form as follows,

$$\boldsymbol{\tau} = 2\mu\mathbf{D}$$

- Where **D** represents the strain-rate tensor and is given by the following relationship,

$$\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\right]$$

- Additionally, the gradient tensor can be decomposed in symmetric and skew parts as follows,

$$\nabla\mathbf{u} = [\mathbf{D} + \mathbf{S}]$$

- Where **S** represents the spin tensor (vorticity), and is given by,

$$\mathbf{S} = \frac{1}{2}\left[\nabla\mathbf{u} - \nabla\mathbf{u}^{\mathrm{T}}\right]$$

- In our notation, **D** represent the symmetric part of the tensor and **S** represents the skew (or anti-symmetric) part of the tensor.

Governing equations – Reynolds averaging

Simplifications of the governing equations of fluid dynamics

- From now on, and only to reduce the amount of algebra, we will use the incompressible, isothermal, Newtonian, governing equations.

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

Governing equations – Reynolds averaging

Conservative vs. non-conservative form of the governing equations

- We presented the governing equations in their conservative form, that is, the vector of conservative variables is inside the derivatives.
- From a mathematical point of view, the conservative and non-conservative form of the governing equations are the same.
- But from a numerical point of view, the conservative form is preferred in CFD. Specially if we are using the finite volume method (FVM).
- The conservative form enforces local conservation as we are computing fluxes across the faces of a control volume.
- The conservative form use flux variables as dependent variables, and the non-conservative form uses the primitive variables as dependent variables.

$$\frac{\partial\left(\rho \mathbf{u}\right)}{\partial t}+\nabla\cdot\left(\rho\mathbf{u}\mathbf{u}\right)=-\nabla p+\nabla\cdot\boldsymbol{\tau}$$

Conservative form

- If you integrate this equation in a control volume, fluxes across the faces will arise.
- The FVM method is based on integrating the governing equations in every control volume.

$$\rho\left(\frac{\partial\left(\mathbf{u}\right)}{\partial t}+\mathbf{u}\cdot\nabla\left(\mathbf{u}\right)\right)=-\nabla p+\nabla\cdot\boldsymbol{\tau}$$

Non-conservative form

Conservative vs. non-conservative form of the governing equations

- Let us recall the following identity,

$$\nabla \cdot (\mathbf{u}\mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u}(\nabla \cdot \mathbf{u})$$

- From the divergence-free constraint $\nabla \cdot \mathbf{u} = 0$ it follows that $\mathbf{u}(\nabla \cdot \mathbf{u})$ is equal to zero. Therefore,

$$\nabla \cdot (\mathbf{u}\mathbf{u}) = \mathbf{u} \cdot \nabla \mathbf{u}$$

- Henceforth, the non-conservative form of the momentum equation (also known as the advective or convective form) is equal to

$$\rho \left(\frac{\partial (\mathbf{u})}{\partial t} + \mathbf{u} \cdot \nabla (\mathbf{u}) \right) = -\nabla p + \nabla \cdot \boldsymbol{\tau}$$

- And is equivalent to the conservative form of the momentum equation (also known as the divergence form),

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau}$$

Incompressible Navier-Stokes using index notation

- The incompressible Navier-Stokes equations can also be written using index notation as follows,

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

$$i = 1, 2, 3$$
$$j = 1, 2, 3$$

- Dust your notes on index notation as from time to time I will change from vector notation to index notation.

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

$$\longleftrightarrow$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j}$$

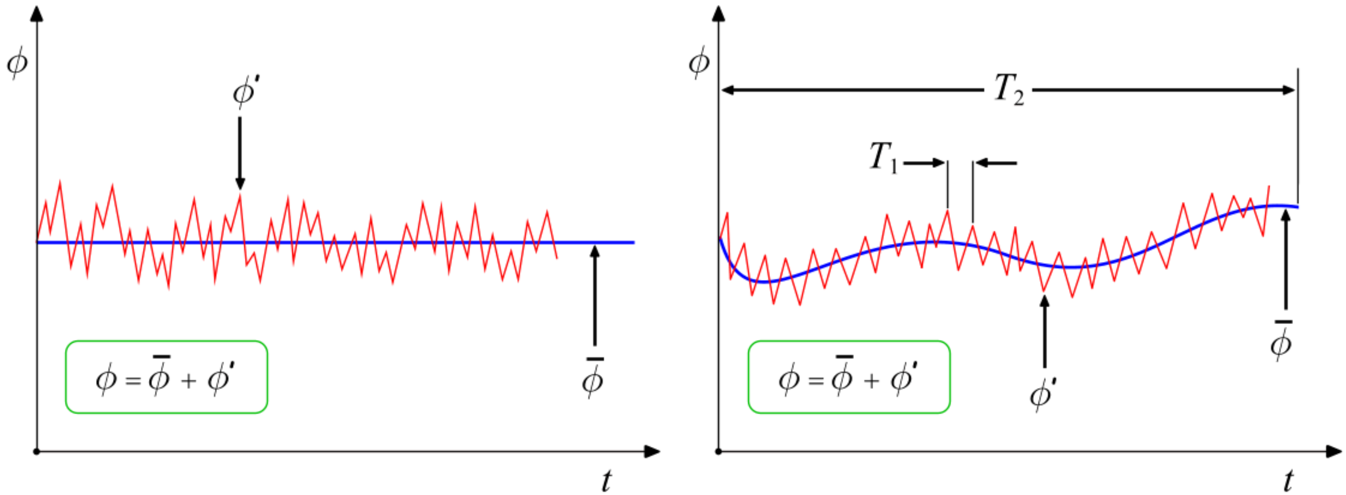
Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. **RANS equations – Reynolds averaging**
3. The Boussinesq hypothesis
4. Sample turbulence models

Governing equations – Reynolds averaging

Instantaneous fluctuations – Removing small scales

Steady mean value | Unsteady mean value



$$\phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\,value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\,part} \qquad \phi(\mathbf{x},t) = \underbrace{\bar{\phi}(\mathbf{x},t)}_{mean\,value} + \underbrace{\phi'(\mathbf{x},t)}_{fluctuating\,part}$$

- We have seen that turbulent flows are characterize by instantaneous fluctuations of velocity, pressure, and all transported quantities.
- In most engineering applications is not of interest resolving the instantaneous fluctuations.
- To avoid the need to resolve the instantaneous fluctuations (or small scales), two methods can be used:
    - Reynolds averaging.
    - Filtering.
- If you want to resolve all scales, you conduct DNS simulations, which are very computationally intensive.

Governing equations – Reynolds averaging

Instantaneous fluctuations – Removing small scales

- Two methods can be used to eliminate the need to resolve the small scales:
  - Reynolds averaging (RANS/URANS):
    - All turbulence scales are modeled.
    - Can be 2D and 3D.
    - Can be steady or unsteady.
  - Filtering (LES/DES):
    - Resolves large eddies.
    - Models small eddies.
    - Intrinsically 3D and unsteady.

- Both methods introduce additional terms in the governing equations that must be modeled (these terms are related to the instantaneous fluctuations).
- The final goal of turbulence modeling is to find the closure equations to model these additional terms (usually a stress tensor).

Governing equations – Reynolds averaging

Overview of turbulence modeling approaches

MODEL

RANS
Reynolds-**A**veraged **N**avier-**S**tokes equations

URANS
Unsteady **R**eynolds-**A**veraged **N**avier-**S**tokes equations

SAS
Scale **A**daptive **S**imulations

DES
Detached **E**ddy **S**imulations

LES
Large **E**ddy **S**imulations

DNS
Direct **N**umerical **S**imulations

SRS
Scale-**R**esolving **S**imulations

Increasing computational cost

Increasing modelling and mathematical complexity

Governing equations – Reynolds averaging

Turbulence modeling – Starting equations

NSE

$$
\left\{
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{S}_u \\
&\frac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) = \nabla \cdot q - \nabla \cdot (p\mathbf{u}) + \boldsymbol{\tau} \mathbf{:} \nabla \mathbf{u} + \mathbf{S}_e
\end{aligned}
\right.
$$

$$+$$

Additional equations to close the system (thermodynamic variables)

Additionally, relationships to relate the transport properties

Additional closure equations for the turbulence models

- Turbulence models equations cannot be derived from fundamental principles.
- All turbulence models contain some sort of empiricism.
- Some calibration to observed physical solutions is contained in the turbulence models.
- Also, some intelligent guessing is used.
- A lot of uncertainty is involved!

Incompressible RANS equations

- Let us write down the governing equations for an incompressible flow.
- When conducting DNS simulations (no turbulence models involved), this is our starting point,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- When using RANS turbulence models, these are the governing equations,

If we retain this term, we talk about **URANS** equations and if we drop it, we talk about **RANS** equations

Reynolds stress tensor
This term requires modeling

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} - \nabla \cdot \left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

Governing equations – Reynolds averaging

Incompressible RANS equations

- The previous set of equations ca be rewritten as,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}^R$$

- Where τ^R is the Reynolds stress tensor, and it can be written as,

$$\tau^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = - \begin{pmatrix} \rho \overline{u'u'} & \rho \overline{u'v'} & \rho \overline{u'w'} \\ \rho \overline{v'u'} & \rho \overline{v'v'} & \rho \overline{v'w'} \\ \rho \overline{w'u'} & \rho \overline{w'v'} & \rho \overline{w'w'} \end{pmatrix}$$

- Notice that the Reynolds stress tensor is not actually a stress, it must be multiplied by density in order to have dimensions corresponding to stresses,

Governing equations – Reynolds averaging

Incompressible RANS equations

- To derive the incompressible RANS equations we need to apply Reynolds averaging to the governing equations.
- Reynolds averaging simple consists in:
    - Splitting the instantaneous value of the primitive variables into a mean component and a fluctuating component (Reynolds decomposition).
    - Averaging the quantities (time average, spatial average or ensemble average).
    - Applying a few averaging rules to simplify the equations.
- When we use Reynolds averaging, we are taking a statistical approach to turbulence modeling.
- When we do DNS, we take a deterministic approach to turbulence modeling.
- Usually, we are interested in the mean behavior of the flow.
- Therefore, by applying Reynolds averaging, we are only solving for the averaged variables and the fluctuations are modeled.

Governing equations – Reynolds averaging

Incompressible RANS equations

- The Reynolds decomposition consists in splitting the instantaneous value of a variable into a mean component and a fluctuating component, as follows,

Instantaneous value

$$\phi(\mathbf{x}, t) = \bar{\phi}(\mathbf{x}) + \phi'(\mathbf{x}, t)$$

Mean component

Fluctuating component

- In our notation, the overbar represents the average (or mean) value, and the prime (or apostrophe) represents the fluctuating part.
- We will use this notation consistently during the lectures.

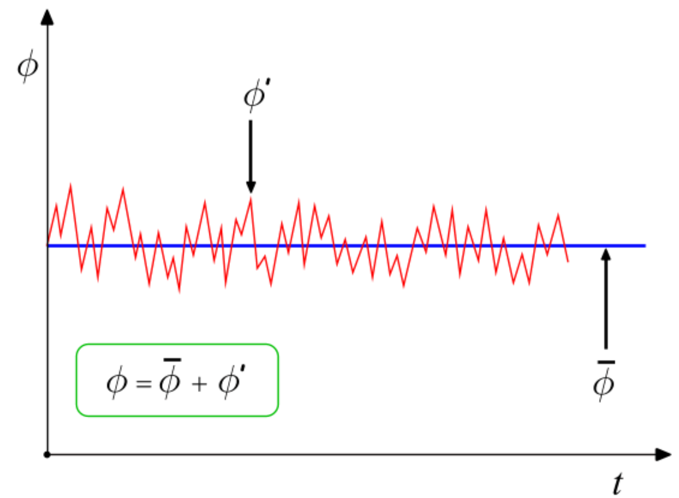
Governing equations – Reynolds averaging

Incompressible RANS equations

- To compute the average (or mean) quantities, we can use time averaging,

$$\bar{\phi}(\mathbf{x}) = \lim_{T\to\infty} \frac{1}{T} \int_{t}^{t+T} \phi(\mathbf{x}, t)\,dt$$

- Here, T represents the averaging interval. This interval must be large compared to the typical time scales of the fluctuations so it will yield to a stationary state.
- Time averaging is appropriate for stationary turbulence or slowly varying turbulent flows, *i.e.*, a turbulent flow that, on average, does not vary much with time.
- Notice that we are not making the distinction between steady or unsteady flow. The time average can be in time or iterative.
- We are only saying that if we take the average between different ranges or values of *t*, we will get approximately the same mean value.



$$\phi(\mathbf{x}, t) = \underbrace{\bar{\phi}(\mathbf{x})}_{mean\ value} + \underbrace{\phi'(\mathbf{x}, t)}_{fluctuating\ part}$$

Incompressible RANS equations

- We can also use spatial averaging and ensemble averaging.
- Spatial averaging is appropriate for homogenous turbulence and is defined as follows,

$$\bar{\phi}(t) = \lim_{V \to \infty} \frac{1}{V} \int_V \phi(\mathbf{x}, t) dV$$

Volume of the domain

- Ensemble averaging is appropriate for unsteady turbulence.

$$\bar{\phi}(\mathbf{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \phi(\mathbf{x}, t)$$

Number of realizations

- In ensemble averaging, the number or realizations (or experiments) must be large enough to eliminate the effects of fluctuations. This type of averaging can be used with steady or unsteady flows.

Governing equations – Reynolds averaging

Incompressible RANS equations

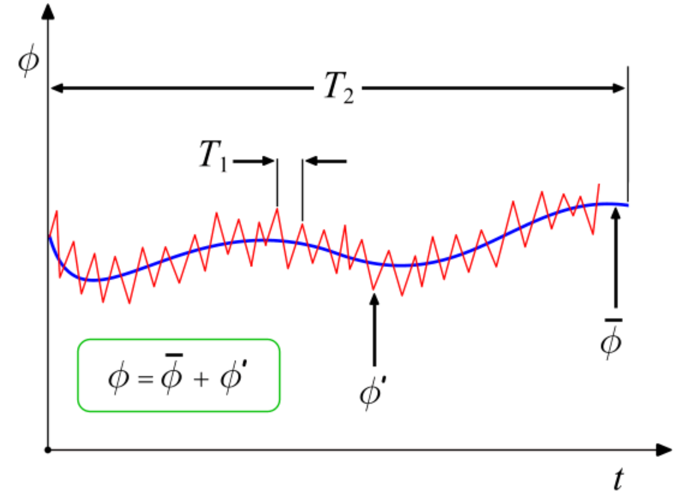
- If the mean quantities varies in time, such as,

$$\phi(\mathbf{x},t) = \bar{\phi}(\mathbf{x},t) + \phi'(\mathbf{x},t)$$

- We simple modify time averaging, as follows,

$$\bar{\phi}(\mathbf{x},t) = \lim_{T\to\infty} \frac{1}{T}\int_{t}^{t+T} \phi(\mathbf{x},t)dt \qquad T_1 << T << T_2$$

- Where T_2 is the time scale characteristic of the slow variations in the flow that we do not wish to regard as belonging to the turbulence.
- In this kind of situations, it might be better to use ensemble averaging.
- However, ensemble averaging requires running many experiments. This approach is better fit for experiments as CFD is more deterministic.
- Ensemble average can also be used when having periodic signal behavior. However, you will need to run for long times in order to take good averages.
- Another approach is the use of phase averaging.



Governing equations – Reynolds averaging

Incompressible RANS equations

- Any of the previous time averaging rules can be used without loss of generality.
- But from this point on, we will consider only time averaging.
- Before continuing, let us recall a few averaging rules that we will use when deriving the RANS equations.

$$\overline{\phi'} = 0,$$

$$\overline{\bar{\phi}} = \bar{\phi},$$

$$\overline{\bar{\phi}} = \overline{\bar{\phi} + \phi'} = \bar{\phi},$$

$$\overline{\phi + \varphi} = \bar{\phi} + \bar{\varphi},$$

$$\overline{\bar{\phi}\varphi} = \overline{\bar{\phi}}\bar{\varphi} = \bar{\phi}\bar{\varphi},$$

$$\overline{\bar{\phi}\varphi'} = \bar{\phi}\overline{\varphi'} = 0,$$

$$\overline{\frac{\partial \phi}{\partial x}} = \frac{\partial \bar{\phi}}{\partial x},$$

$$\begin{aligned}
\overline{\phi\varphi} &= \overline{(\bar{\phi} + \phi')(\bar{\varphi} + \varphi')} \\
&= \overline{\bar{\phi}\bar{\varphi} + \bar{\phi}\varphi' + \bar{\varphi}\phi' + \phi'\varphi'} \\
&= \overline{\bar{\phi}\bar{\varphi}} + \overline{\bar{\phi}\varphi'} + \overline{\bar{\varphi}\phi'} + \overline{\phi'\varphi'} \\
&= \bar{\phi}\bar{\varphi} + \overline{\phi'\varphi'},
\end{aligned}$$

$$\overline{\phi'^2} \neq 0,$$

$$\overline{\phi'\varphi'} \neq 0,$$

$$\overline{\int \phi ds} = \int \bar{\phi} ds$$

Incompressible RANS equations

- Let us recall the Reynolds decomposition for the primitive variables of the incompressible Navier-Stokes equations (NSE),

$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x}) + \mathbf{u}'(\mathbf{x},t),$$
$$p(\mathbf{x},t) = \bar{p}(\mathbf{x}) + p'(\mathbf{x},t)$$

- By substituting the previous equations into the incompressible (NSE), using the previous averaging rules, and doing some algebra, we arrive to the incompressible RANS/URANS equations,

If we retain this term, we talk about **URANS** equations and if we drop it, we talk about **RANS** equations

Reynolds stress tensor
This term requires modeling
$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}^R$$

Governing equations – Reynolds averaging

Incompressible RANS equations

- The RANS equations are very similar to the starting equations.

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}^R$$

RANS/URANS equations

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

NSE with no turbulence models (DNS)

- The differences are that all quantities has been averaged (the overbar over the primitive variables).
- And the appearance of the Reynolds stress tensor τ^R.
- Notice that the Reynolds stress tensor is not actually a stress, it must be multiplied by density in order to have dimensions corresponding to stresses,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

Governing equations – Reynolds averaging

Incompressible RANS equations

- The Reynolds stress tensor τ^R arises from the Reynolds averaging and it can be written as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

$$\mathbf{uu} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{bmatrix}$$

- It basically correlates the velocity fluctuations.
- In CFD we do not want to resolve the velocity fluctuations as it requires very fine meshes and small time-steps.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled.
- The rest of the terms appearing in the governing equations, can be computed from the mean flow.
- Notice that the Reynolds stress tensor is symmetric.

Governing equations – Reynolds averaging

Incompressible RANS equations

- The Reynolds stress tensor τ^R, is the responsible for the increased mixing and larger wall shear stresses. Remember, increased mixing and larger wall shear stresses are properties of turbulent flows.
- The RANS/URANS approach to turbulence modeling requires the Reynolds stresses to be appropriately modeled.
- The question now is, how do we model the Reynolds stress tensor τ^R?
- It is possible to derive its own governing equations (six new equations as the tensor is symmetric).
- This approach is known as Reynolds stress models (RSM), which we will briefly address in Lecture 6.
- Probably, this is the most physically sounded RANS model (RSM or Reynolds stress model) as it avoids the use of hypothesis/assumptions to model this term.
- However, it is much simpler to model the Reynolds stress tensor.
- The most widely hypothesis/assumption used to model the Reynolds stress tensor is the Boussinesq hypothesis, that we will study in next section.

Governing equations – Reynolds averaging

Incompressible RANS equations

- We just outlined the incompressible RANS equations.
- The compressible RANS equations are similar. To derive them, we use Favre average (which can be seen as a mass-weighted averaging) and a few additional averaging rules.
- If we drop the time derivative in the governing equations, we are dealing with steady turbulence.
- On the other hand, if we keep the time derivative, we are dealing with unsteady turbulence.
- If you can afford it, ensemble averaging is recommended. Have in mind that CFD is deterministic, so you should start each realization using different initial conditions and boundary conditions fluctuations to obtain different outcomes.
- The derivation of the LES equations is very similar, but instead of using averaging, we filter the equations in space, and we solve the temporal scales.
- LES/DES models are intrinsically unsteady and three-dimensional.
- We will address LES/DES methods in Lecture 10.

Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. ~~RANS equations – Reynolds averaging~~
3. **The Boussinesq hypothesis**
4. Sample turbulence models

The Boussinesq hypothesis

- The RANS/URANS approach to turbulence modeling requires the Reynolds stress tensor τ^R to be appropriately modeled.

$$\tau^R = \tau_{ij}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = -\begin{pmatrix} \rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'} \\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'} \\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'} \end{pmatrix}$$

- Remember, we do not want to resolve the instantaneous fluctuations.
- Even if it is possible to derive governing equations for the Reynolds stress tensor τ^R (six new equations as the tensor is symmetric), it is much simpler to model this term.
- The approach of deriving the governing equations for the Reynolds stress tensor τ^R is known as Reynolds stress model (RSM).
- Probably, this is the most physically sounded RANS model (RSM or Reynolds stress model) as it avoids the use of hypothesis/assumptions to model this term.

The Boussinesq hypothesis

- We will address the RSM model in Lecture 6.
- If you are curious, this is how the constitutive equations look like,

$$\underbrace{\frac{\partial \tau_{ij}^R}{\partial t}}_{1} + \underbrace{\bar{u}_k \frac{\partial \tau_{ij}^R}{x_k}}_{2} = \underbrace{-\left(\tau_{ik}^R \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk}^R \frac{\partial \bar{u}_i}{\partial x_k}\right)}_{3} + \underbrace{2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}}_{4} + \dots$$

$$\dots + \underbrace{\frac{1}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}}\right)}_{5} + \underbrace{\frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tau_{ij}^R}{\partial x_k}\right)}_{6} + \underbrace{\frac{\partial}{\partial x_k}\left(\overline{u_i' u_j' u_k'}\right)}_{7}$$

1. Transient stress rate of change term.
2. Convective term.
3. Production term.
4. Dissipation rate.
5. Turbulent stress transport related to the velocity and pressure fluctuations.
6. Rate of viscous stress diffusion (molecular).
7. Diffusive stress transport resulting from the triple correlation of velocity fluctuations.

We get 6 new equations, but we also generate 22 new unknowns.

$$\overline{u_i' u_j' u_k'} \quad \rightarrow \quad 10 \text{ unknowns}$$

$$2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} \quad \rightarrow \quad 6 \text{ unknowns}$$

$$\frac{1}{\rho}\left(\overline{u_i' \frac{\partial p'}{\partial x_j}} + \overline{u_j' \frac{\partial p'}{\partial x_i}}\right) \quad \rightarrow \quad 6 \text{ unknowns}$$

The Boussinesq hypothesis

- Modeling the Reynolds stress tensor is a much easier approach.
- A common approach used to model the Reynolds stress tensor τ^R, is to use the Boussinesq hypothesis.
- This approach was proposed by Boussinesq in 1877.
- He stated that the Reynolds stress tensor is proportional to the mean strain rate tensor, multiplied by a constant (which we will call turbulent eddy viscosity).
- The Boussinesq hypothesis is somehow similar to the hypothesis taken when dealing with Newtonian flows, where the viscous stresses are assumed to be proportional to the shear stresses, therefore, to the velocity gradient.
- The Boussinesq approximation reduces the turbulence modeling process from finding the six turbulent stresses in the RSM model to determining an appropriate value for the turbulent eddy viscosity μ_T.
- By the way, do not confuse the Boussinesq approximation used in turbulence modeling with the completely different concept found in natural convection (or buoyancy-driven flows).

The Boussinesq hypothesis

- A common approach used to model the Reynolds stress tensor τ^R, is to use the Boussinesq hypothesis.
- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T \overline{\mathbf{D}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho k \mathbf{I}$$

$\overline{\mathbf{D}}^R$	$\rightarrow$ Reynolds averaged strain-rate tensor.	k	$\rightarrow$ turbulent kinetic energy.
$\mathbf{I}$	$\rightarrow$ identity matrix (or Kronecker delta).	μ_T	$\rightarrow$ turbulent eddy viscosity.

- At the end of the day we want to determine the turbulent eddy viscosity.
- Each turbulence model will compute this quantity in a different way.
- Remember, the turbulent eddy viscosity μ_T is not a fluid property, it is a property needed by the turbulence model.

The Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T\overline{\mathbf{D}}^R - \frac{2}{3}\rho k\mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho k\mathbf{I}$$

$$\overline{\mathbf{D}}^R = \frac{1}{2}\left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

$$k = \frac{1}{2}\overline{\mathbf{u}'\cdot\mathbf{u}'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is equivalent to the Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T \overline{\mathbf{D}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho k \mathbf{I}$$

This term represent normal stresses, therefore, is analogous to the pressure term that arises in the viscous stress tensor

- The term circled in the Boussinesq hypothesis, is added in order for the Boussinesq approximation to be valid when traced.
- That is, the trace of the right-hand side must be equal to the trace of the left-hand side,

$$-\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)^{\mathrm{tr}} = \tau_{ii} = -2\rho k$$

- Hence, it is consistent with the definition of turbulent kinetic energy

$$k = \frac{1}{2}\overline{\mathbf{u}'\cdot\mathbf{u}'} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The Boussinesq hypothesis

- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean strain rate tensor (therefore the mean velocity gradient), as follows,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T \overline{\mathbf{D}}^R - \frac{2}{3}\rho k \mathbf{I} = \mu_T \left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho k \mathbf{I}$$

This term represent normal stresses, therefore, is analogous to the pressure term that arises in the viscous stress tensor

- In order to evaluate the turbulent kinetic energy, usually a governing equation for k is derived and solved.
- Typically two-equations models include such an option, as we will see in Lecture 6.
- The term circled in the Boussinesq hypothesis can be ignored if there is no governing equation for k.
- Closures based on the Boussinesq approximation are known as eddy viscosity models (EVM).

The Boussinesq hypothesis

- As previously mentioned, the Boussinesq approximation lies in the belief that the Reynolds Stress tensor behaves in a similar fashion as the Newtonian viscous stress tensor.
- In spite of the theoretical weakness of the Boussinesq approximation, it does produce reasonable results for a large number of flows.
- The main disadvantage of the Boussinesq hypothesis as presented (linear model), is that it assumes that the turbulent eddy viscosity is an isotropic scalar quantity, which is not strictly true.
- Summary of shortcomings of the Boussinesq approximation,
    - Poor performance in flows with large extra strains, *e.g.*, curved surfaces, strong vorticity, swirling flows.
    - Rotating flows, *e.g.*, turbomachinery, wind turbines.
    - Highly anisotropic flows and flows with secondary motions, *e.g.*, fully developed flows in non-circular ducts.
    - Non-local equilibrium and flow separation, *e.g.*, airfoil in stall, dynamic stall.

- Many EVM models has been developed and improved along the years so they address the shortcomings of the Boussinesq approximation.
- EVM models are the cornerstone of turbulence modeling.

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- Recall the incompressible RANS equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}(\nabla p) + \nu \nabla^2 \bar{\mathbf{u}} + \frac{1}{\rho}\nabla \cdot \boldsymbol{\tau}^R \qquad \text{where} \qquad \boldsymbol{\tau}^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right)$$

- By using the Boussinesq approximation, we can write the governing equations as follows,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}\overbrace{(\mu + \mu_t)}^{\text{Effective viscosity}} \nabla \bar{\mathbf{u}}\right]$$

Turbulent viscosity

Normal stresses arising from the Boussinesq approximation

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- Or using index notation, the RANS equations can be written as follows,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j x_j} + \frac{1}{\rho}\frac{\partial \tau_{ij}^R}{\partial x_j}$$

- By using the Boussinesq approximation,

$$\tau_{ij}^R = -\rho \overline{u_i' u_j'} = 2\mu_t S_{ij} - \frac{2}{3} k \delta_{ij}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

- We can write the governing equations as follows,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0$$

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_i \bar{u}_j}{x_j} = -\frac{1}{\rho}\left[\frac{\left(\partial \bar{p} + \partial \frac{2}{3}\rho k\right)}{\partial x_i}\right] + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \mu_t\right)\frac{\partial \bar{u}_i}{\partial x_j}\right]$$

The Boussinesq hypothesis

Final touches to the incompressible RANS equations

- The problem now reduces to computing the turbulent eddy viscosity μ_T in the momentum equation.

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}(\mu + \mu_t)\nabla \bar{\mathbf{u}}\right]$$

- This can be done by using any of the models that we will study in Lecture 6.
  - Zero equation models.
  - One equation models.
  - Two equation models.
  - Three, four, five, … , equation models.
  - Reynolds stress models.
  - And so on.

The Boussinesq hypothesis

Relationship for the turbulent eddy viscosity

- In most turbulence models, a relationship for the turbulent eddy viscosity is derived using dimensional arguments (as we have seen so far and will study later).
    - This can be done by using any combination of dimensional groups, that is, velocity, length, time, etc. In the end, we should have viscosity units.
- This relationship can be corrected later or validated based on empirical and physical arguments, *e.g.*, asymptotic analysis, canonical solutions, analytical solutions, consistency with experimental measurements, and so on.
- It is also possible the use numerical arguments to correct, calibrate, and validate the relationship. To achieve this end, we rely on scale resolving simulations (most of the time DNS simulations).
- Regardless of the approach used, we see a recurring behavior. Specifically, eddy viscosity and length scale are all related on the basis of dimensional arguments.
- Historically, dimensional analysis has been one of the most powerful tools available for deducing and correlating properties of turbulent flows.
- However, we should always be aware that while dimensional analysis is extremely useful, it unveils nothing about the physics underlying its implied relationships.

Roadmap to Lecture 5

1. ~~Governing equations of fluid dynamics~~
2. ~~RANS equations – Reynolds averaging~~
3. ~~The Boussinesq hypothesis~~
4. **Sample turbulence models**

Sample turbulence models

RANS equations – EVM equations

- By using the Boussinesq approximation in the incompressible RANS equations, we obtain the following set of equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = -\frac{1}{\rho}\left(\nabla \bar{p} + \frac{2}{3}\rho \nabla k\right) + \nabla \cdot \left[\frac{1}{\rho}(\mu + \mu_t)\nabla \bar{\mathbf{u}}\right]$$

- The problem now reduces to computing the turbulent eddy viscosity in the momentum equation.
- Let us explore two closure models, the $k-\epsilon$ model and $k-\omega$ the model.

Sample turbulence models

$k-\epsilon$ Turbulence model governing equations

- It is called $k-\epsilon$ because it solves two additional equations for modeling the turbulence, namely, the turbulent kinetic energy k and the turbulence dissipation rate ϵ.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \epsilon + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}} - C_{\epsilon_2} \frac{\epsilon^2}{k} + \nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]$$

- With the following closure coefficients,

$$C_{\epsilon_1} = 1.44 \qquad C_{\epsilon_2} = 1.92 \qquad C_\mu = 0.09 \qquad \sigma_k = 1.0 \qquad \sigma_\epsilon = 1.3$$

- And closure relationships,

$$\nu_t = \frac{C_\mu k^2}{\epsilon} \qquad \omega = \frac{\epsilon}{C_\mu k} \qquad l = \frac{C_\mu k^{3/2}}{\epsilon}$$

Sample turbulence models

$k-\epsilon$ Turbulence model governing equations

- The closure equations correspond to the standard $k-\epsilon$ model.
- They have been manipulated so there are no terms including velocity fluctuations, besides the Reynolds stress tensor and the turbulence dissipation rate.
- The Reynolds stress tensor is modeled using the Boussinesq approximation.
- The turbulence dissipation rate is modeled using a second transport equation.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \overbrace{\tau^R : \nabla \bar{\mathbf{u}}}^{\text{Production}} - \underset{\text{Dissipation}}{\epsilon} + \overbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right]}^{\text{Diffusion}}$$

$$\nabla_t \epsilon + \nabla \cdot (\bar{\mathbf{u}} \epsilon) = C_{\epsilon_1} \underbrace{\frac{\epsilon}{k} \tau^R : \nabla \bar{\mathbf{u}}}_{\text{Production}} - \underset{\text{Dissipation}}{C_{\epsilon_2} \frac{\epsilon^2}{k}} + \underbrace{\nabla \cdot \left[\left(\nu + \frac{\nu_t}{\sigma_\epsilon} \right) \nabla \epsilon \right]}_{\text{Diffusion}}$$

Sample turbulence models

$k-\omega$ Turbulence model governing equations

- It is called $k-\omega$ because it solves two additional equations for modeling the turbulence, namely, the turbulent kinetic energy k and the turbulence specific dissipation rate ω.

$$\nabla_t k + \nabla \cdot (\bar{\mathbf{u}} k) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* k \omega + \nabla \cdot \left[(\nu + \sigma^* \nu_t) \nabla k \right]$$

$$\nabla_t \omega + \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \omega^2 + \nabla \cdot \left[(\nu + \sigma \nu_t) \nabla \omega \right]$$

- With the following closure coefficients,

$$\alpha = 5/9 \qquad \beta = 3/40 \qquad \beta^* = 9/100 \qquad \sigma = 1/2 \qquad \sigma^* = 1/2$$

- And closure relationships,

$$\nu_t = \frac{k}{\omega} \qquad \epsilon = \beta^* \omega k \qquad l = \frac{k^{1/2}}{\omega}$$

Sample turbulence models

$k-\omega$ Turbulence model governing equations

- The closure equations correspond to the Wilcox (1998) $k-\omega$ model.
- They have been manipulated so there are no terms including velocity fluctuations, besides the Reynolds stress tensor.
- The Reynolds stress tensor is modeled using the Boussinesq approximation.

$$\nabla_t k + \nabla\cdot(\bar{\mathbf{u}}k) = \overbrace{\tau^R : \nabla\bar{\mathbf{u}}}^{\text{Production}} - \overset{\text{Dissipation}}{\beta^* k\omega} + \overbrace{\nabla\cdot\left[\left(\nu+\sigma^*\nu_t\right)\nabla k\right]}^{\text{Diffusion}}$$

$$\nabla_t \omega + \nabla\cdot(\bar{\mathbf{u}}\omega) = \alpha\frac{\omega}{k}\underbrace{\tau^R : \nabla\bar{\mathbf{u}}}_{\text{Production}} - \underset{\text{Dissipation}}{\beta\omega^2} + \underbrace{\nabla\cdot\left[\left(\nu+\sigma\nu_t\right)\nabla\omega\right]}_{\text{Diffusion}}$$

Sample turbulence models

Final remarks

- The previous EVM models, are probably the most widely used ones.
- The standard $k - \epsilon$ is a high Reynolds number model (wall modeling), and the $k - \omega$ is wall resolving model (low Reynolds number).
- By inspecting the closure equations of the $k - \epsilon$ model, we can evidence that the turbulent kinetic energy and dissipation rate must go to zero at the correct rate in order to avoid turbulent viscosity production close to walls.
- Instead, the $k - \omega$ model does not suffer of this problem as the turbulence specific dissipation rate is proportional to $\omega \propto y^{-2}$. Therefore, the specific dissipation rate close to the walls is usually a large value.
- Remember, you need to define initial conditions and boundary conditions when using turbulence models.
- By the way, some models can be very sensitive to initial conditions.
- We addressed turbulence estimates in Lecture 4. We will revisit this subject in the next lectures.

Sample turbulence models

Short description of some RANS turbulence models

Model	Short description
Spalart-Allmaras	This is a one equation model. Suitable for external aerodynamics, tubomachinery and high speed flows. Good for mildly complex external/internal flows and boundary layer flows under pressure gradient (e.g. airfoils, wings, airplane fuselages, ship hulls). Performs poorly with flows with strong separation.
Standard k–epsilon	This is a two equation model. Very robust and widely used despite the known limitations of the model. Performs poorly for complex flows involving severe pressure gradient, separation, strong streamline curvature. Suitable for initial iterations, initial screening of alternative designs, and parametric studies. Can be only used with wall functions.
Realizable k–epsilon	This is a two equation model. Suitable for complex shear flows involving rapid strain, moderate swirl, vortices, and locally transitional flows (e.g. boundary layer separation, massive separation, and vortex shedding behind bluff bodies, stall in wide-angle diffusers, room ventilation). It overcome the limitations of the standard k-epsilon model.
Standard k–omega	This is a two equation model. Superior performance for wall-bounded boundary layer, free shear, and low Reynolds number flows compared to models from the k-epsilon family. Suitable for complex boundary layer flows under adverse pressure gradient and separation (external aerodynamics and turbomachinery).
SST k–omega	This is a two equation model. Offers similar benefits as the standard k–omega. Not overly sensitive to inlet boundary conditions like the standard k–omega. Provides more accurate prediction of flow separation than other RANS models. Can be used with and without wall functions. Probably the most widely used RANS model.