Turbulence and CFD models: Theory and applications

Roadmap to Lecture 3

1. Turbulence modeling – Scales of turbulence
   From Kolmogorov scales to Taylor microscales, to integral scales
2. Energy spectrum and energy cascade.
   Integral length scale and grid length scale
3. Turbulence near the wall - Law of the wall
4. A glimpse to a turbulence model

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Turbulence modeling – Scales of turbulence. From Kolmogorov scales to Taylor microscales, to integral scales.

- We are going to derive a few relations.
- Many of the derivations are based on dimensional analysis. So, I invite you to dust your notes.
- Let me remind you a few derived (and base) quantities that we will use.

Base quantity	Dimensional units	SI units
Length	L	m
Mass	M	kg
Time	T	s

Derived quantity	Symbol	Dimensional units	SI units
Velocity	-	LT^{-1}	m/s
Density	ρ	ML^{-3}	kg/m^3
Kinematic viscosity	ν	L^2T^{-1}	m^2/s
Dynamic viscosity	μ	$ML^{-1}T^{-1}$	kg/m-s
Energy dissipation rate per unit mass	ϵ	L^2T^{-3}	m^2/s^3
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Wavenumber	κ	L^{-1}	1/m
Energy spectral density per wavenumber	$E(\kappa)$	L^3/T^2	m^3/s^2

Turbulence modeling – Scales of turbulence

- Let us recall our definition of turbulence.
  - Unsteady, aperiodic motion in which all transported quantities fluctuate in space and time.
  - The instantaneous fluctuations are random both in space and time. Therefore, difficult to resolve.
  - Turbulent flows contains a wide range of eddy sizes (scales):
    - Large eddies derives their energy from the mean flow and are anisotropic. These eddies are unstable and they break-up into smaller eddies.
    - At the scales of the smallest eddies, the turbulent energy is dissipated. The behavior of the small eddies is more universal in nature.
    - In between the large and small eddies, there are some intermediate scales (that will not address for the moment).
    - The turbulent kinetic energy is transferred from the largest eddies to the smallest ones.
- So the question is, how can we determine those scales, in particular the smallest scales?
- Also, how the energy is transferred from the large eddies to the smallest ones? What is the mechanism?
- These questions and more are answered using Kolmogorov's universal equilibrium theory (K41).

Turbulence modeling – Scales of turbulence

- Kolmogorov's universal equilibrium theory (K41) was originally stated in the form of three hypothesis, which are summarized hereafter:
    - **Kolmogorov's hypothesis of local isotropy.** The small-scale turbulent motions are statistically isotropic.
    - **Kolmogorov's first similarity hypothesis.** The small-scale motions have a universal form that is uniquely determined by ν and ϵ.
    - **Kolmogorov's second similarity hypothesis.** The statistics of the motions between the large and small-scales have a universal form that is uniquely determined by ϵ, independent of ν.
- As a consequence of these hypotheses, the velocity and time scales decrease as the eddies decrease.
- Starting from here, let us derive the Kolmogorov scales and the energy spectrum relationship.
- For a complete description of the hypotheses, the interested reader should refer to reference [1].

[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Turbulence modeling – Scales of turbulence

- Summarizing the previous hypotheses, and according to Kolmogorov's universal equilibrium theory (K41), the motion at the smallest scales should depend only upon:
  - The rate at which the larger eddies supply energy,

$$\epsilon = -\frac{dk}{dt}$$

Turbulent kinetic energy

  - The kinematic viscosity ν.

- Having established that ϵ have the following dimensional units L^2T^{-3}, and ν have the following dimensional units L^2T^{-1}, we can derive the Kolmogorov's scales,

$$\eta \rightarrow \text{Length scale}$$
$$\tau \rightarrow \text{Time scale}$$
$$\upsilon \rightarrow \text{Velocity scale}$$

Turbulence modeling – Scales of turbulence

- By using dimensional analysis and the similarity hypotheses (and a lot of intuition and maybe good luck), Kolmogorov derived the following relations that determine the smallest scales in turbulence (Kolmogorov scales),

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$$

Length scale

$$\tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}$$

Time scale

$$\upsilon_\eta = (\nu\epsilon)^{1/4}$$

Velocity scale

- These scales are indicative of the smallest eddies, that is, the scales at which the energy is dissipated.
- Remember, turbulence is a continuum phenomenon; therefore, the Kolmogorov length scale is much larger than any molecular length scale.
- By the way, by simple inspection you can verify that the dimensional groups in the Kolmogorov scales all match.

Turbulence modeling – Scales of turbulence

- From the Kolmogorov scales, we can derive the following relations,

$$Re_\eta = \frac{\eta u_\eta}{\nu} = 1 \qquad \text{and} \qquad \epsilon = \nu \left(\frac{v_\eta}{\eta} \right)^2 = \frac{\nu}{\tau_\eta^2}$$

> - To arrive to this relation look at Kolmogorov's first similarity hypothesis.
> - Remember to check the dimensional groups

- Where Re_η is the Kolmogorov Reynolds number.
- For the dissipation rate ϵ, notice that the following relation provides a consistent characterization of the velocity gradients of the dissipative eddies,

$$\frac{u_\eta}{\eta} = \frac{1}{\tau_\eta}$$

- The fact that the Kolmogorov Reynolds number is equal to 1, is consistent with the notion that the energy cascade proceeds to smaller and smaller scales until the Reynolds number is small enough for dissipation due to viscosity to be effective.
- At the Kolmogorov Reynolds number, viscous effects dominate over convective effects. Therefore, these small eddies are dissipated at a rate ϵ.

Turbulence modeling – Scales of turbulence

- From the Kolmogorov turbulence scales, it can be seen that for large Reynolds number (turbulent flows) the length, time, and velocity scales of the smallest eddies are small compared to those of the largest eddies.
- Using the subscript $_0$ to denote the largest scales and the Kolmogorov scales, we can derive the following relations,

Largest eddies →

Smallest eddies →

$$\frac{l_0}{\eta} \sim Re_T^{3/4} \qquad \frac{\tau_0}{\tau_\eta} \sim Re_T^{1/2} \qquad \frac{u_0}{v_\eta} \sim Re_T^{1/4}$$

- Where the size of largest eddies l_0 is comparable to the flow scale (*i.e.*, it depends on the geometry), and their characteristic velocity u_0 is on the order of the mean flow.
- And Re_T is the turbulence Reynolds number or the Reynolds number related to the integral scales (which is about one or two orders of magnitude lower than the characteristic Reynolds number),

The Reynolds number of the largest eddies is on the order of the characteristic Reynolds number of the system, and as it is high (turbulent flow), the effects of viscosity are small.

$$Re_T = \frac{k^{1/2} l_0}{\nu} \quad \text{where} \quad k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

Turbulence modeling – Scales of turbulence

- In the relationship l_0/η, the length scale l_0 associated with the energy containing eddies ($l_0 >> \eta$), can be quantified with the turbulence model or using correlations (it is in the order of the system scale).
- For high Reynolds number (turbulent flows), dimensional analysis suggests, and measurements confirm, that k can be expressed in terms of ϵ and l_0 (turbulence length scale) as follows [1],

$$k \sim (\epsilon l_0)^{2/3} \implies \epsilon \sim \frac{k^{3/2}}{l_0} \implies l_0 \sim \frac{k^{3/2}}{\epsilon}$$

- Where k is the turbulent kinetic energy or TKE. We will talk more about TKE later.
- The TKE can be computed as follows. Notice that it is related to the velocity fluctuations and it is anisotropic (all components are different).

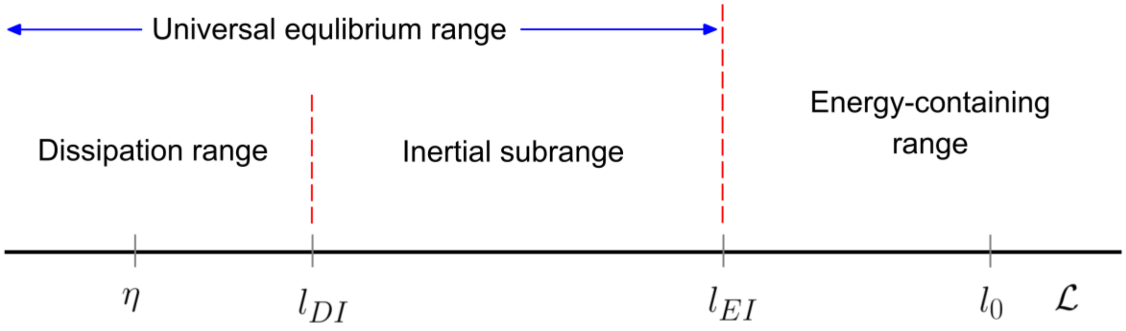
$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The overbar means time-average

[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.

Turbulence modeling – Scales of turbulence

- So far, we addressed the scales in the energy-containing range (large eddies) and the scales in the dissipation range (Kolmogorov).
- Between these two ranges there are many eddies that are too small to behave as integral length scales and too large to behave as Kolmogorov eddies.
- This range is known as the inertial range where the motion of the eddies are determined by inertial effects (viscous effects are negligible).
- In this range, the second Kolmogorov ‘s similarity hypothesis is valid.
- The eddies found in this range are characterized by the Taylor microscales.



- l_{EI} represents the end of the energy-containing range and the beginning of the inertial sub-range.
- l_{DI} represents the beginning of the dissipation range and the end of the inertial sub-range.
- The energy is transferred at a constant rate from l_{EI} to l_{DI} .
- 4/5 of the energy is contained in the energy-containing range.
- 1/5 of the energy is contained in the inertial subrange.
- The extreme l_{EI} is characterized by TKE.
- The extreme l_{DI} is characterized by dissipation.
- Taylor scales can be seen as hybrid eddies.

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Turbulence modeling – Scales of turbulence

- Let us define the Taylor microscales.

$$\lambda = \left(\frac{10 \nu k}{\epsilon} \right)^{1/2}$$

Length scale

$$\tau_\lambda = \left(\frac{15 \nu}{\epsilon} \right)^{1/2}$$

Time scale

$$u_\lambda = \frac{\lambda}{\tau_\lambda}$$

Velocity scale

- Remember, these scales are contained between the integral scales and the Kolmogorov scales (inertial subrange).
- The Taylor microscales ratio between the largest scales and smallest scales can be computed as follows,

$$\frac{\lambda}{l_0} = \sqrt{10} Re_T^{-1/2}$$

$$\frac{\lambda}{\eta} = \sqrt{10} Re_T^{1/4}$$

Recall that,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu}$$

Note: the constant values might change from author to author, nevertheless, they are close.

Turbulence modeling – Scales of turbulence

- Because the Taylor microscale is generally too small to characterize large eddies and too large to characterize small eddies, it has generally been ignored in most turbulence modeling research [1].
- The Taylor scale Reynolds number Re_λ is defined follows,

$$Re_\lambda = \frac{u'\lambda}{\nu} \qquad \text{where} \qquad k = \frac{2}{3}u'^2$$

- The Taylor scale Reynolds number is related to the turbulent Reynolds number as follows,

$$Re_\lambda = \left(\frac{20}{3}Re_T\right)^{1/2}$$

- We can also relate the timescale of Taylor microscales to the Kolmogorov scales,

$$\frac{\lambda}{u'} = \left(\frac{15\nu}{\epsilon}\right)^{1/2} = \sqrt{15}\tau_\eta \qquad \text{where} \qquad k = \frac{2}{3}u'^2$$

[1] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

Turbulence modeling – Scales of turbulence

- Some additional relationships,

$$\epsilon = \frac{15\nu u'^2}{\lambda^2}$$

$$\lambda = \left(\frac{10\nu k}{\epsilon}\right)^{1/2} = \sqrt{10}\eta^{2/3} l_0^{1/3}$$

$$u_\lambda = \frac{\lambda}{\tau_\lambda} = \left[\left(\frac{10k\nu}{\epsilon}\right)\left(\frac{\epsilon}{15\nu}\right)\right]^{1/2} = \left(\frac{2k}{3}\right)^{1/2}$$

Where,

$$k = \frac{2}{3}u'^2$$

- It might appear that the previous Taylor relations were pulled out of thin air, we will not go into details on the derivation, because they are not used very often.
- In any case, the interested reader should refer to references [1, 2, 3, 4]. Remember to always check the dimensions.
- From time to time, you will find the Taylor scales used to characterize grid turbulence.

References:
[1] S. Pope. Turbulent Flows, Cambridge University Press, 2000.
[2] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.
[3] P. Davidson. Turbulence. An Introduction for Scientists and Engineers. Oxford University Press, 2015
[4] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.

Turbulence modeling – Implications of scales

- The previous relationships indicate that the range of turbulent scales may span orders of magnitude for high Reynolds number flows.
- Since the Kolmogorov length scale η is much smaller that the large or integral scales (*i.e.,* wing chord, channel height, blockage ratio) associated with the flow of interest, it is easy to see that in numerical simulations a large amount of grid points/cells would be required to fully simulate turbulent flows.
- For example, in a direct numerical simulation (DNS),where all scales of turbulence are resolved, the relationship,

$$\frac{l_0}{\eta} \sim Re_T^{3/4}$$

- Implies that the number of grid points/cells in one direction is directly proportional to,

$$Re_T^{3/4}$$

- Thus, in a DNS simulation the meshing requirements scales proportional to $Re_T^{9/4}$ (to resolve all dimensions), or approximately proportional to Re_T^{3} for a single time step.

Turbulence modeling – Implications of scales

- And the number of time steps N_t needed to satisfy a Courant condition of CFL < 1, is on the order of,

Total duration of the simulation

$$N_t \sim \frac{T}{\Delta t} \sim \frac{T}{\eta/u} \sim \frac{T}{l_0/u} Re^{3/4}$$

$$\Delta t \sim \Delta x/u \sim \eta/u$$

Recall that,

$$Re_T = \frac{k^{1/2} l_0}{\nu} = \frac{k^2}{\epsilon \nu}$$

- That is, the number of time steps is in the order of $\mathcal{O}(Re_T^{3/4})$
- And the number of operations required for a DNS simulation is approximately proportional to $N_{xyz}^3 N_t$ (where N_{xyz} is the number of grid points/cells in every dimension).
- So the computing time scales proportional to Re_T^3.
- Again, these numbers are huge. Even with todays most advanced supercomputers.
- And this is without considering the energy consumption costs.
- The previous are just estimates that are not written in cement, but they give a good idea of the computational cost of resolving all turbulent scales.

Turbulence modeling – Turbulent kinetic energy

- So far, we used here and there the turbulent kinetic energy k or TKE, without given a formal definition.
- TKE is the kinetic energy per unit mass of the fluctuating turbulent velocity (u', v', w'). Recall that the dimensional units of TKE are L^2T^{-2}.
- TKE is associated with the eddies in turbulent flows and can be computed as follows:

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

The overbar means time-average

- For high Reynolds number (turbulent flows), dimensional analysis suggests, and measurements confirm, that k can be expressed in terms of ϵ and l_0 (turbulence length scale) as follows [1],

$$k \sim (\epsilon l_0)^{2/3} \implies \epsilon \sim \frac{k^{3/2}}{l_0}$$

- Where l_0 is a length scale associated with the energy containing eddies ($l_0 >> \eta$), or integral length scales, which can be quantified with the turbulence model or using correlations.

[1] G. I. Taylor. Statistical theory of turbulence. Proceedings of the Royal Society of London. 1935.

Turbulence modeling – Turbulent kinetic energy

- If you are wondering the origin of the TKE equation, recall the equation of the kinetic energy of an object or moving material volume,

$$K = \frac{1}{2} m v^2$$

- If you compute K for the fluctuating velocity and divide by the mass (per unit mass), you obtain the previous TKE equation, that is,

$$k = \frac{1}{2} \left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right)$$

- As we will see later, TKE can also be calculated from the turbulence model or from the Reynolds stress tensor.

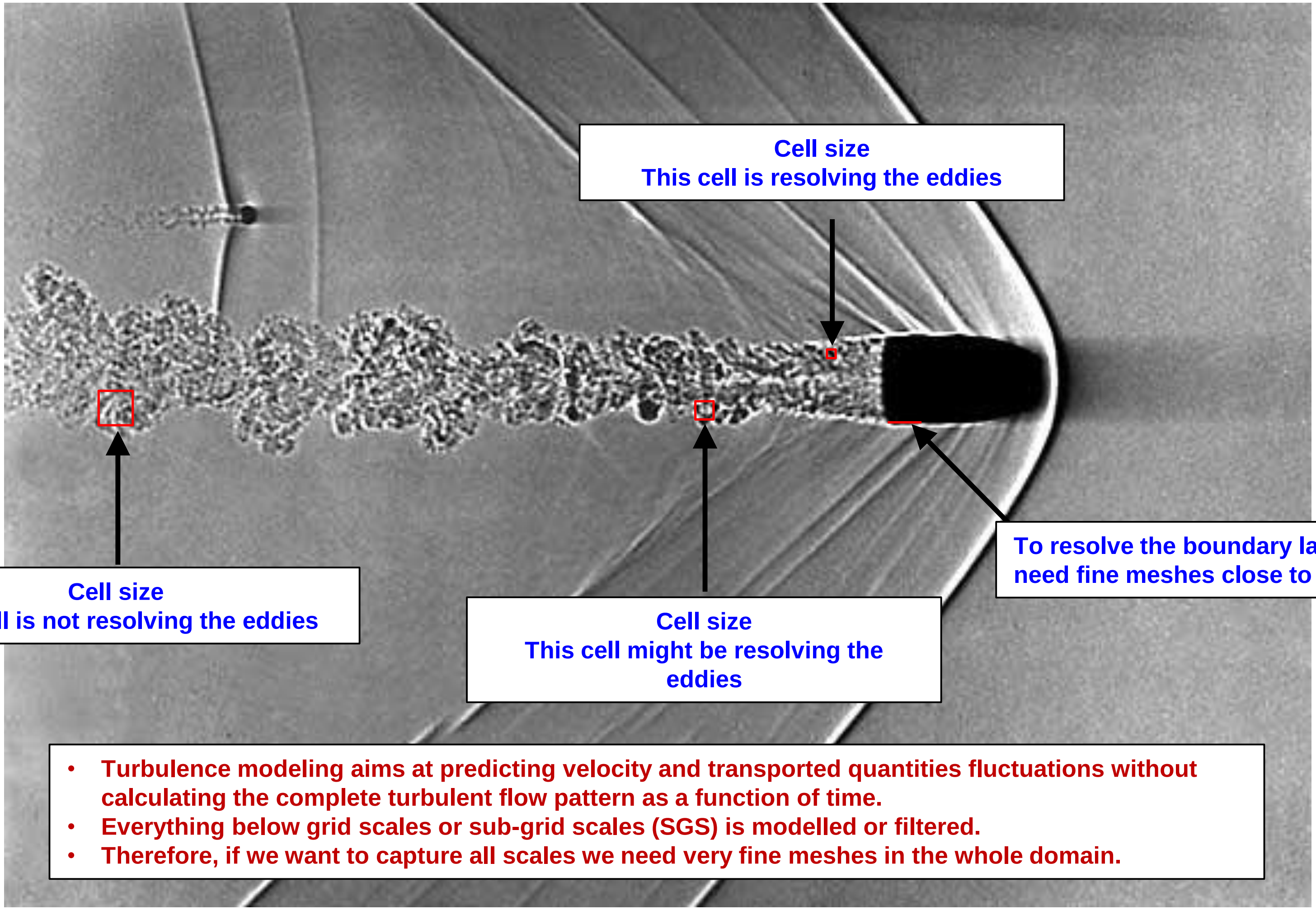
Turbulence modeling – Grid requirements

- The grid requirements of turbulence modeling in CFD can be summarize as follows:
  - DNS simulations requires no modeling, but it demands resolution from the large scales all the way through at least the beginning of the dissipation scales. This results in a grid scaling proportional to Re_T^3, or worse.
  - LES simulations requires modeling of part of the inertial sub-range and into the beginning of the dissipation scales. The amount of required modeling is set by the grid resolution but is unlikely that the grid will scale worse than Re_T^2.
    - Even if this requirement appears to be high, LES simulation are starting to become affordable in modern supercomputers.
  - RANS simulations requires modeling of everything from the integral scales into the dissipation range. As a consequence, the grid scaling is a weak function of the Reynolds number.

Turbulence modeling – Grid requirements

- As you can see from the previous requirements, RANS simulations are very affordable (steady and unsteady).
- RANS simulations are the workhorse of turbulence modeling in industrial applications.
    - Steady state RANS simulations will remain the dominant simulation method for turbulent flows for many years.
- RANS models are accurate, robust, fast, mesh insensitive, and valid for a wide range of physics.
- Nevertheless, the use of scale resolving simulations (SRS) for industrial applications is predicted.
    - It is expected that LES and DES will become more affordable in the years to come.
- DNS is only used in research and for low Reynolds number.
    - DNS simulations are used to calibrate RANS models.

Turbulence modeling – Grid requirements



- Turbulence modeling aims at predicting velocity and transported quantities fluctuations without calculating the complete turbulent flow pattern as a function of time.
- Everything below grid scales or sub-grid scales (SGS) is modelled or filtered.
- Therefore, if we want to capture all scales we need very fine meshes in the whole domain.

Bullet at Mach 1.5

Photo credit: Andrew Davidhazy. Rochester Institute of Technology.

Turbulence modeling – Scales of turbulence. From Kolmogorov scales to Taylor microscales, to integral scales.

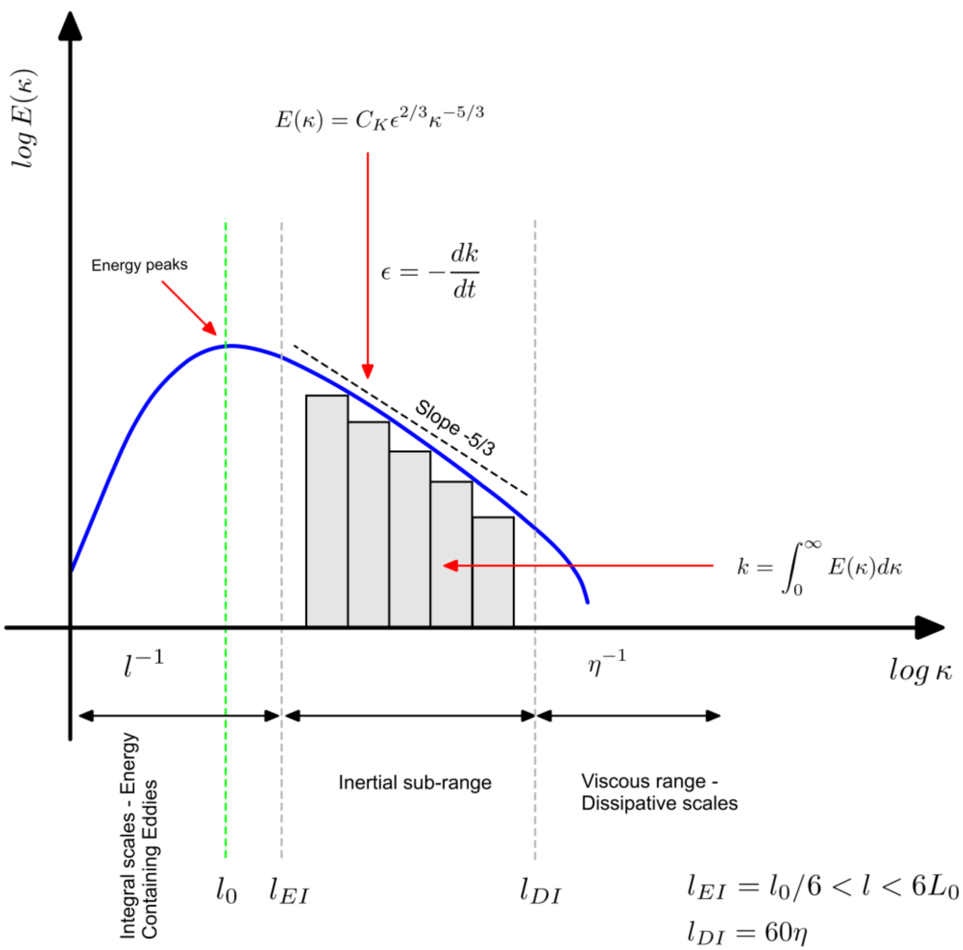
- As you can see, resolving all turbulence scales in CFD (in space and time) requires a formidable amount of computational power and user time.
- To this, you need to add the IO overhead, and the qualitative and quantitative post-processing, which can be as expensive as the simulations.
- Therefore, the importance of using turbulence models to alleviate the incredible requirements of resolving all turbulence scales.

Roadmap to Lecture 3

1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales, to integral scales~~
2. **Energy spectrum and energy cascade. Integral length scale and grid length scale**
3. Turbulence near the wall - Law of the wall
4. A glimpse to a turbulence model

Energy spectrum and energy cascade. Integral length scale and grid length scale

Energy spectrum and energy cascade

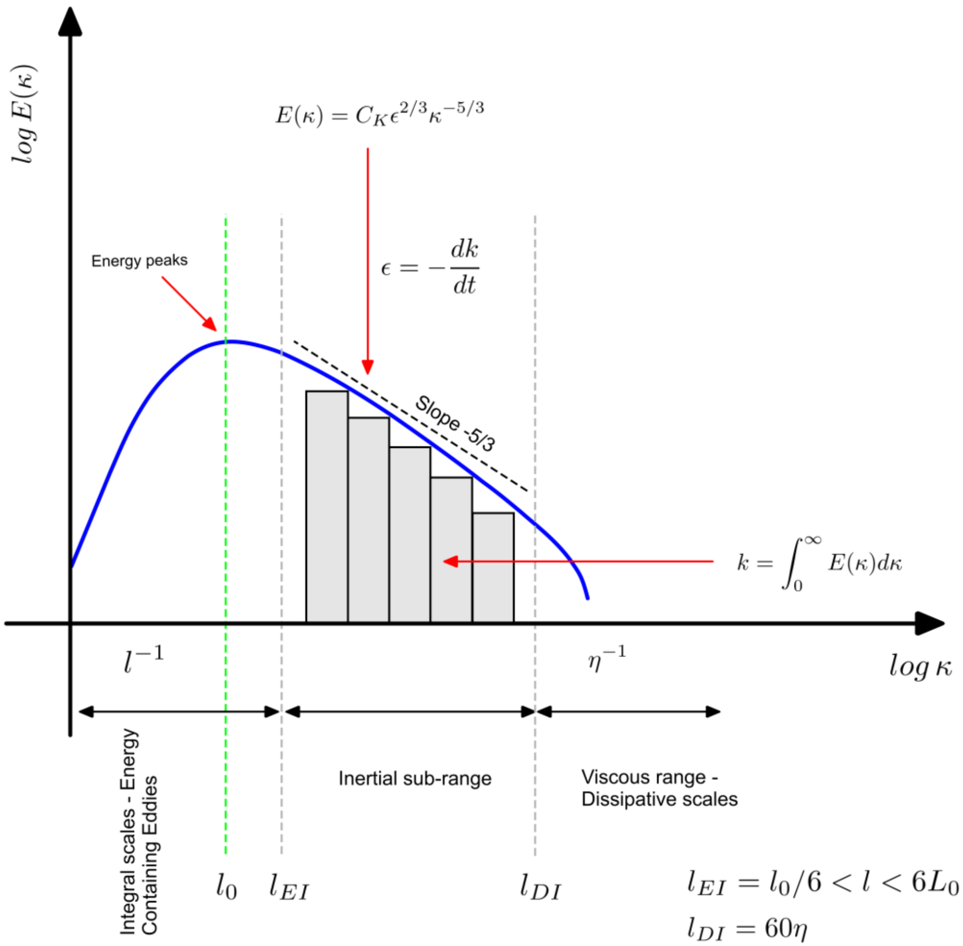


- The existence of a wide separation of scales is a central assumption Kolmogorov made as part of his universal equilibrium theory.
- Kolmogorov hypothesized that there is a range of eddy sizes between the largest and smallest for which the cascade process is independent of the statistics of the energy containing eddies and of the effect of viscosity.
- As a consequence, a range of wavenumbers exists in which the energy transferred by inertial effects, wherefore $E(\kappa)$ depends only upon ϵ and κ.
- On dimensional grounds, Kolmogorov concluded that,

$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- These relationship is sometimes known as the Kolmogorov -5/3 law.

Energy spectrum and energy cascade

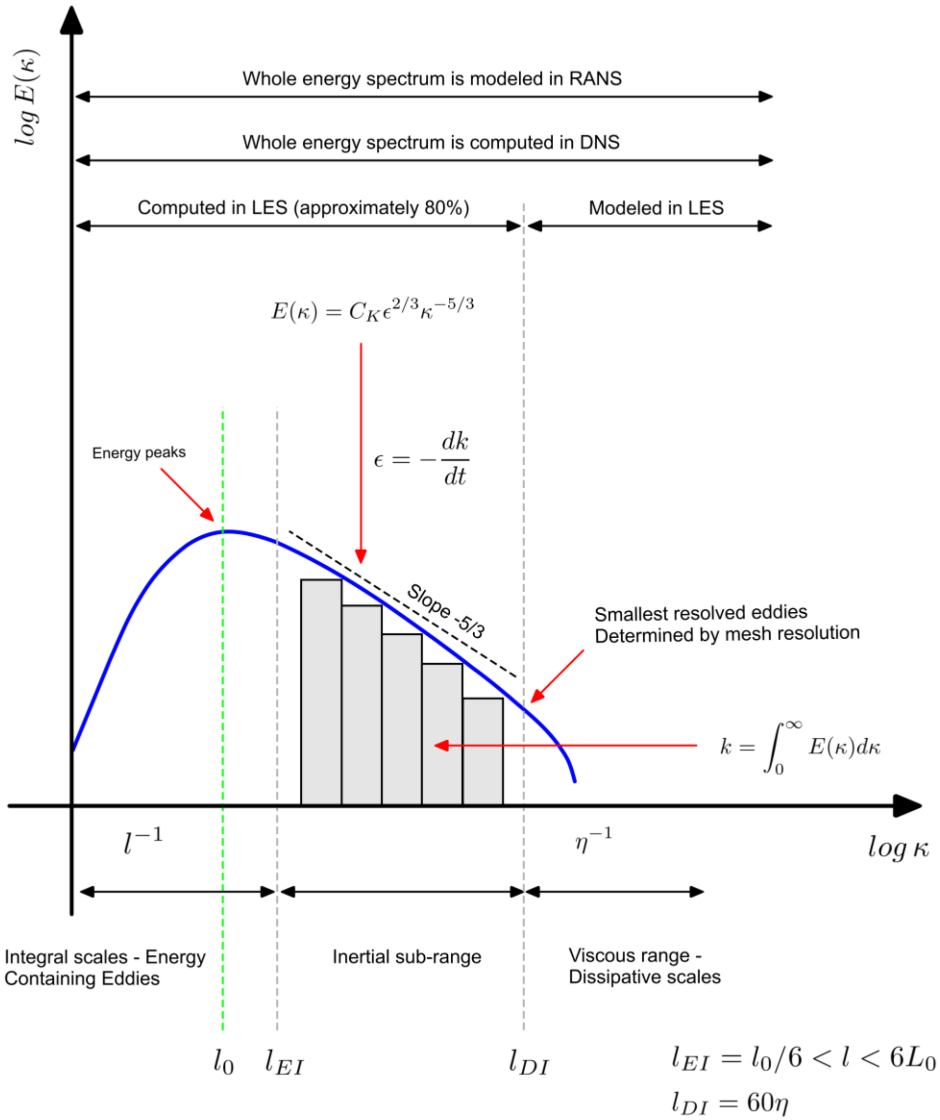


$$E(\kappa) = C_K \epsilon^{2/3} \kappa^{-5/3}$$

- The energy spectrum equation $E(\kappa)$ is fundamental in turbulence modeling.
- In this equation, C_K is the Kolmogorov constant, which has been found to be between 0.4 and 0.6 [1, 2].
- The exponent of κ (or the wavenumber) in $E(\kappa)$, represents the slope of the inertial sub-range (-5/3).
- The existence of this inertia sub-range has been verified by many experiments and numerical simulations [1, 2].
- Kolmogorov derived the equation of $E(\kappa)$ using dimensional analysis and good intuition.
- Have in mind that different relations could be derived using different hypothesis.

[1] K. R. Sreenivasan. On the universality of the Kolmogorov constant. Physics of Fluids 7, 2778 (1995).
[2] P. K. Yeung. On the universality of the Kolmogorov constant in numerical simulations of turbulence. ICASE Report No. 97-64, 1997.

Energy spectrum and energy cascade



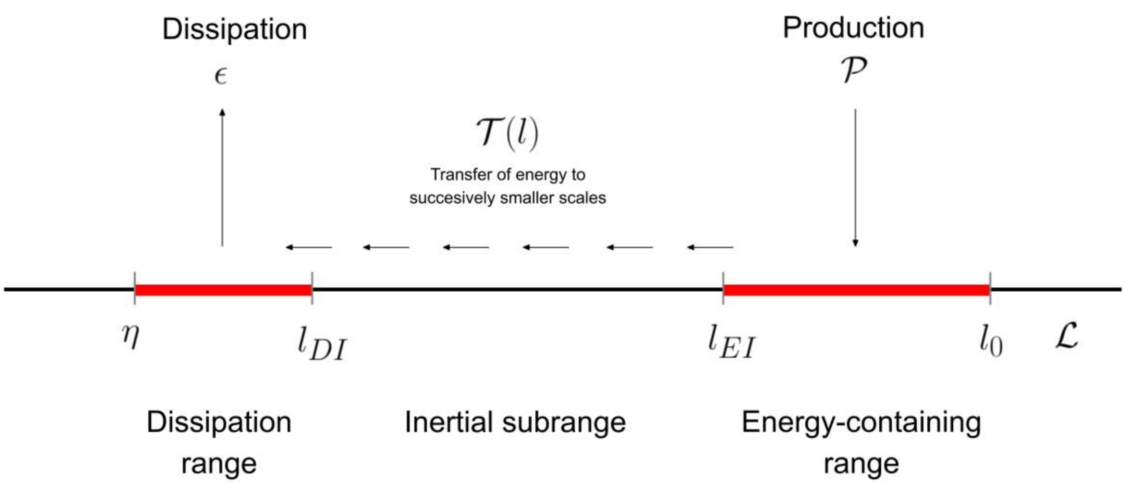
- Notice that this kind of graph is local.
- It will be different for each and every point in the domain.
- In the x axis the wave number is plotted,

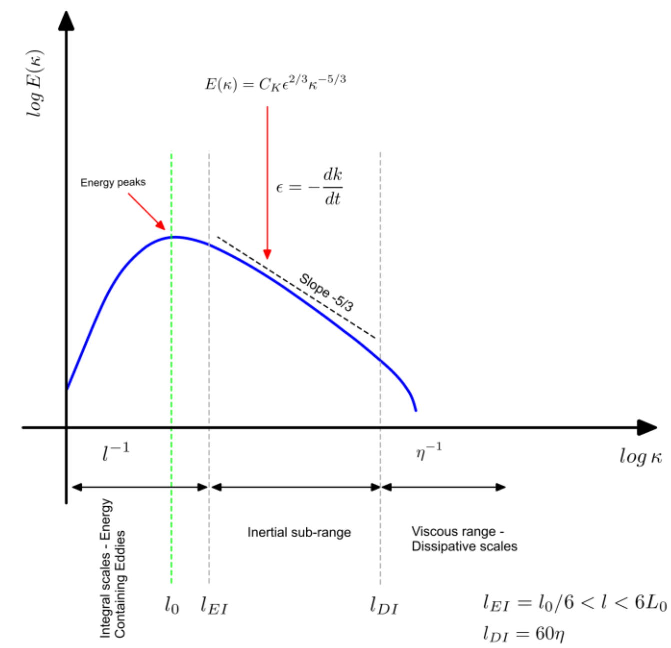
$$\kappa = \frac{2\pi}{l} = \frac{2\pi f}{U}$$

- The energy spectrum density or energy spectrum, $E(\kappa)$ is related to the Fourier transform of k.
- The turbulent power spectrum represents the distribution of the turbulent kinetic energy k across the various length scales.
- It is a direct indication of how energy is dissipated with eddies size.
- The mesh resolution determines the fraction of the energy spectrum directly resolved.
- Remember, in CFD eddies cannot be resolved down to the molecular dissipation limit.

Energy spectrum and energy cascade

$$\mathcal{T}_{EI} \equiv \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \epsilon$$



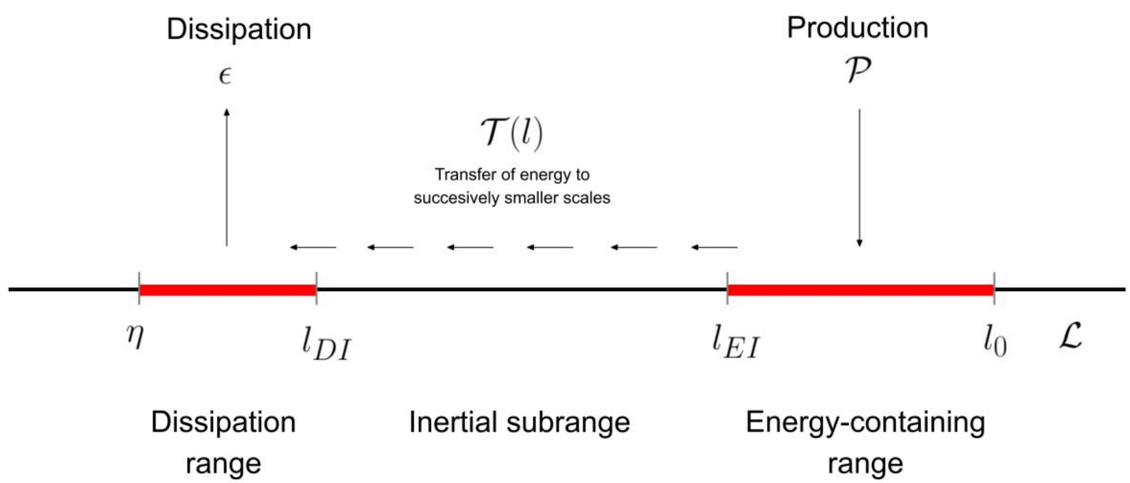


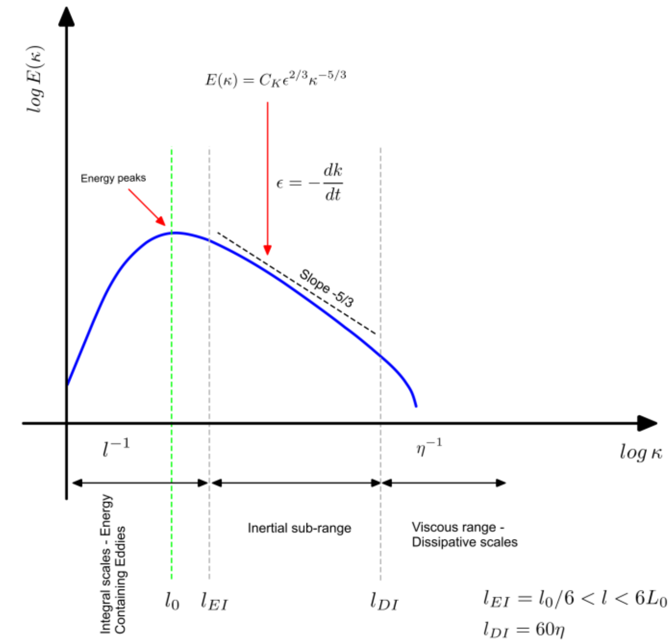
- l_{EI} represents the end of the energy-containing range and the beginning of the inertial sub-range.
- l_{DI} represents the beginning of the dissipation range and the end of the inertial sub-range.
- Under equilibrium conditions, turbulence production is equal to turbulence dissipation.
- The energy is transferred at a constant rate from l_{EI} to l_{DI} .
- The extension of the -5/3 law (the inertial sub-range region) is larger for large Reynolds number.
- At low Reynolds numbers, it is difficult to distinct between l_{EI} and l_{DI}.

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and energy cascade

$$\mathcal{T}_{EI} \equiv \mathcal{T}(l_{EI}) = \mathcal{T}(l) = \mathcal{T}_{DI} = \mathcal{T}(l_{DI}) = \epsilon$$



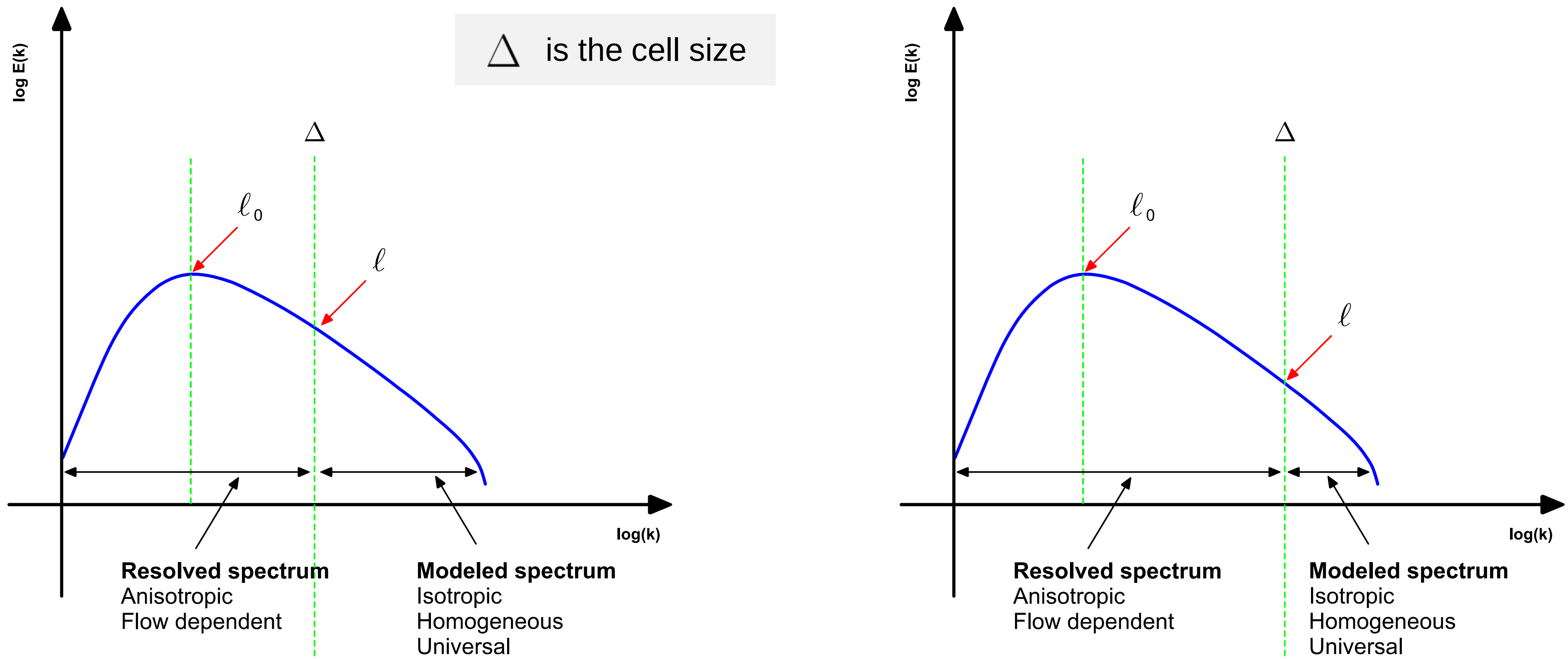


- Note that dissipation takes place at the end of the sequence of this process (l_{DI}).
- The rate of dissipation ϵ is determined, therefore, by the first process in the sequence (l_{EI}), which is the transfer of energy from the largest eddies.
- The energy dissipation rate per unit mass ϵ is given by the following equation, which comes from the transport equation of the turbulent kinetic energy (which we will derived in Lectures 5 and 6),

$$\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k}}$$

Figure adapted from: S. Pope. Turbulent Flows, Cambridge University Press, 2000.

Energy spectrum and energy cascade



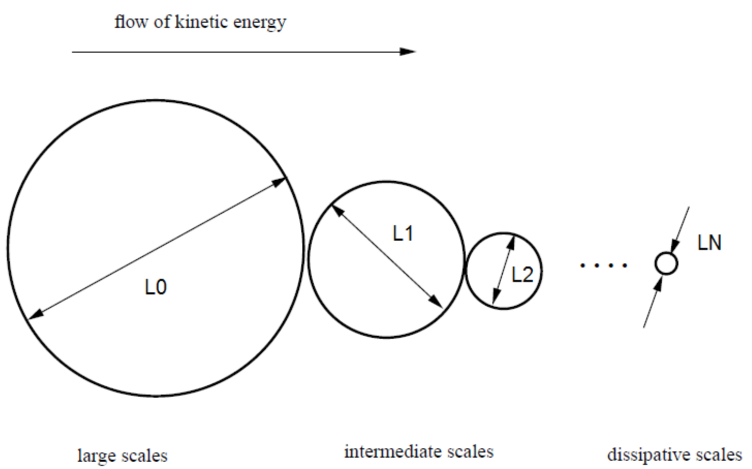
Coarse mesh – More energy is modeled

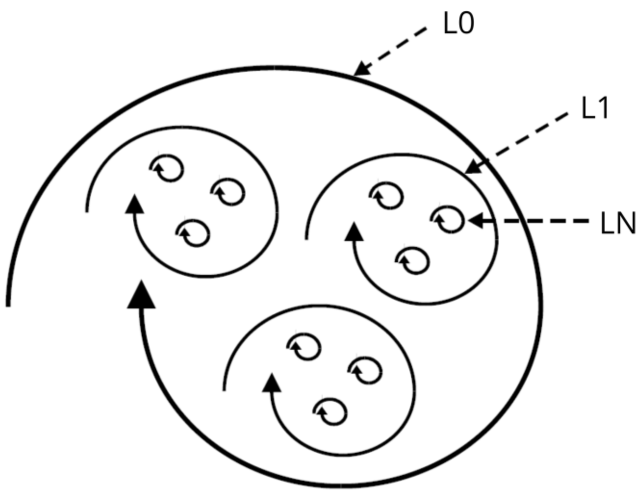
Fine mesh – Less energy is modeled

- The finer the mesh the less energy that is being modeled.
- Turbulent kinetic energy peaks at integral length scale l_0.
- In SRS simulations, this scale must be sufficiently resolved.

Energy spectrum and energy cascade. Integral length scale and grid length scale

Energy spectrum and energy cascade





- The energy-containing eddies are denoted by L0.
- L1 and L2 denotes the size of the eddies in the inertial sub-range such that L2 < L1 < L0.
- LN is the size of the dissipative eddies.
- The large, energy containing eddies transfer energy to smaller eddies via vortex stretching.
- Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.
- Large eddies derive their energy from the mean flow.
- The size and velocity of large eddies are on the order of the mean flow.
- Large eddies (L0) are anisotropic; whereas, small eddies (LN) are isotropic.
- The eddies in the inertial sub-range become more isotropic as energy is transferred from large eddies to small (dissipative) eddies.

References:
S. Pope. Turbulent Flows. Cambridge University Press. 2014.

Energy spectrum and energy cascade

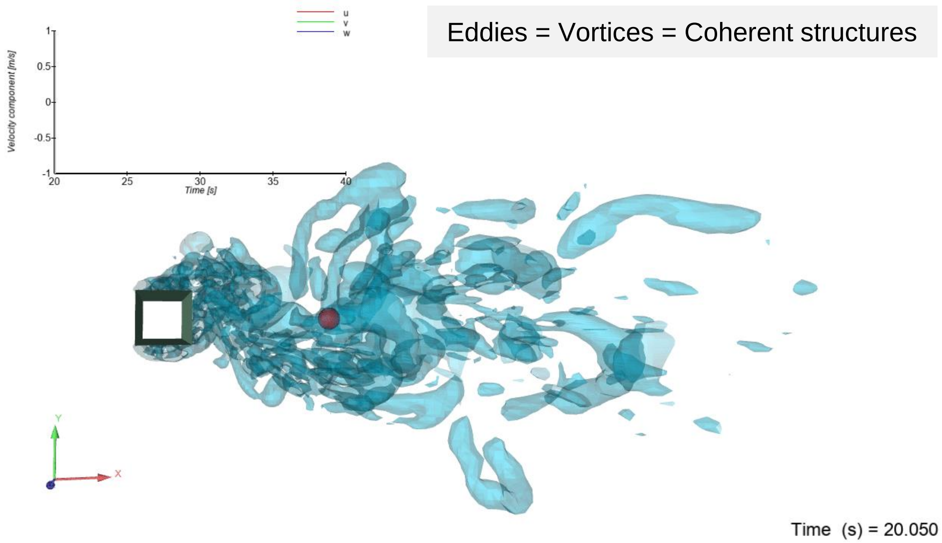
- Energy spectrum and Richardson's rhyme [1],

> *"Big whorls have little whorls,*
> *which feed on their velocity,*
> *and little whorls have lesser whorls,*
> *and so on to viscosity"*

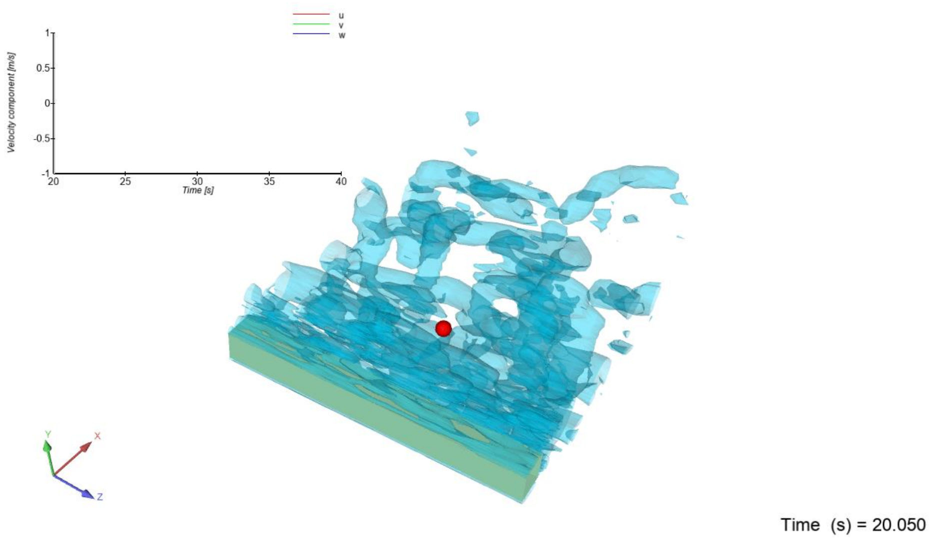
[1] L. F. Richardson. "Weather Prediction by Numerical Process". Cambridge University Press, 1922.

Energy cascade in action

Eddies = Vortices = Coherent structures



http://www.wolfdynamics.com/training/turbulence/image19.gif



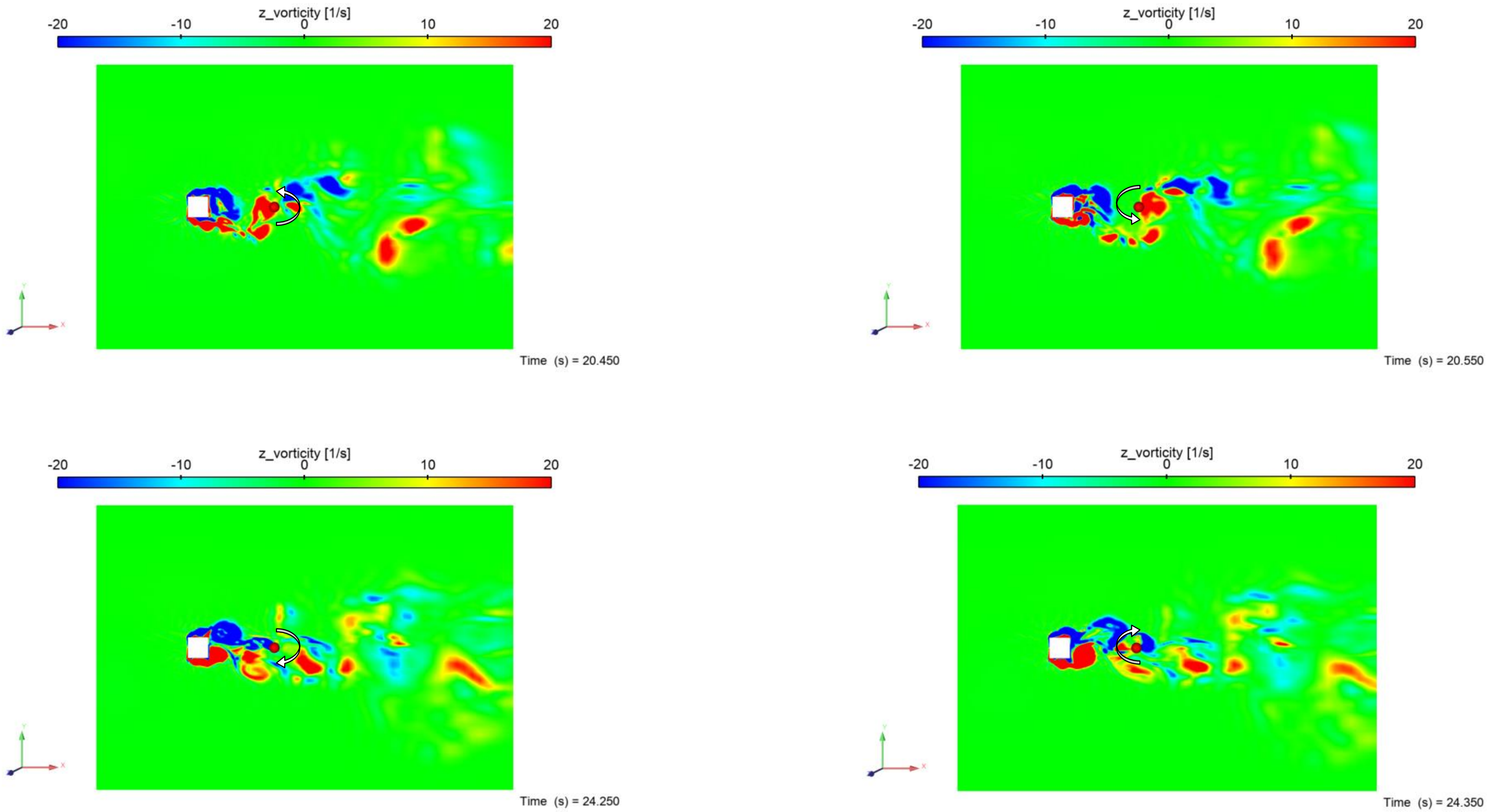
http://www.wolfdynamics.com/training/turbulence/image20.gif

- To plot the energy spectrum, we need to sample the velocity field in a location behind the wake (the red sphere in this case).
- Then, by using signal processing (FFT), we can convert physical space into frequency space.
- We will address this type of post-processing in Lectures 7 and 8.

- **From these animations, we can notice that:**
- Large eddies:
  - They are energy-containing and extract their energy from the mean flow.
  - Their velocity is on the other of the mean flow.
  - Their size is on the order of the mean flow or the obstacle the flow over.
  - They are anisotropic and unstable. They break-up into smaller eddies.
  - Their frequency is low compare the small eddies.

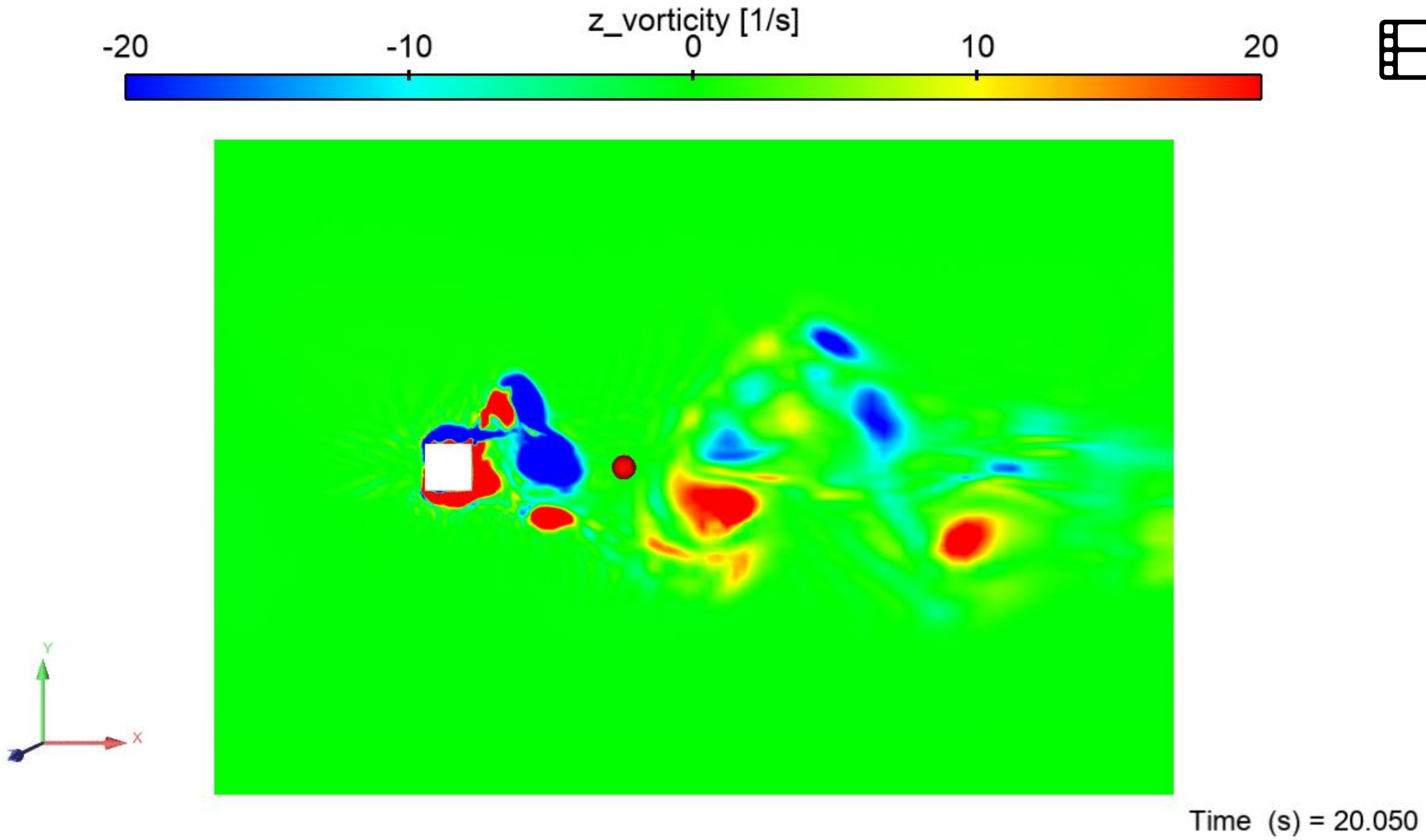
- Small eddies:
  - Smallest eddies convert kinetic energy into thermal energy via viscous dissipation.
  - Their behavior is more universal in nature.
  - Their frequency is high.

Energy cascade in action



- All the information passing over the probe (sampling location) can be used to compute the energy spectrum (and more).

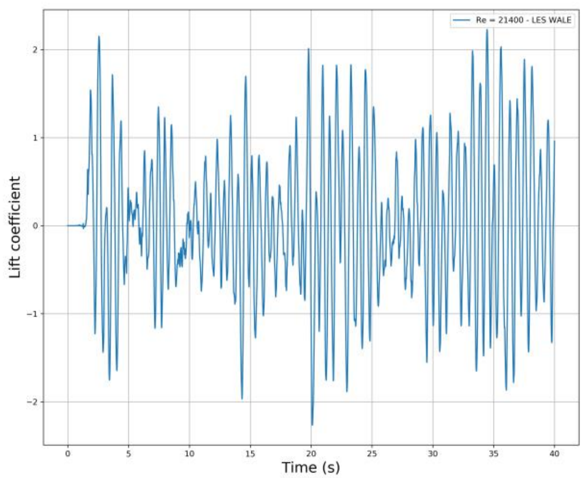
Energy cascade in action

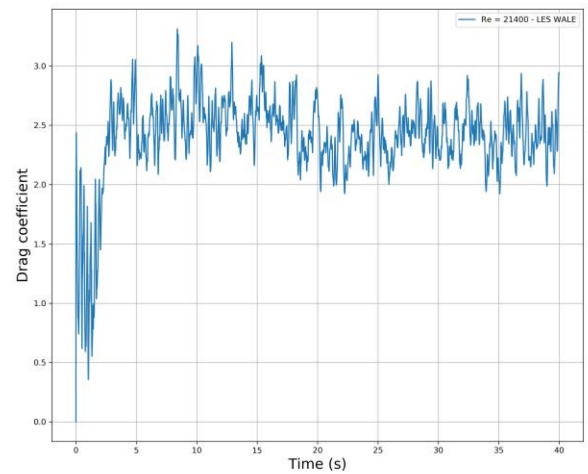


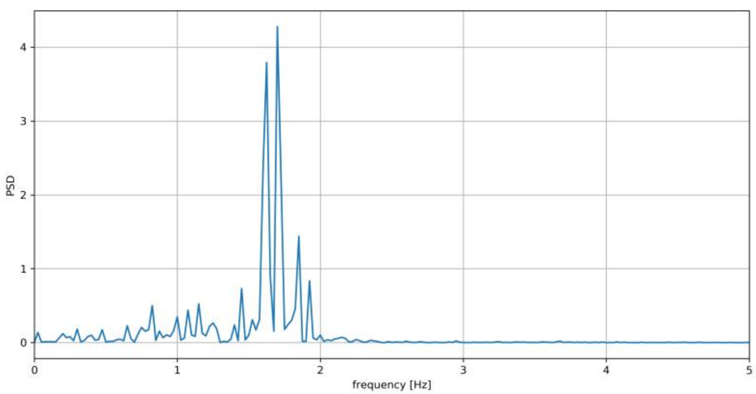
Vorticity contours – The velocity field is being sampled at the red sphere.

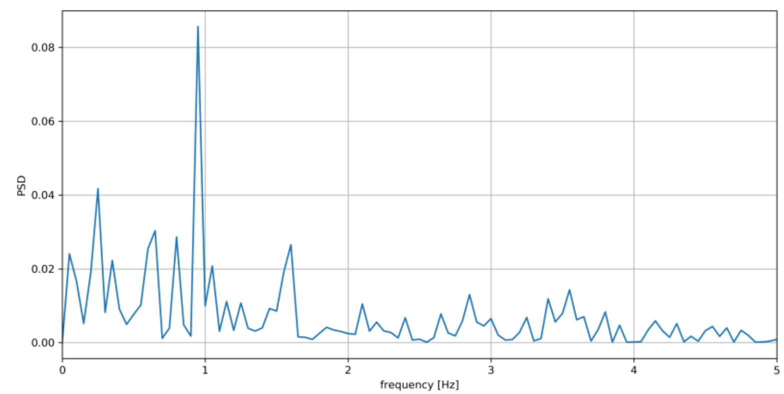
http://www.wolfdynamics.com/training/turbulence/image21.gif

Energy cascade in action



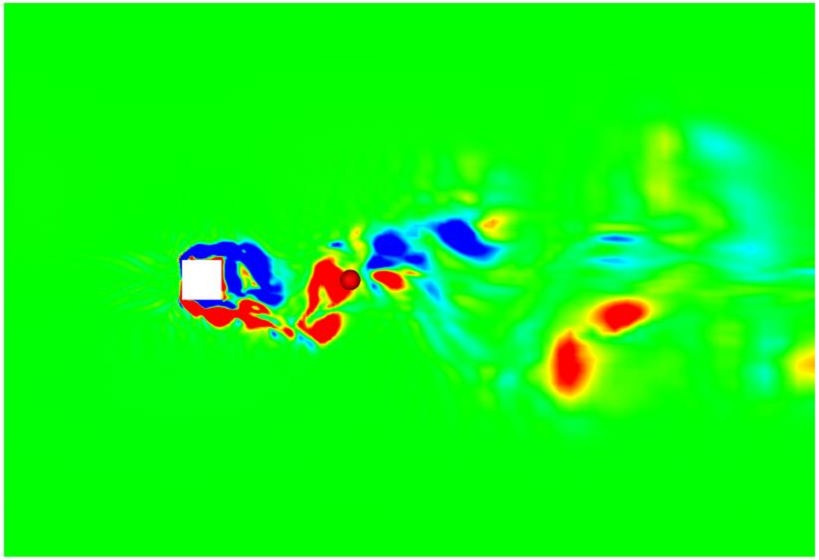


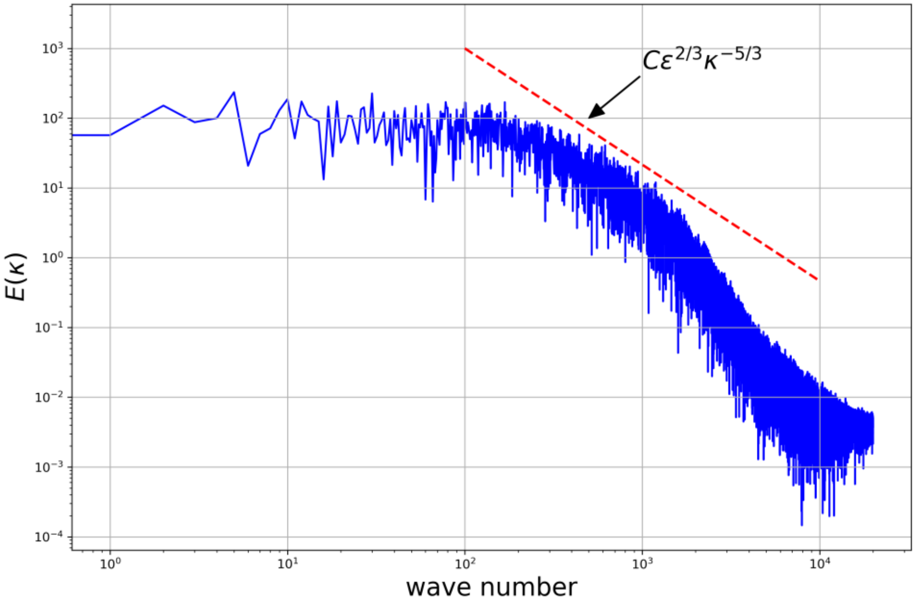




- Plot of the lift and drag coefficient signals (top figures).
- The signal, can be used to compute a dominant frequency (bottom figures).
- Remember, when computing the descriptive statistics you should not consider the initial transient.

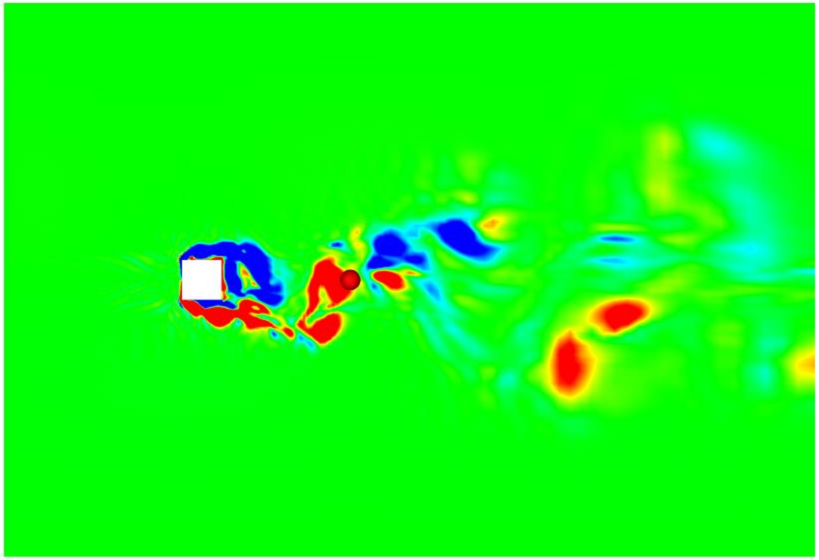
Energy cascade in action

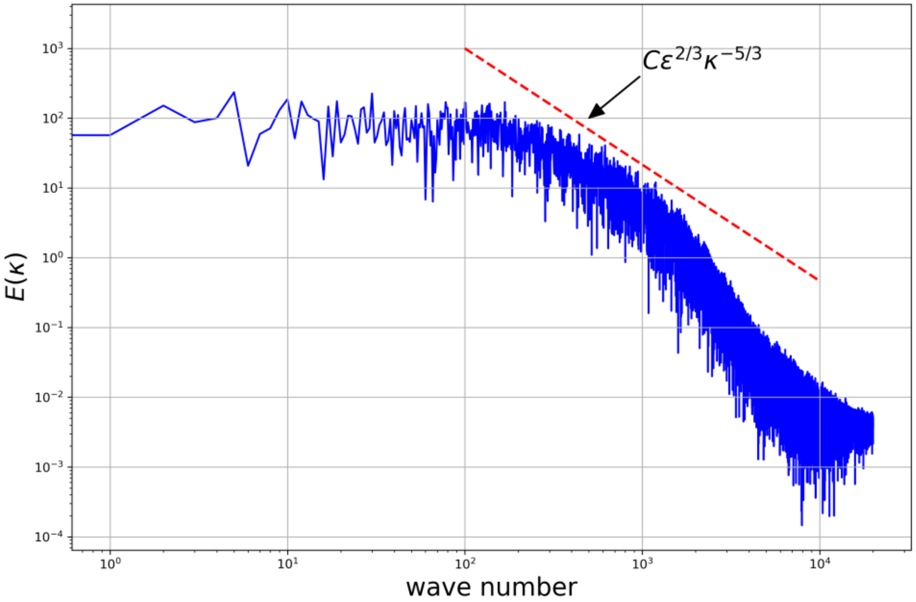




- Plot of the energy spectrum (right figure).
- Remember, this plot is local. To obtain this plot, you will need to measure the velocity fluctuations at a given location, in this case, the red sphere in the left figure.
- The energy spectrum (right figure) represents the distribution of the TKE across the various wavenumbers. The wavenumber κ is proportional to the inverse of the eddy size l.
- The plot shows that the TKE peaks at largest scales (or small wavenumbers). Then, TKE rapidly decreases as the eddy sizes are smaller (large wavenumbers). And in the end, TKE is dissipated at the smallest scales. This is the energy cascade.
- In the plot, the inertial sub-range is compared against the -5/3 law (red line with a slope of -5/3).

Energy cascade in action



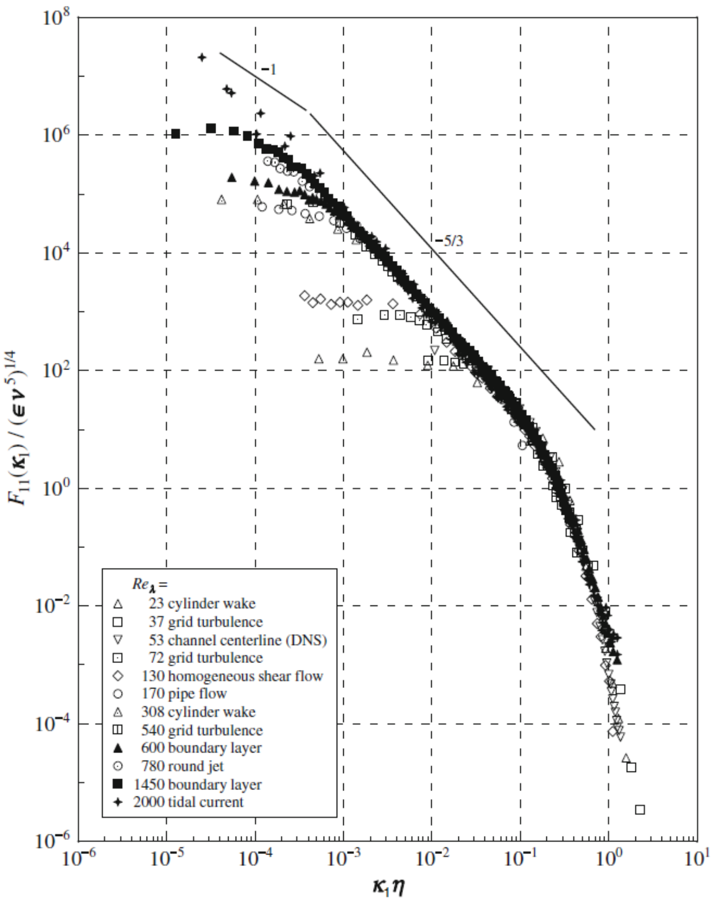


- Large wavenumbers, indicates small eddies with large frequency. Conversely, small wavenumbers are an indication of large eddies with low frequency.
- Have in mind that the relationship $l \propto k^{-1}$ should be treated as an order of magnitude approximation.
- In this plot, it is difficult to distinguish between k^{-1} and $2\pi k^{-1}$ (the wavelength of the Fourier component).
- In the energy spectrum plot, the energy should not accumulate or increase at large wavenumbers. If this happens, it is an indication that the mesh is too coarse or there are issues with the turbulence model.
- As a general note, in typical engineering applications the smallest eddies are about 0.1 to 0.01 mm and have frequencies in the order of 10 kHz.

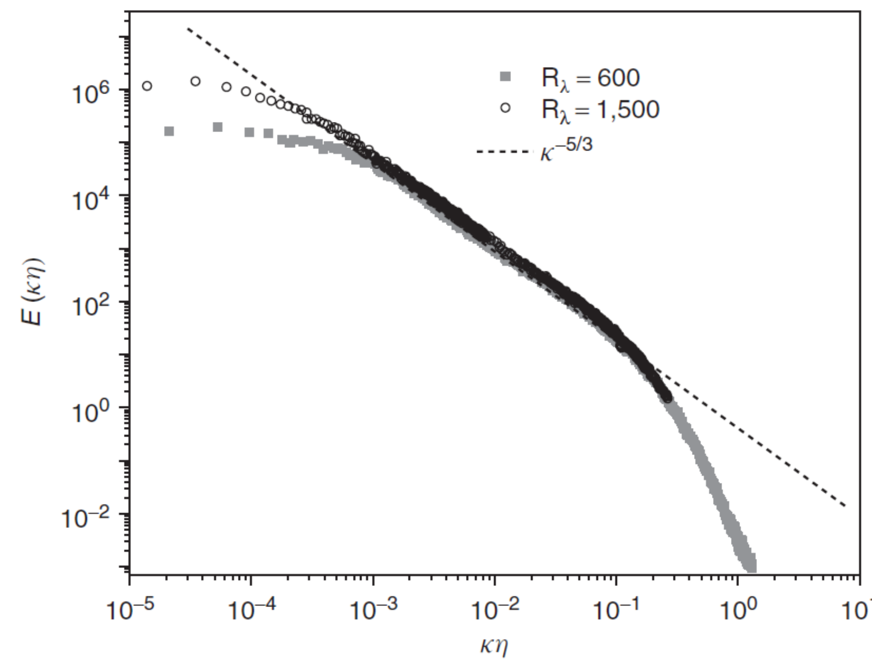
Validity of the Kolmogorov theory (K41)

- Kolmogorov’s theory is an asymptotic theory, it has been shown to work well in the limit of very high Reynolds numbers.
- Kolmogorov’s theory assumes that the energy cascade is one way, from large eddies to small eddies.
- However, experimental studies have shown that energy is also transferred from smaller scales to larger scales (a process called backscatter), albeit at a much lower rate.
- Nevertheless, the dominant energy transfer is always for large eddies to small eddies.
- The theory assumes that turbulence at high Reynolds numbers is completely random. However, large scale coherent structures may form.
- It also assumes that the smallest scales are very isotropic. Note that in reality, they are elongated structures with a small degree of anisotropy (which means that the eddies have forgotten their initial anisotropic state).
- Kolmogorov’s theory has been confirmed using experiments and large-scale simulations (DNS or direct numerical simulations).

Validity of the Kolmogorov theory (K41)



One-dimensional spectra scaled with respect to the microstructure from various turbulent flows. This demonstrates the universal character of the microstructure and illustrates the −5/3 behavior of the inertial subrange [1].



Experimental spectra measured in the boundary layer of the NASA Ames 80 x 100 foot wind tunnel [2].

References:
[1] F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.
[2] P. Durbin, B.A. Pettersson-Reif. Statistical Theory and Modeling for Turbulent Flow. Wiley, 2011.

Energy spectrum and mesh resolution

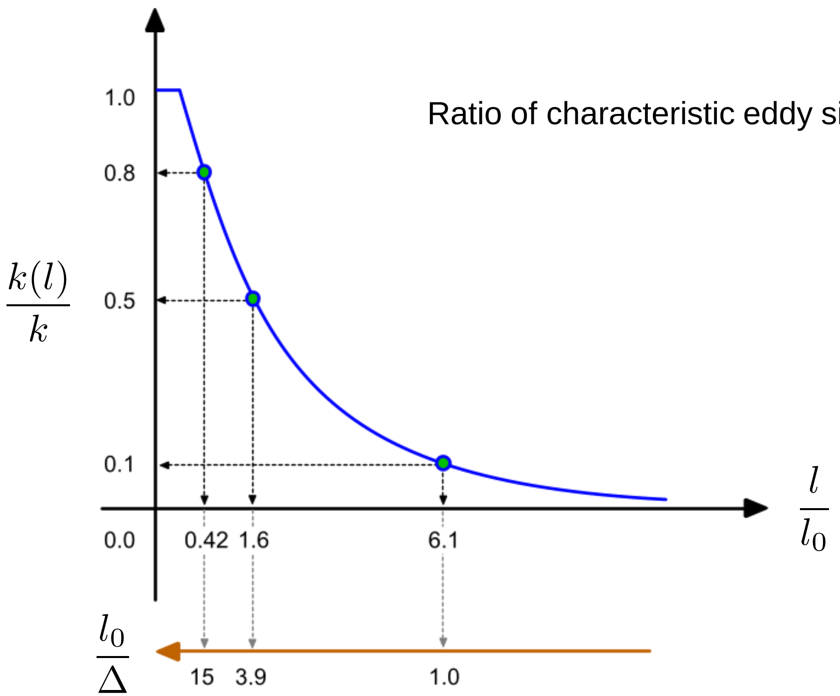
- DNS simulations are quite expensive, they require a lot of grid points/cells in order to resolve all the turbulent scales (in space and time).
- As we have seen, in a DNS simulation the gridding/meshing requirements scales proportional to $Re_T^{9/4}$ or approximately proportional to Re_T^3 for a single time step.
- And every time step should be sufficiently resolved in time (CFL condition less than 1, and the ideal value should be less than 0.5).
- Those are a lot of grid points/cells and time-steps.
- To avoid the extremely high computational requirements of DNS simulations, we can use large-eddy simulations (LES).
- A good LES simulation, aims at resolving 80% of the energy spectrum.
- If the mesh requirements of a LES are still too high, we can do a VLES (very large eddy simulation) where we aim at resolving 50% of the energy spectrum.
- Another alternative are detached-eddy simulations (DES). In DES, we use RANS close to walls and LES in the far field.
- LES, VLES, and DES are commonly called scale-resolving simulations (SRS).

Energy spectrum and mesh resolution

- If SRS requirements are still too high, which is the case for most of the industrial applications requiring quick turn-around times, we can use RANS/URANS models.
- In RANS/URANS simulations the whole energy spectrum is modeled.
- RANS/URANS also heavily relies in wall functions and incentive y^+ near the wall treatment.
- The meshing requirements of RANS/URANS should be sufficiently to capture well integral scales l_0 and model/resolve the boundary layer (according to the near the wall treatment).
- As it can be seen from this discussion, meshing requirements are driven by the turbulent scales we would like to resolve. Meshing depends on the turbulence modeling approach taken.
- SRS simulations requires fine meshes to resolve the space and time scales.
- RANS/URANS can use coarse meshes, as all the scales are being modeled.
- In the next lecture, we will address how to compute estimates of the turbulence scales and leverage this information to generate the mesh.

Energy spectrum and mesh resolution

- So far, we have seen that the mesh resolution determines the fraction of the turbulent kinetic energy directly resolved.
- Let us suppose that we want to resolve 80% of the turbulent kinetic energy *k(l)* in a LES simulation. Then, approximately 15 cells will be needed across the integral length scale l_0.
- In the same way, if you would like to solve 50% of the turbulent kinetic energy *k(l)* (VLES), you will need approximately 4 cells across the integral length scale l_0.



Cumulative turbulent kinetic energy against lengths scale of eddies

Ratio of integral length scale to grid length scale

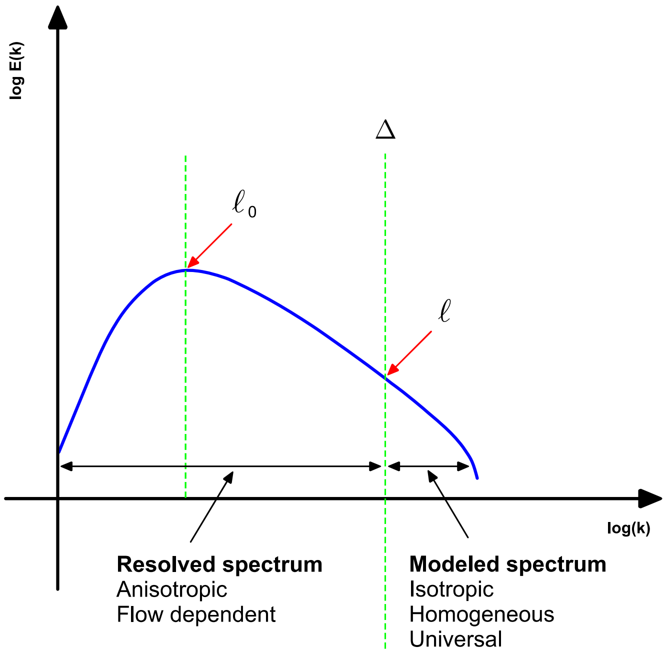
Ratio of characteristic eddy size to integral length scale

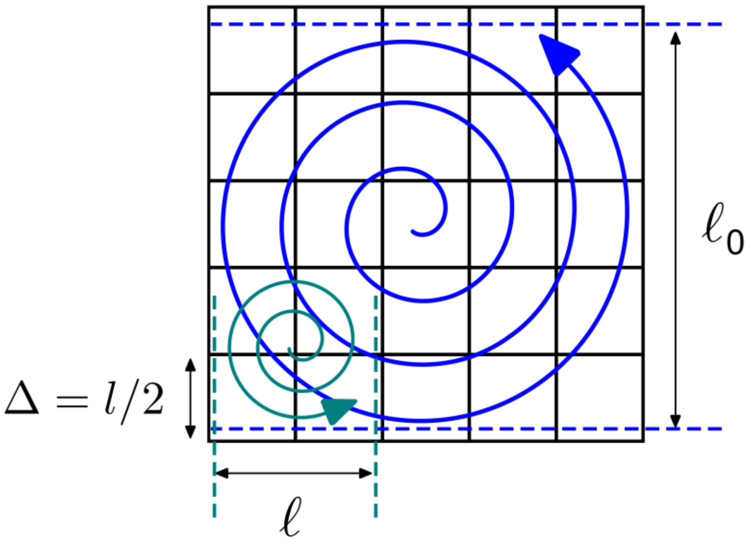
	l/l_0	l_0/Δ
$k(l) = 0.1k$ (10%)	6.1	1.0
$k(l) = 0.5k$ (50%)	1.6	3.9
$k(l) = 0.8k$ (80%)	0.42	15
$k(l) = 0.9k$ (90%)	0.16	38

Characteristics turbulent kinetic energy and length scales of the energy spectrum. For a rigorous explanation of these results, please refer to Turbulent Flows by S. Pope

Energy spectrum and mesh resolution

- In LES simulations, it is a good practice to have at least 10-15 cells across the integral length scale l_0.
- If you are interested in VLES, 4 to 6 cells per integral length scale l_0 are recommended. The same requirement applies to RANS/URANS.
- To resolve an eddy with a length scale l, where l is the smallest scales that can be resolved with the grid or Δ , and $l << l_0$. At least a couple of cells need to be used in each direction.
- Remember, eddies cannot be resolved down to the molecular dissipation limit (it is too expensive).





Integral length scale and grid length scale

- The integral length scale l_0 can be roughly estimate as follows,
  - Based on a characteristic length, such as the size of a bluff body or pipe diameter.
  - From correlations.
  - From experimental results.
  - From a precursor RANS simulation.
- Remember, turbulent kinetic energy peaks at integral length scale l_0.
- Therefore, these scales must be sufficiently resolved in LES/DES simulations, or captured (be able to track) in RANS/URANS simulations.
- After identifying the integral scales, you can cluster enough cells in the domain regions where you expect to find the integral scales (or large eddies).
- In other words, put enough cells in the wake or core of the flow.
- In RANS/URANS/VLES simulations, it is acceptable to use a minimum of 5 cells across integral length scales.
- LES simulations have higher requirements.

Integral length scale and grid length scale

- The integral length scales l_0 can be computed from a precursor RANS simulation.
- To compute the integral scales, you need to compute the turbulent kinetic energy k, dissipation rate ϵ, and specific dissipation rate ω.
- Therefore, you need to use a two-equation turbulence models:
  - $k - \epsilon$ family models.
  - $k - \omega$ family models.
- Depending on the model selected, you can compute l_0 as follows,

$$l_0 = \frac{k^{1.5}}{\epsilon} \quad \text{or} \quad l_0 = \frac{k^{0.5}}{C_\mu \omega} \quad \text{where} \quad C_\mu = 0.09$$

- Again, if you check the dimensions you will see that the base dimensions match.

Derived quantity	Symbol	Dimensional units	SI units
Energy dissipation rate per unit mass	ϵ	L^2T^{-3}	m^2/s^3
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Specific dissipation rate	ω	T^{-1}	$1/s$

Integral length scale and grid length scale

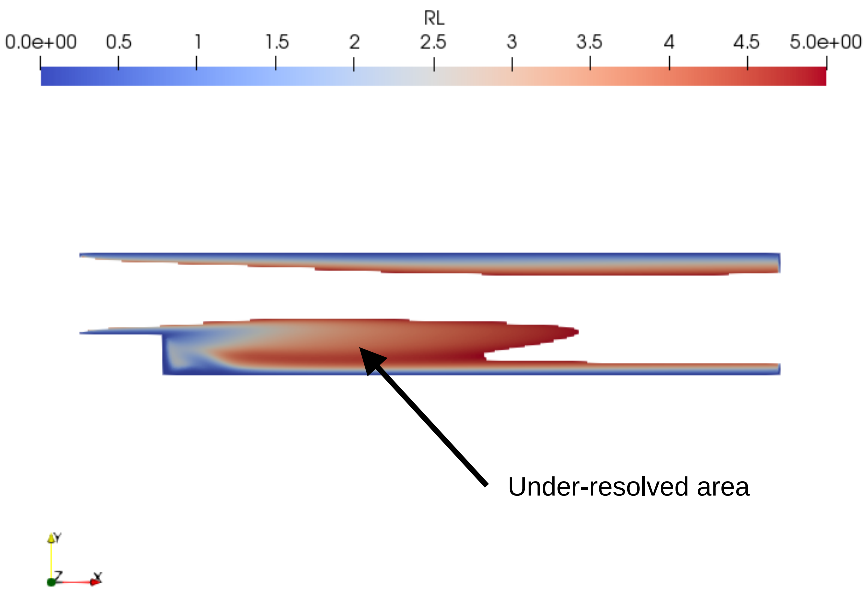
- The ratio of integral length scale to grid length scale R_l can be computed as follows,

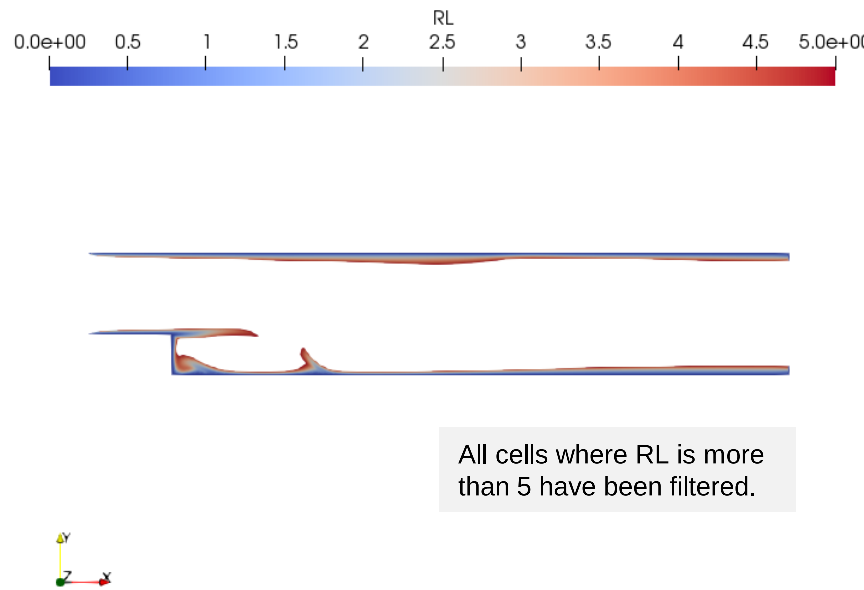
$$R_l = \frac{l_0}{\Delta}$$ where Δ can be approximated as follows $$\Delta \approx \sqrt[3]{\text{cell volume}}$$

This approximation is accurate if the aspect ratios are modest (less than 1.2)

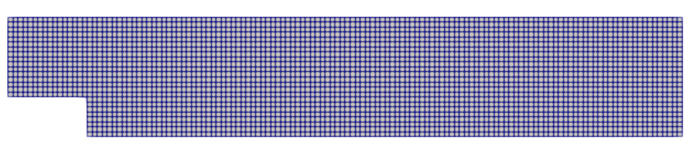
- The recommended value of R_l is about $R_l > 5 - 10$.
- Where 5 should be considered the lowest limit of resolution (for RANS/URANS and VLES) and 10 is the desirable lower limit (for LES/DES).
- Higher values can be used if computer power and time constrains permits.
- This is a very rough estimate, which is likely problem dependent.
- Remember, in well resolved LES simulations equal mesh resolution should be provided in all directions.

Integral length scale and grid length scale





All cells where RL is more than 5 have been filtered.

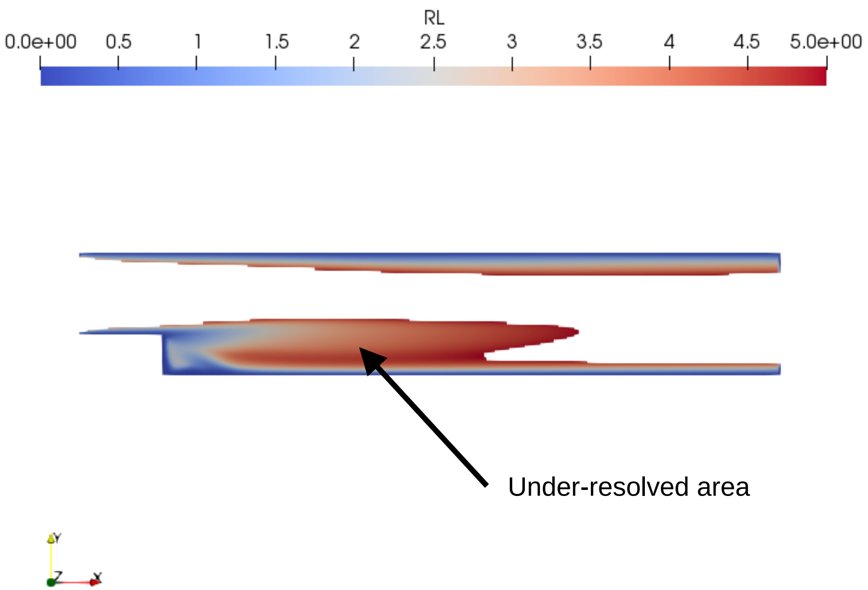


Coarse mesh

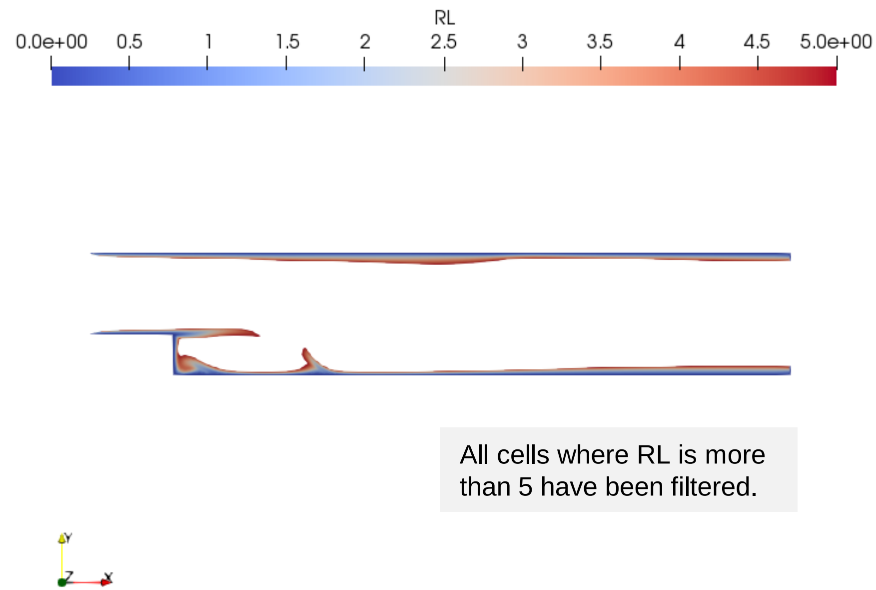
Fine mesh

Energy spectrum and energy cascade. Integral length scale and grid length scale

Integral length scale and grid length scale



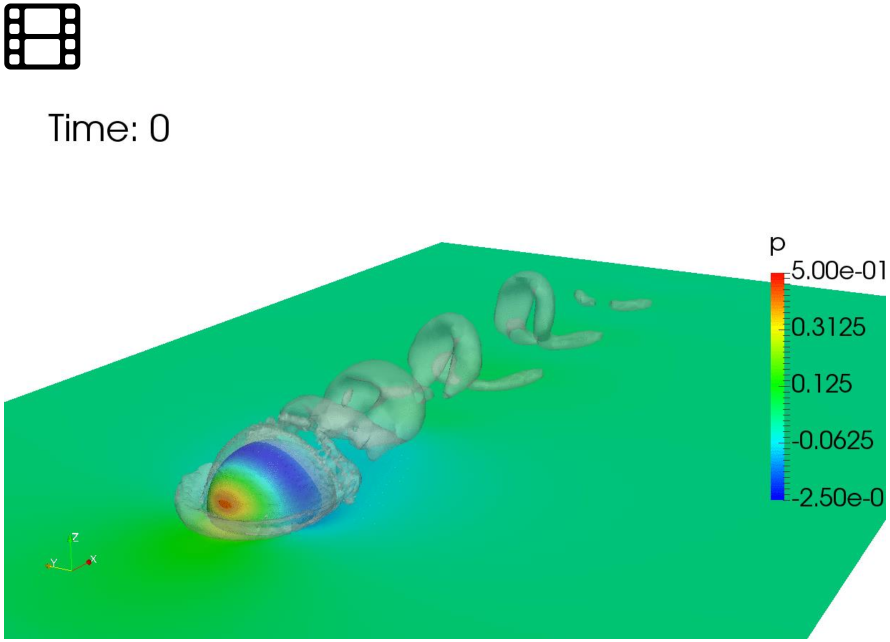
Coarse mesh



Fine mesh

- To identify integral length scales and grid length scales you can plot contours of these quantities at different locations/planes in the domain.
- The lowest limit of R_l can be clipped so that the well resolved areas do not appear. In this case we are showing the cells where $0 < R_l < 5$.
- Under-resolved areas (the areas shown), will need finer meshes or local mesh adaption.
- Near-wall regions always pose challenges. In these areas is better to quantify the y$^+$ value.

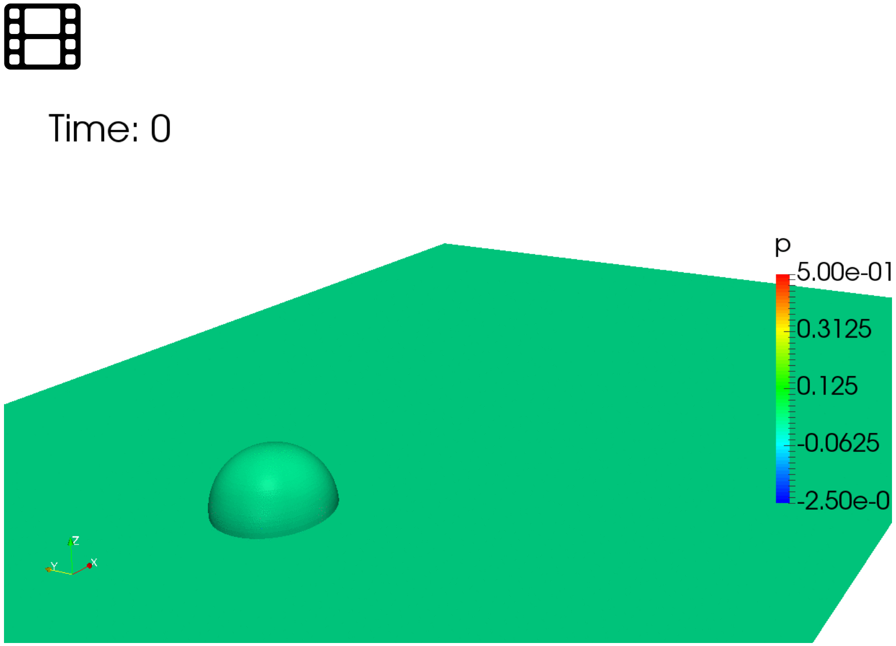
Turbulence modeling – Space discretization



Coarse mesh

http://www.wolfdynamics.com/training/turbulence/image7.gif

- The vortices are dissipated due to numerical diffusion (low mesh resolution).
- This case uses as initial conditions the outcome of a steady simulation.
- Due to the low mesh resolution, it is quite difficult to onset the instability if you start from a uniform initial condition.



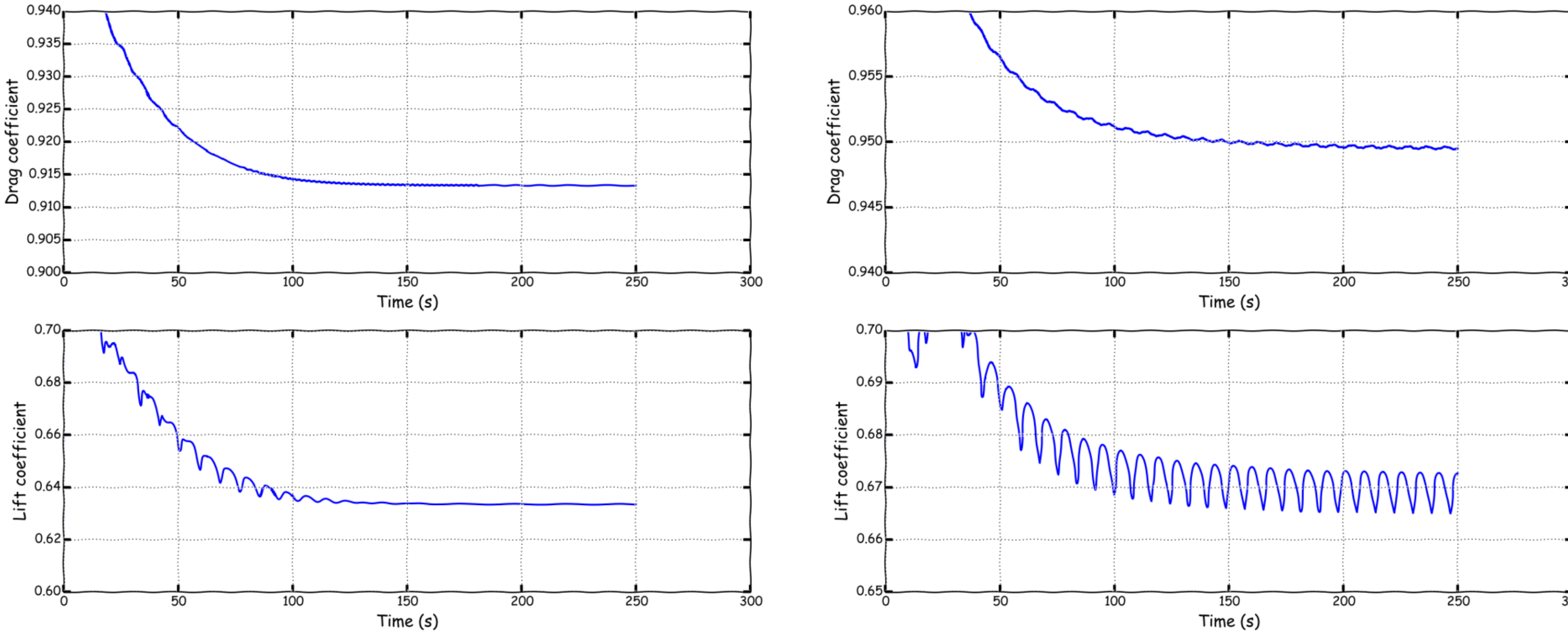
Fine mesh

http://www.wolfdynamics.com/training/turbulence/image8.gif

- The fine mesh captures the small spatial scales that the coarse mesh does not manage to resolve.
- Even with uniform initial conditions, the mesh captures the special scales without numerical dissipation.

Vortices visualized using Q criterion.

Turbulence modeling – Space discretization



Coarse mesh

- The coarse mesh does not capture small spatial scales; hence, they add numerical diffusion to the solution.
- You will have the impression that you have arrived to a steady state.

Fine mesh

- The fine mesh captures the small spatial scales that the coarse mesh does not manage to resolve.

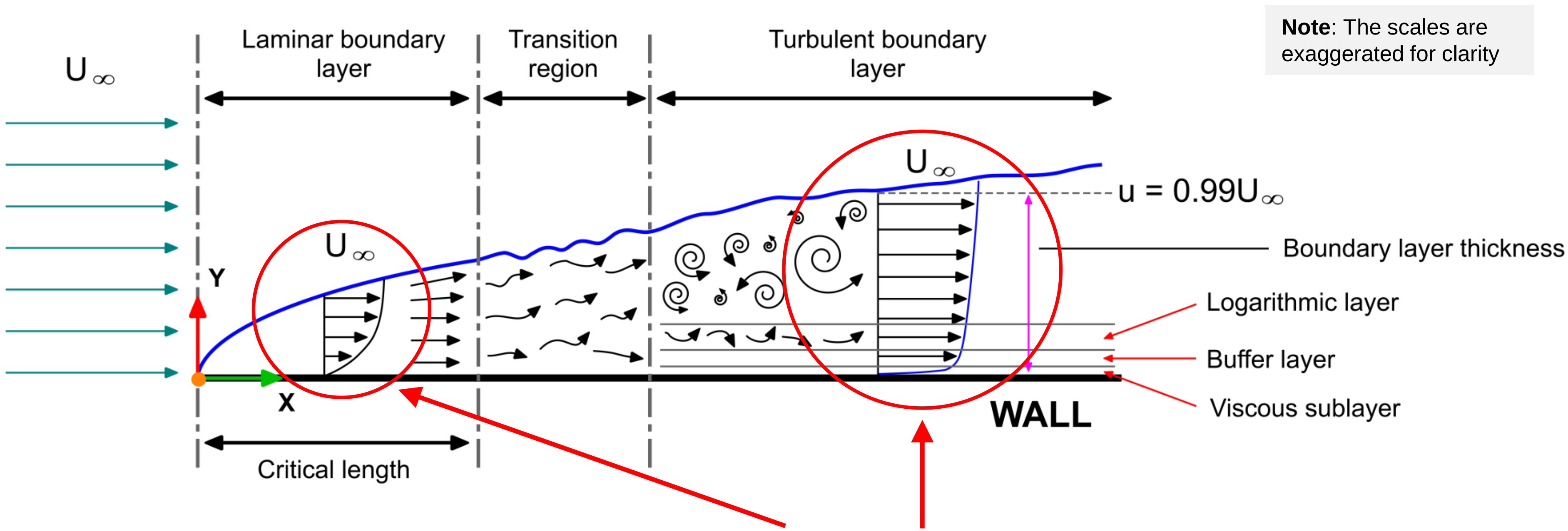
Drag and lift coefficient signals – Unsteady simulations

Roadmap to Lecture 3

1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales, to integral scales~~
2. ~~Energy spectrum and energy cascade. Integral length scale and grid length scale~~
3. **Turbulence near the wall - Law of the wall**
4. A glimpse to a turbulence model

Turbulence near the wall - Law of the wall

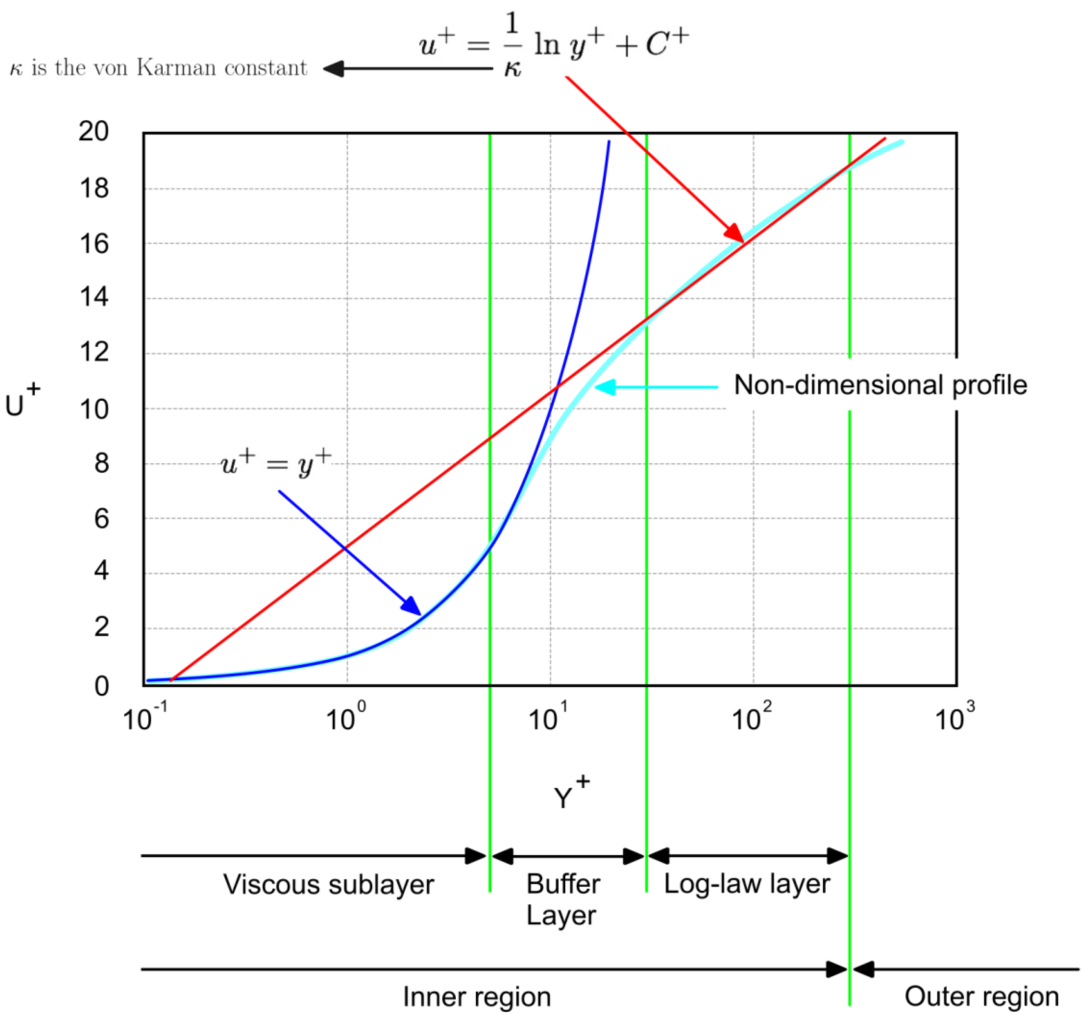
Turbulence near the wall – Boundary layer



Actual profile - Physical velocity profile

- Near walls, in the boundary layer, the velocity changes rapidly.
- In turbulence modeling in CFD, the most important zones are the viscous sublayer and the log-law layer.
- The buffer layer is the transition layer which we try to avoid as much as possible.
- Turbulence modeling in CFD requires different considerations depending on whether you solve the viscous sublayer, model the log-law layer, or solve the whole boundary layer (including the buffer zone).

Turbulence near the wall – Law of the wall



- The law of the wall, is one of the most famous empirically determined relationships in turbulent flows near solid boundaries.
- Measurements show that, for both internal and external flows, the streamwise velocity in the flow near the wall varies logarithmically with distance from the surface.
- This behavior is known as law of the wall (also known as logarithmic law of the wall or log-law layer), as is stated as follows,

$$u^+ = \frac{1}{\kappa} \ln y^+ + C^+$$

where

$$\kappa \approx 0.41 \quad C^+ \approx 5.0$$

Reported values of C^+ can go anywhere from 4.5 to 5.5
Reported values of κ can go anywhere from 0.36 to 0.42

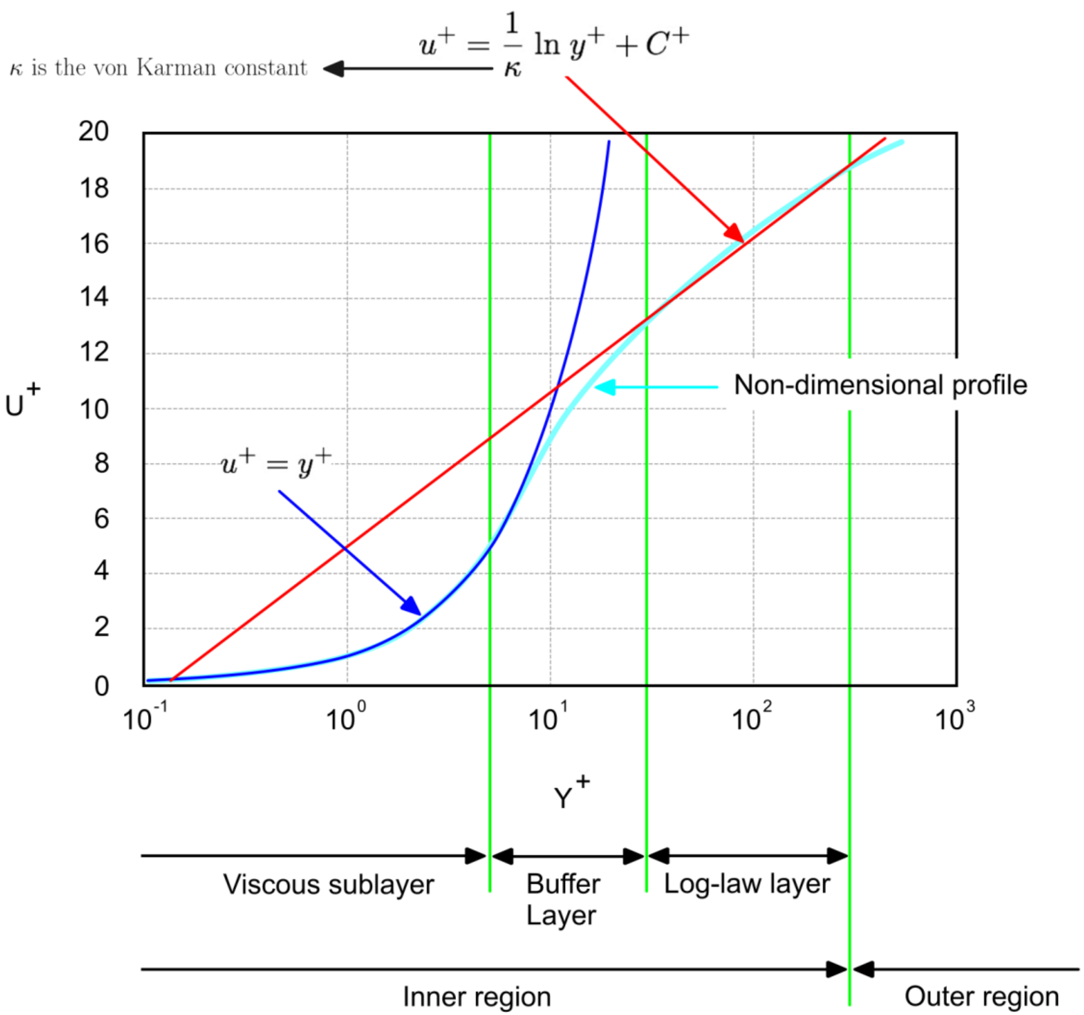
- The law of the wall (log-law layer) is valid for values of y^+ equal to,

$$30 < y^+ < 300$$

- At this point it only rest defining y^+ and u^+.

Turbulence near the wall - Law of the wall

Turbulence near the wall – Definition of y^+ and u^+



$$y^+ = \frac{\rho \times U_\tau \times y}{\mu} = \frac{U_\tau \times y}{\nu}$$

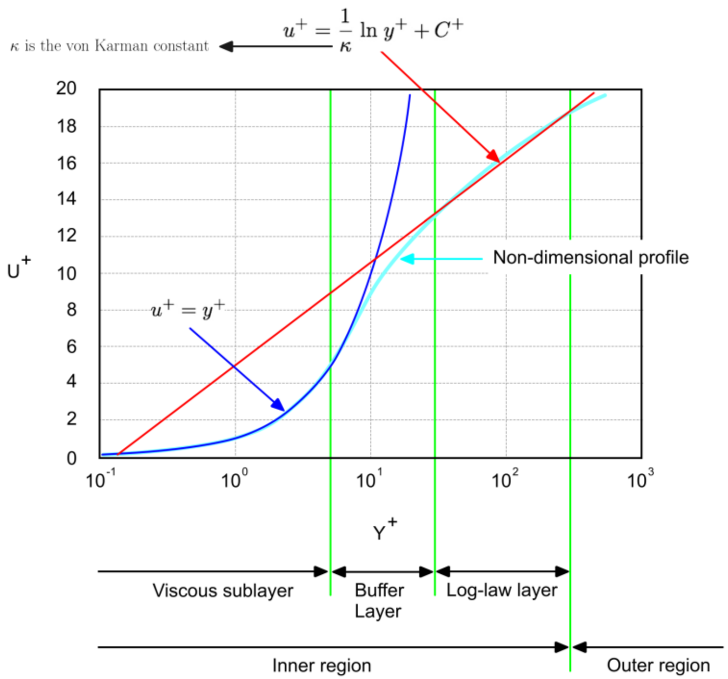
$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

$$u^+ = \frac{U}{U_\tau}$$

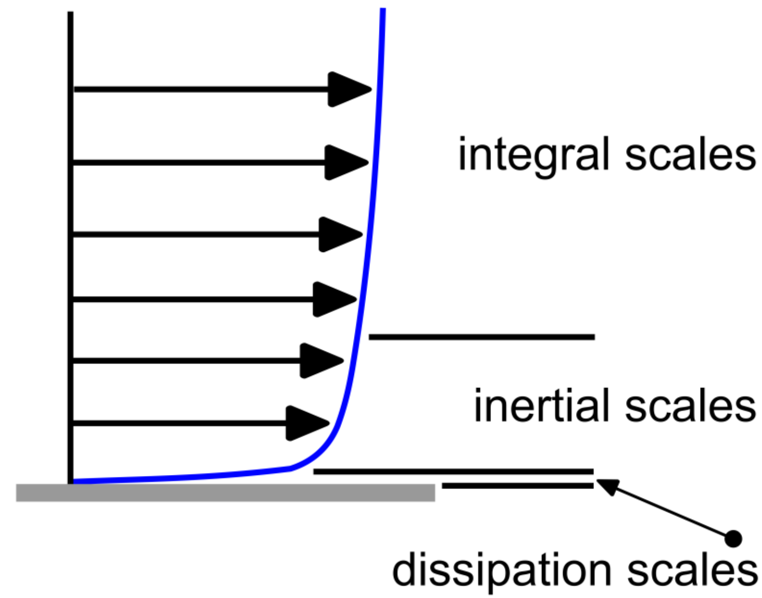
Where y is the distance normal to the wall, U_τ is the shear velocity, and u^+ relates the mean velocity to the shear velocity

- y^+ or wall distance units is a very important concept when dealing with turbulence modeling.
- Remember this definition as we are going to use it a lot.

Non-dimensional profile against physical velocity profile



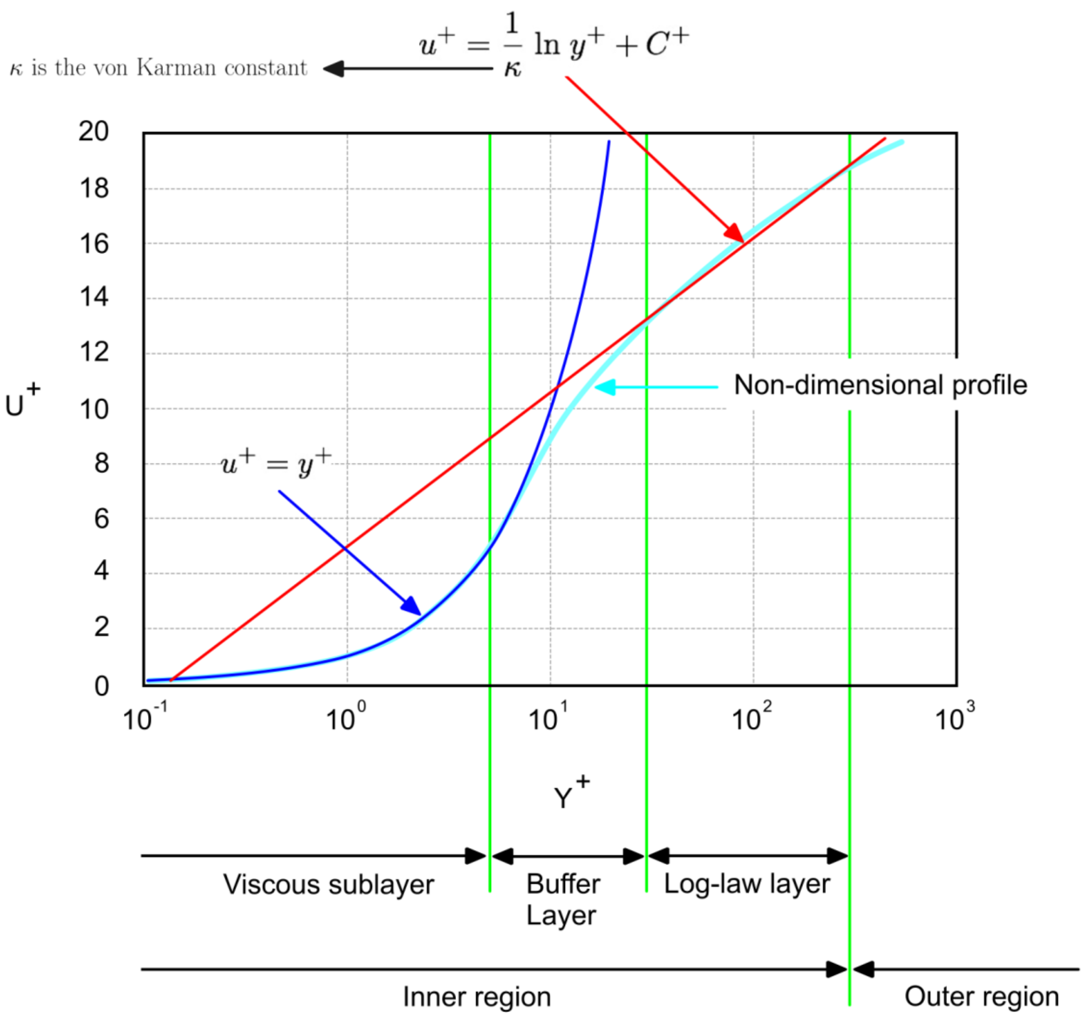
Non-dimensional profile



Physical velocity profile – Actual profile

- The use of the non-dimensional velocity u^+ and non-dimensional distance from the wall y^+, results in a predictable boundary layer profile for a wide range of flows.
- Under standard working conditions this profile is the same, however, under non-equilibrium conditions (production and dissipation of turbulent kinetic energy not balanced), rough walls, porous media, buoyancy, viscous heating, strong pressure gradients, and so on, the profile might be different.
- While the non-dimensional velocity profile is the same for many flows, the physical velocity profile is different.

Turbulence near the wall – Relations according to y^+ value



$$30 < y^+ < 300$$

$$u^+ = \frac{1}{\kappa} \ln y^+ + C^+$$

$$\kappa \approx 0.41 \quad C^+ \approx 5.0$$

$$y^+ < 6$$

$$u^+ = y^+$$

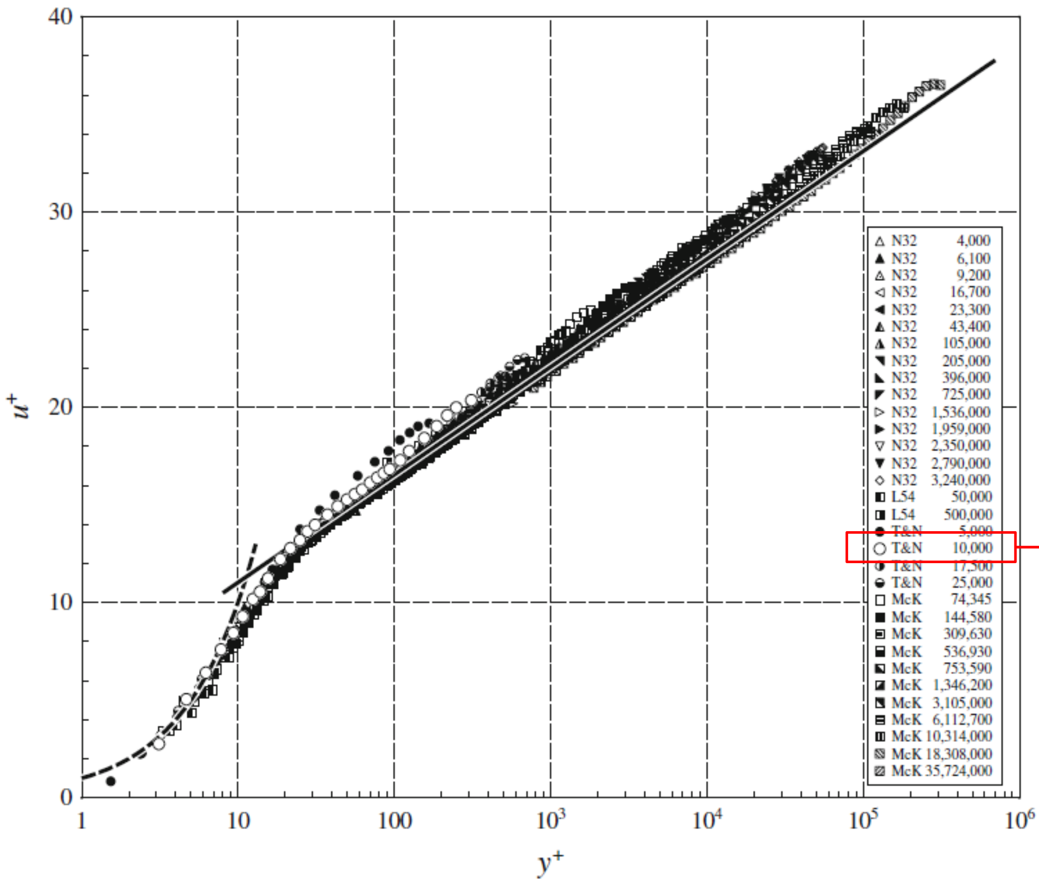
$$6 < y^+ < 30$$

$$u^+ \neq y^+$$

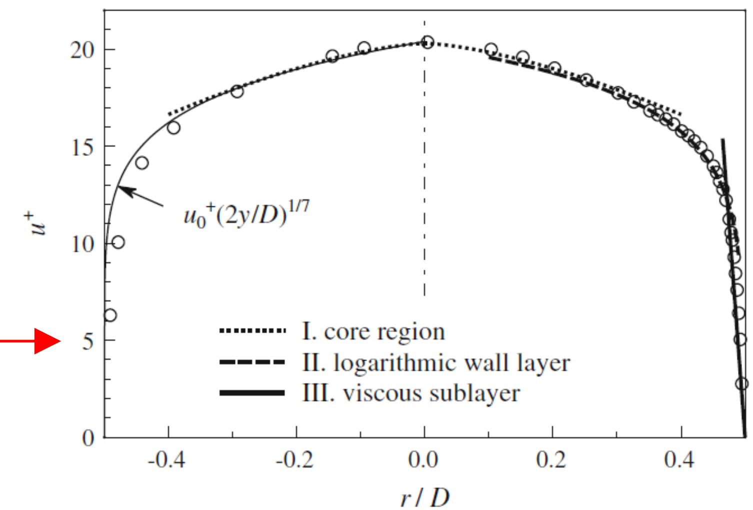
$$u^+ \neq \frac{1}{\kappa} \ln y^+ + C^+$$

Note: the range of y^+ values might change from reference to reference but roughly speaking they are all close to these values.

Turbulence near the wall – Experimental data



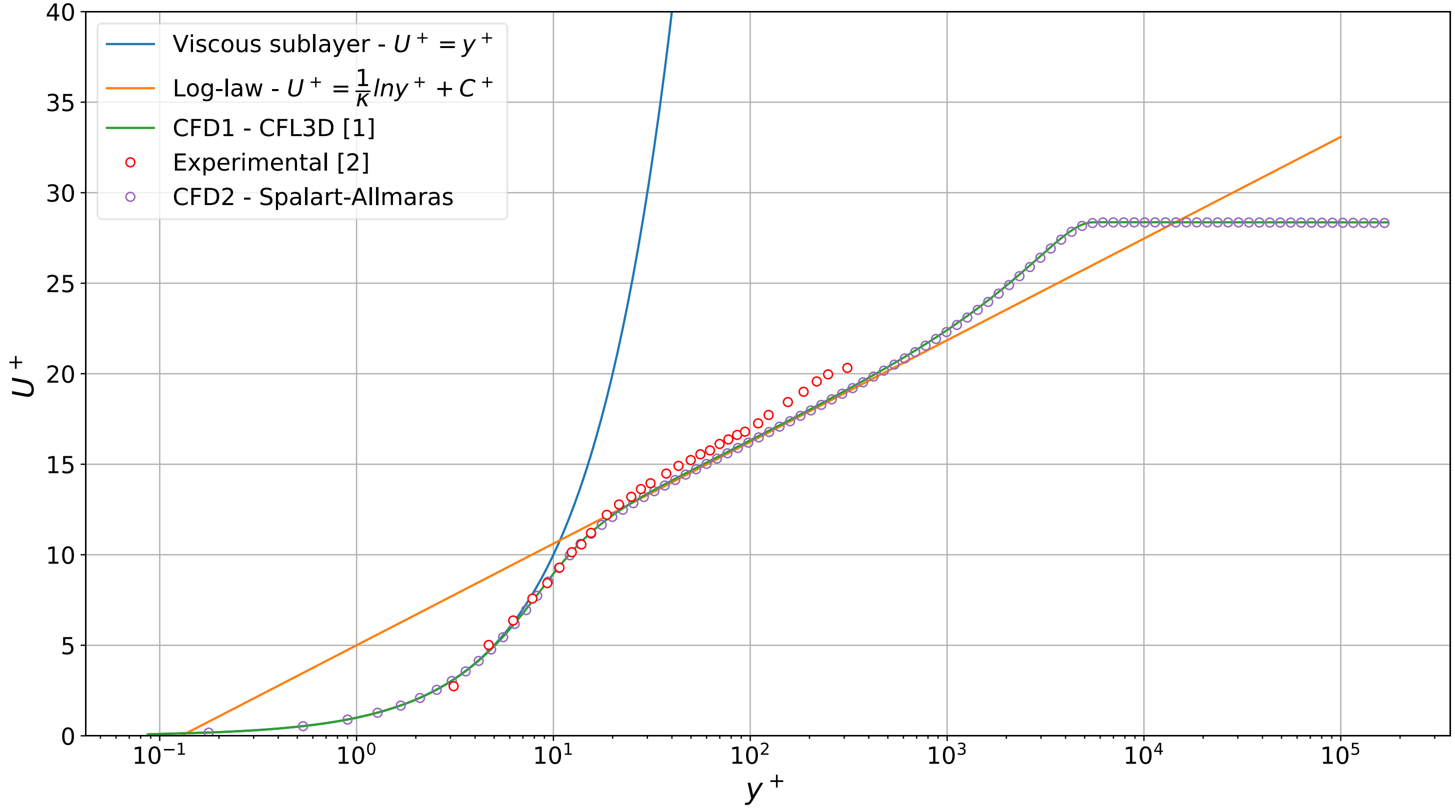
Dimensionless mean velocity profile u⁺ as a function of the dimensionless wall distance y⁺ for turbulent pipe flow with Reynolds numbers between 4000 and 3600000.



The dimensionless mean velocity profile u⁺ of a turbulent flow in a pipe at Re = 10000 as a function of the distance r from the centerline of a pipe with diameter D, compared to experimental data.

References:
F. Nieuwstadt, B. Boersma, J. Westerweel. Turbulence. Introduction to Theory and Applications of Turbulent Flows. Springer, 2016.

Turbulence near the wall – Relations according to y^+ value

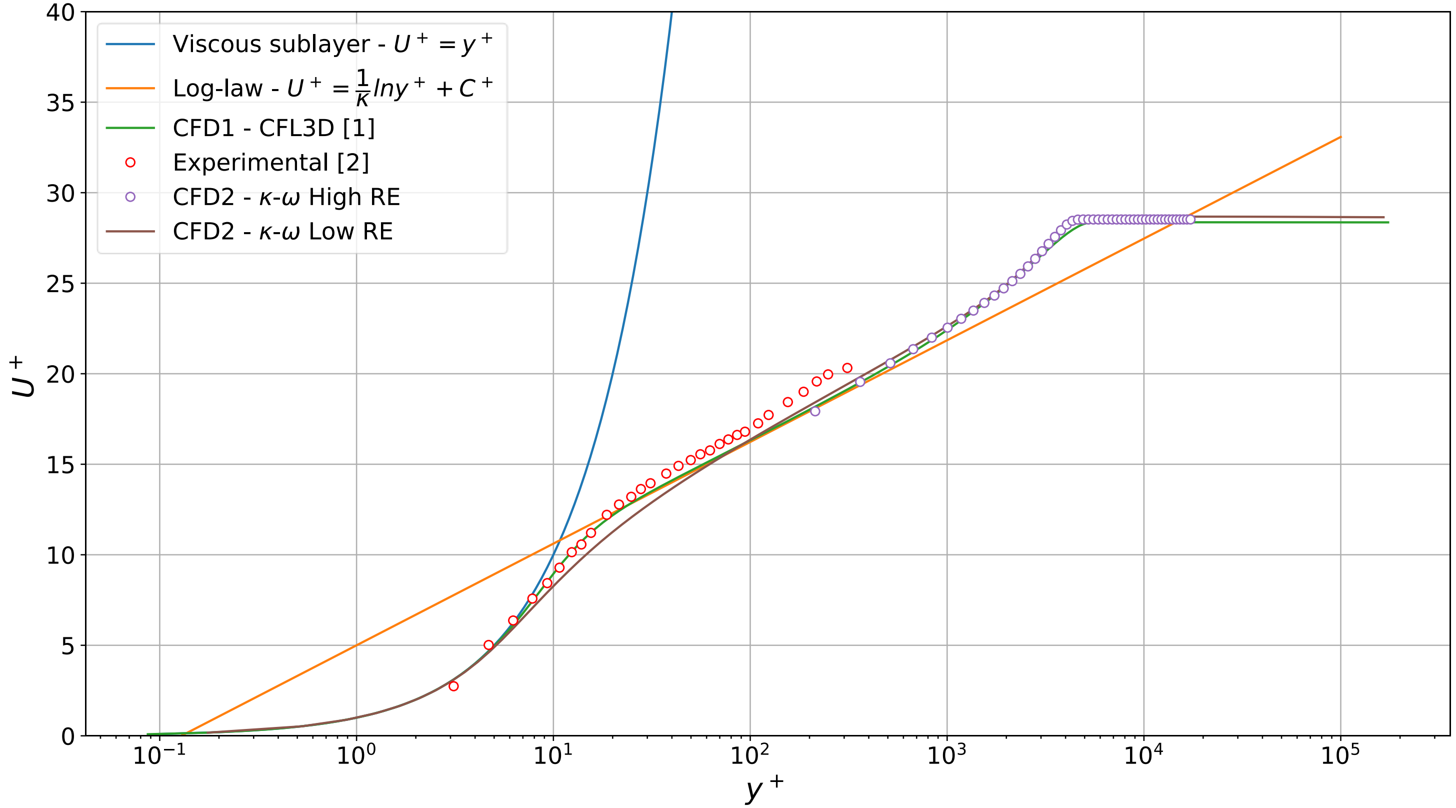


- Plot of the non-dimensional velocity profile.
- Notice that all cases plotted correspond to different physics and Reynolds numbers.

References:
[1] https://turbmodels.larc.nasa.gov
[2] J. M. J. den Toonder and F. T. M. Nieuwstadt. Reynolds number effects in a turbulent pipe flow for low to moderate Re. Physics of Fluids 9, 3398 (1997).

Turbulence near the wall - Law of the wall

Turbulence near the wall – Relations according to y⁺ value



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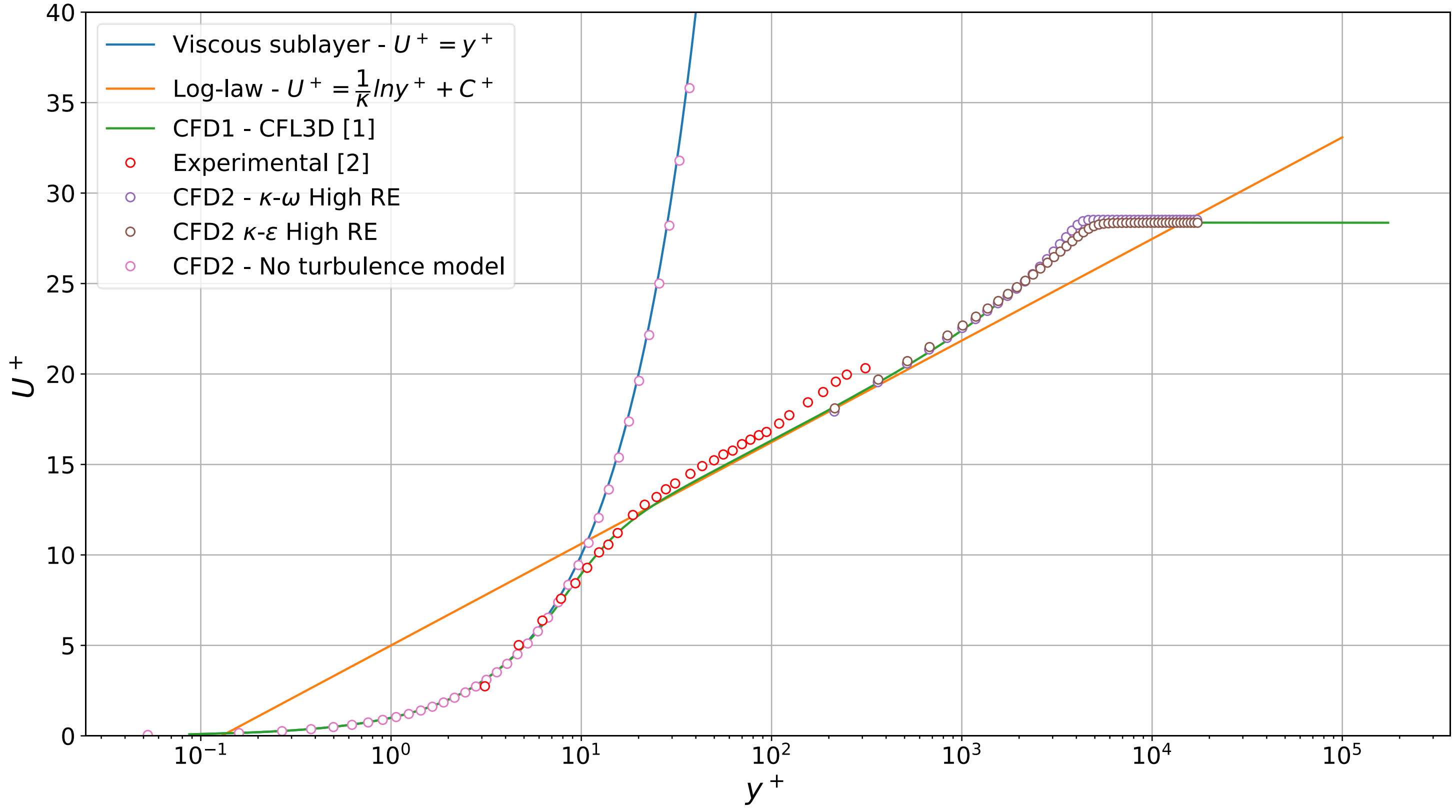

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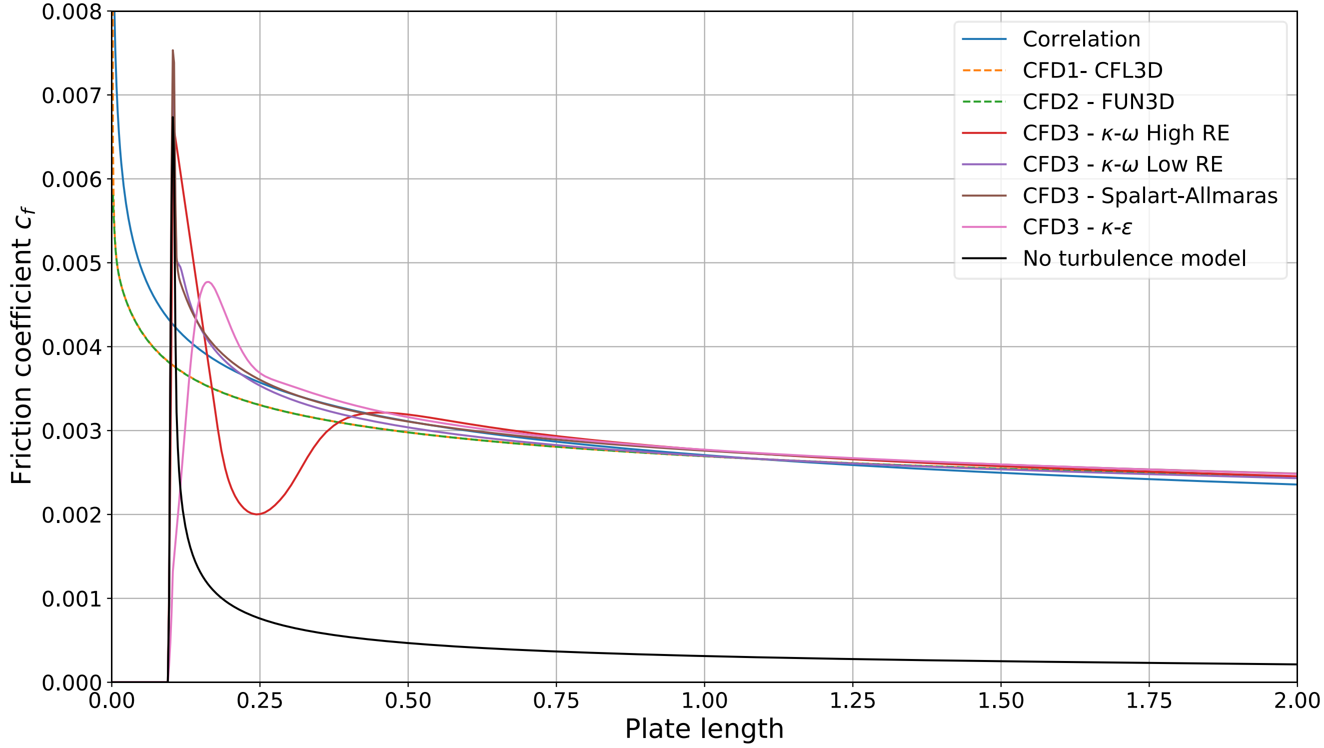
Turbulence near the wall - Law of the wall

Turbulence near the wall – Relations according to y^+ value



- Plot of friction coefficient in function of length for the flat plate case [1].
- Notice that the case where we did not use turbulence model (DNS simulation), it highly under predicts the friction coefficient value.
- The importance of using a turbulence model.

References:
https://turbmodels.larc.nasa.gov/flatplate.html

Turbulence near the wall - Law of the wall

Turbulence near the wall – Relations according to y+ value

- From the non-dimensional u^+ vs y^+ plots, it is possible to fit a function that covers the entire laminar and turbulent regimes.
- The most widely known *"universal"* velocity profile is Spalding's law [1], which is essentially a fit of the laminar, buffer and logarithmic regions of the boundary layer,

$$y^+ = u^+ + \frac{1}{E}\left[e^{\kappa u^+} - 1 - \frac{\kappa u^+}{1!} - \frac{(\kappa u^+)^2}{2!} - \frac{(\kappa u^+)^3}{3!} - \frac{(\kappa u^+)^4}{4!}\right]$$

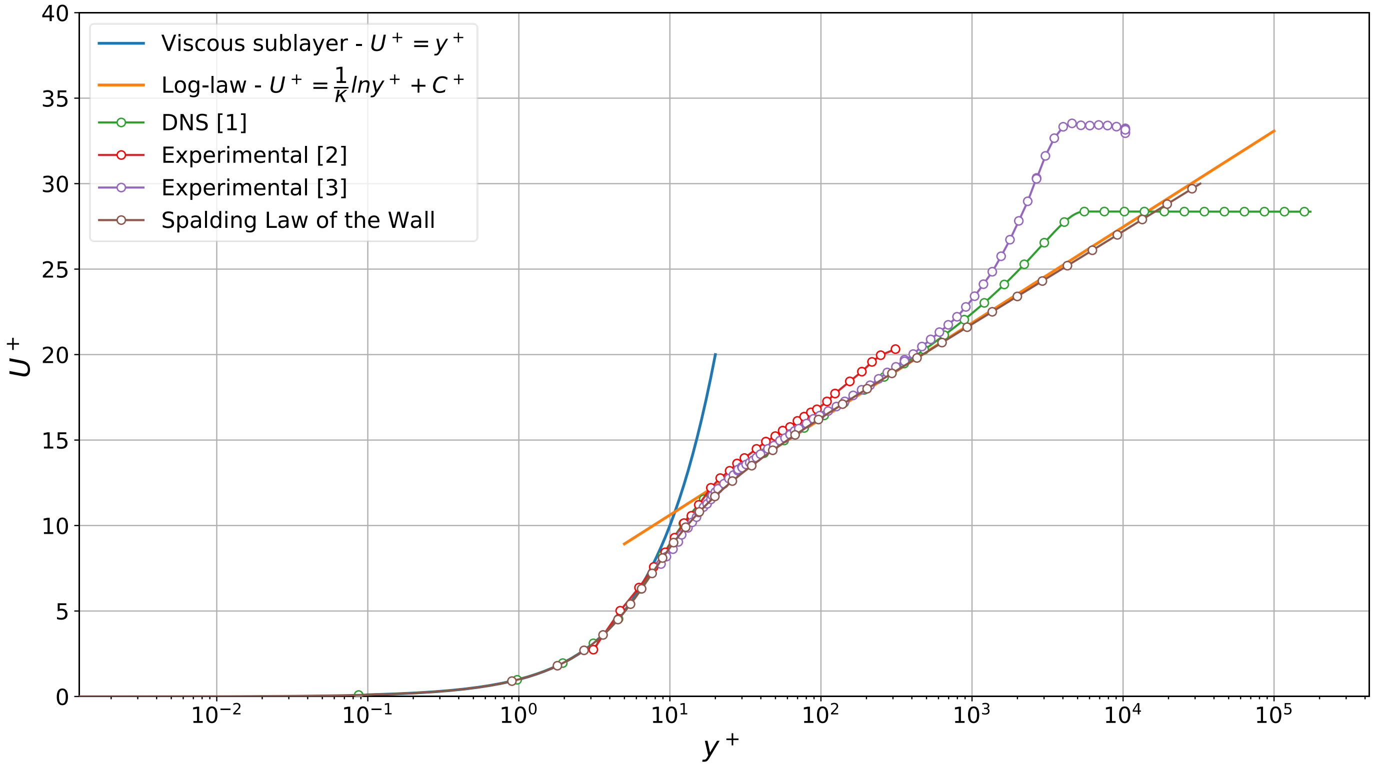
$$\frac{1}{E} = e^{-\kappa C^+}$$

- Here, κ is the von Karman constant and *E* is another constant needed to fit the well known logarithmic law.
- Reported values of C^+ can go anywhere from 4.5 to 5.5.
- Reported values of κ can go anywhere from 0.36 to 0.42
- Reported values of *E* can go anywhere from 8.5 to 9.3.

[1] Spalding. A single formula for the law of the wall. J. of Applied Mechanics. 1961.

Turbulence near the wall – Relations according to y+ value

- Comparison of the Spalding's law against the Log-law, experimental results, and numerical results.
- The following values were used to plot Spalding's law, $\kappa = 0.42 \qquad E = 9.1$

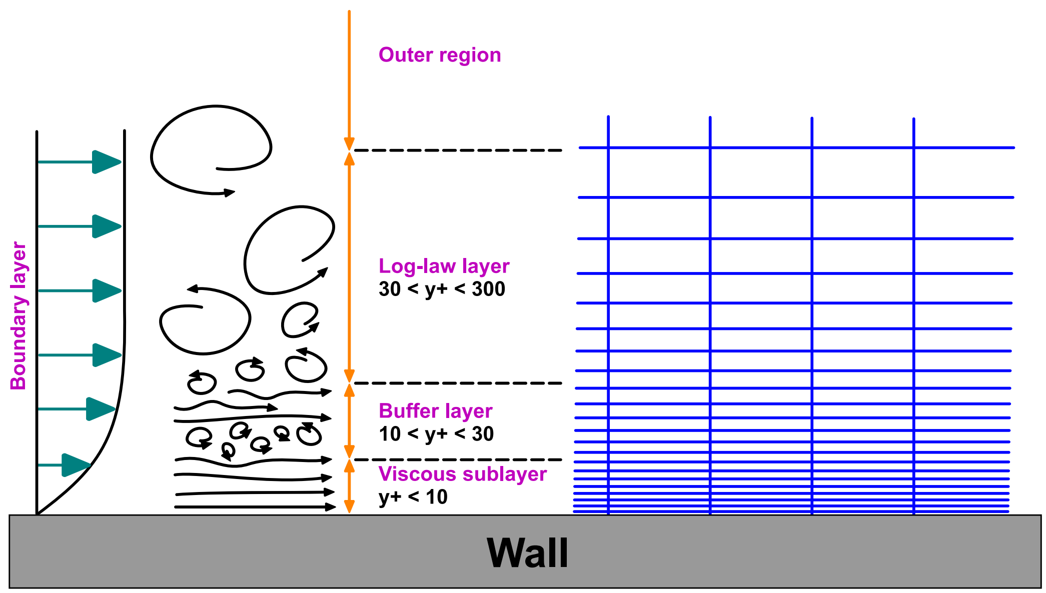


References:
[1] https://turbmodels.larc.nasa.gov
[2] J. M. J. den Toonder and F. T. M. Nieuwstadt. Reynolds number effects in a turbulent pipe flow for low to moderate Re. Physics of Fluids 9, 3398 (1997).
[3] https://www.flow.kth.se/flow-database/experimental-data-1.791818 (APG database)

Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

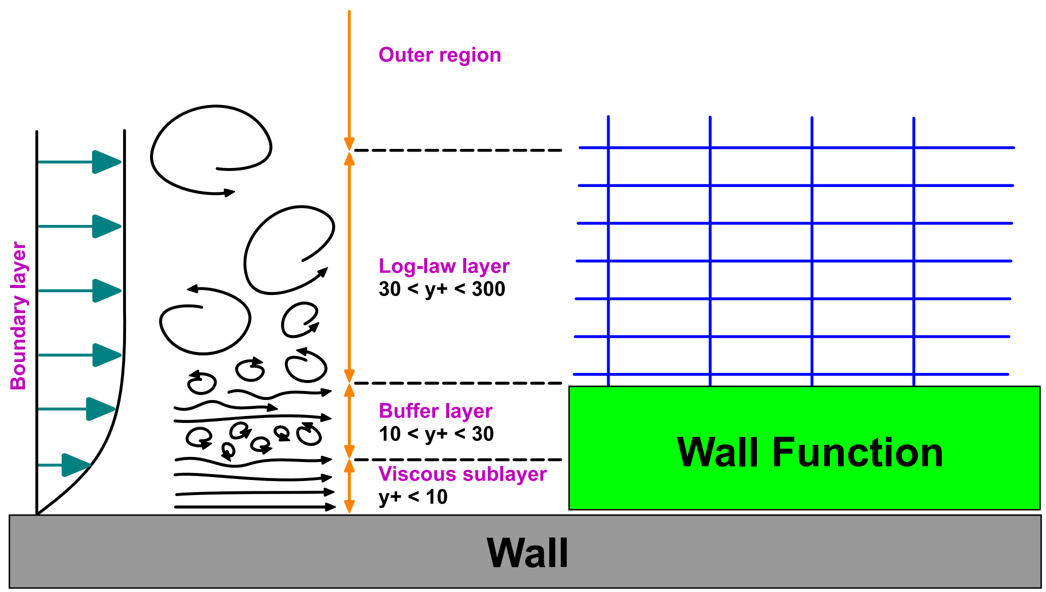
- When dealing with wall turbulence, we need to choose a near-wall treatment.
- If you want to resolve the boundary layer up to the viscous sub-layer you need very fine meshes close to the wall.
- In terms of y^+, you need to cluster at least 6 to 10 layers at $y^+ < 10$. You need to properly resolve the velocity profile.
- But for good accuracy, usually you will use 15 to 30 layers.
- This is the most accurate approach, but it is computationally expensive.



Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

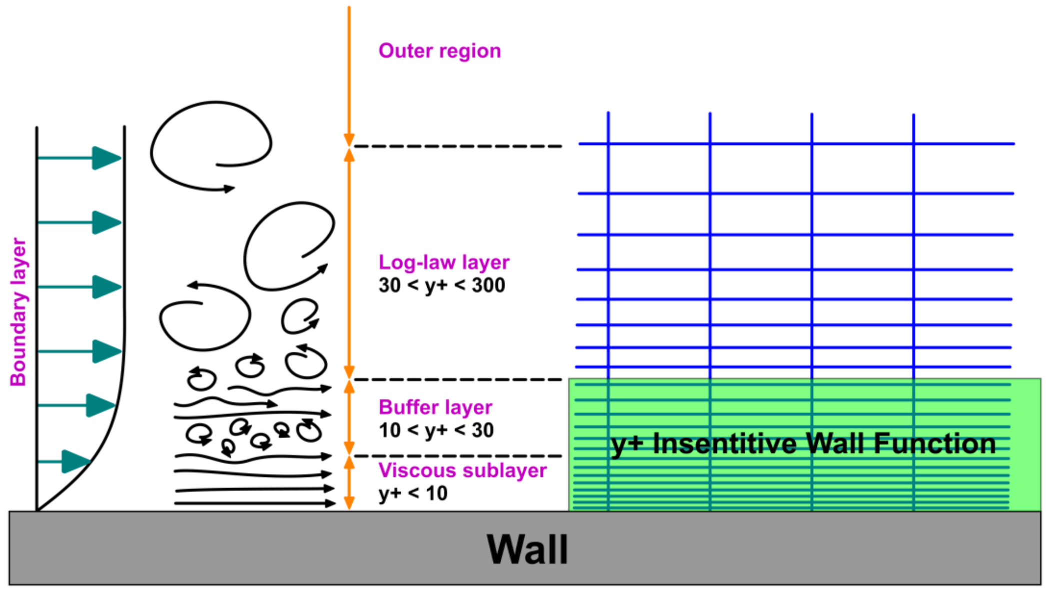
- When dealing with wall turbulence, we need to choose a near-wall treatment.
- If you are not interested in resolving the boundary layer up to the viscous sub-layer, you can use wall functions.
- In terms of y^+, wall functions will model everything below $y^+ < 30$ or the target y^+ value.
- This approach use coarser meshes, but you should be aware of the limitations of the wall functions.
- You will need to cluster at least 5 to 10 layers to resolve the profiles (U and k).



Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

- When dealing with wall turbulence, we need to choose a near-wall treatment.
- You can also use the y^+ insensitive wall treatment (sometimes known as continuous wall functions or scalable wall functions). This kind of wall functions are valid in the whole boundary layer.
- In terms of y^+, you can use this approach for values between $1 \leq y^+ \leq 300$.
- This approach is very flexible as it is independent of the y^+ value, but is not available in all turbulence models
- You also should be aware of the limitations this wall treatment method.



Turbulence near the wall - Law of the wall

Near-wall treatment and wall functions

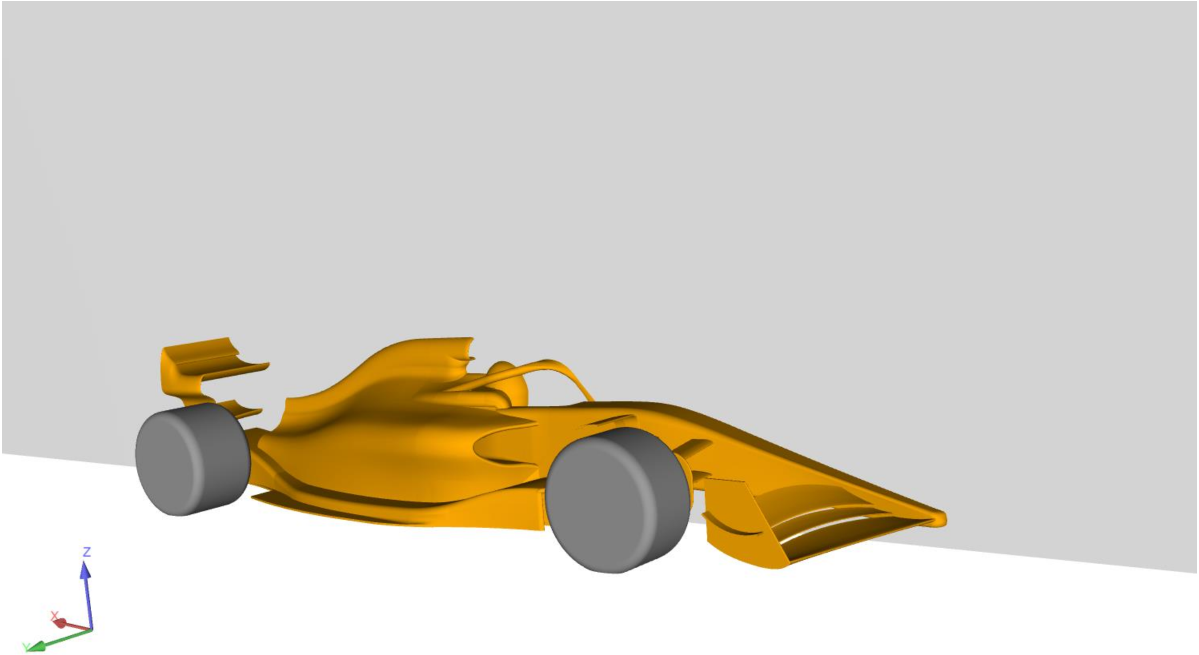
- Generally speaking, wall functions is the approach to use if you are more interested in the mixing in the outer region, rather than the forces on the wall.
- If accurate prediction of forces or heat transfer on the walls are key to your simulation (aerodynamic drag, turbomachinery blade performance, heat transfer) you might not want to use wall functions.
- The wall function approach is also known as high-RE (HRN).
- The approach where you do not use wall functions is known as low-RE (LRN).
- Wall functions should be avoided if $10 < y^+ < 30$. This is the transition region, and wall function do not perform very well here (nobody knows what is going on in this region).
- The low-RE approach is computational expensive as it requires clustering a lot cells near the walls.
- To get good results with LRF, you will need to cluster at least 10 layers for $y^+ < 6$. But values up to $y^+ < 10$ are acceptable. It is primordial solving the velocity profile.

Near-wall treatment and wall functions

- If you do not have any restrictions in the near-wall treatment, use wall functions.
- Wall functions can be used in RANS, DES and LES.
- You can also use them with moving walls.
- If you are doing LES, it is highly recommended to use wall functions. Otherwise, your meshing requirements will be very similar to DNS.
- In practice, maintaining a fixed value of y^+ in the cells next to the walls throughout the domain is very challenging. In this cases, you should monitor the average value.
- Maintaining a value of $y^+ > 30$ when using wall functions during grid refinement studies can be difficult and problematic. Remember, wall functions are not accurate when $y^+ < 30$.
- Grid refinement studies are a common practice in CFD and a recommended best practice. Therefore, wall treatments that are insensitive to y^+ values are preferred.

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count

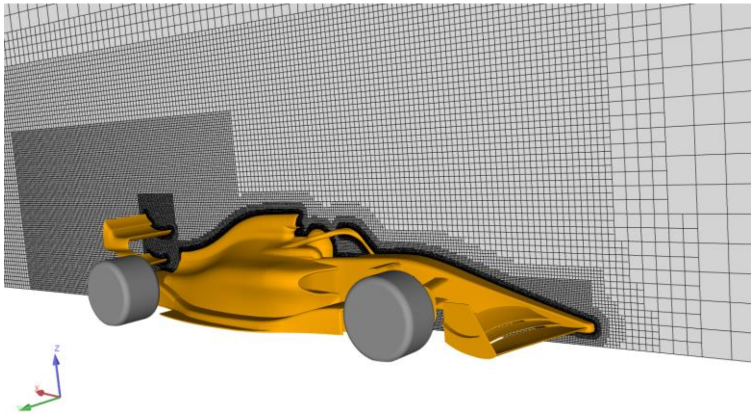


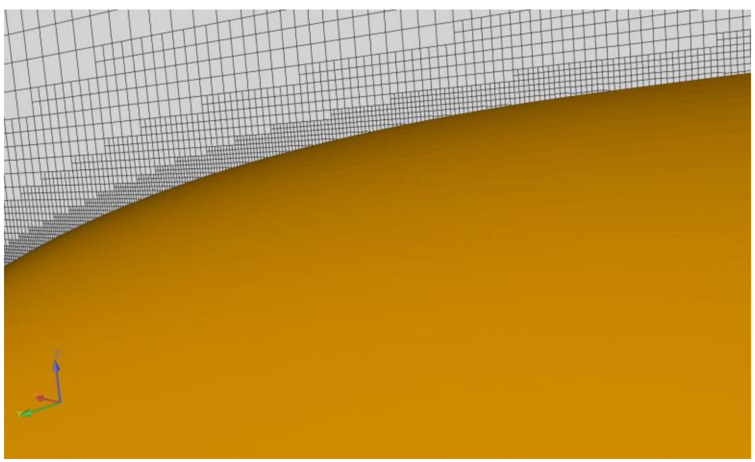
	Mesh 1	Mesh 2
Number of cells	57 853 037	111 137 673

- Can you guest the difference between the meshes?

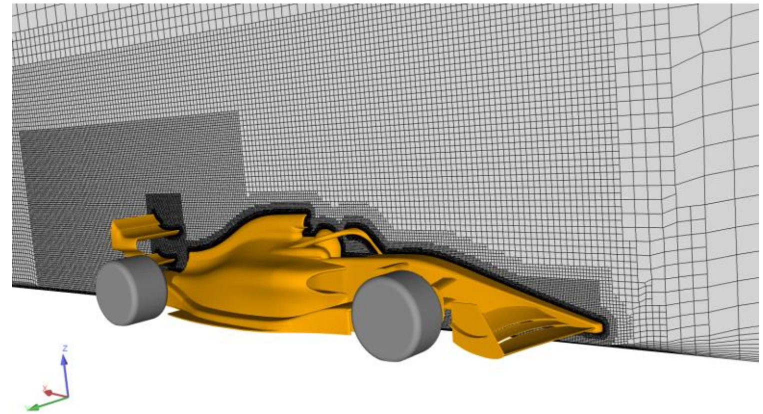
Turbulence near the wall - Law of the wall

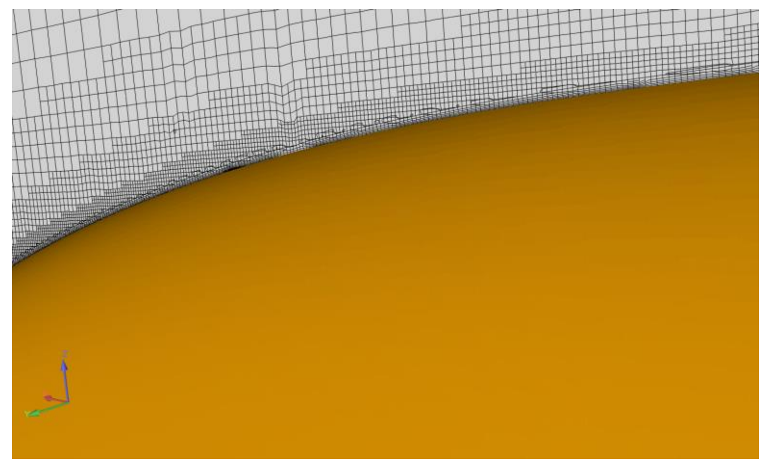
Influence of near-wall treatment in cell count





Average y^+ approximately 60



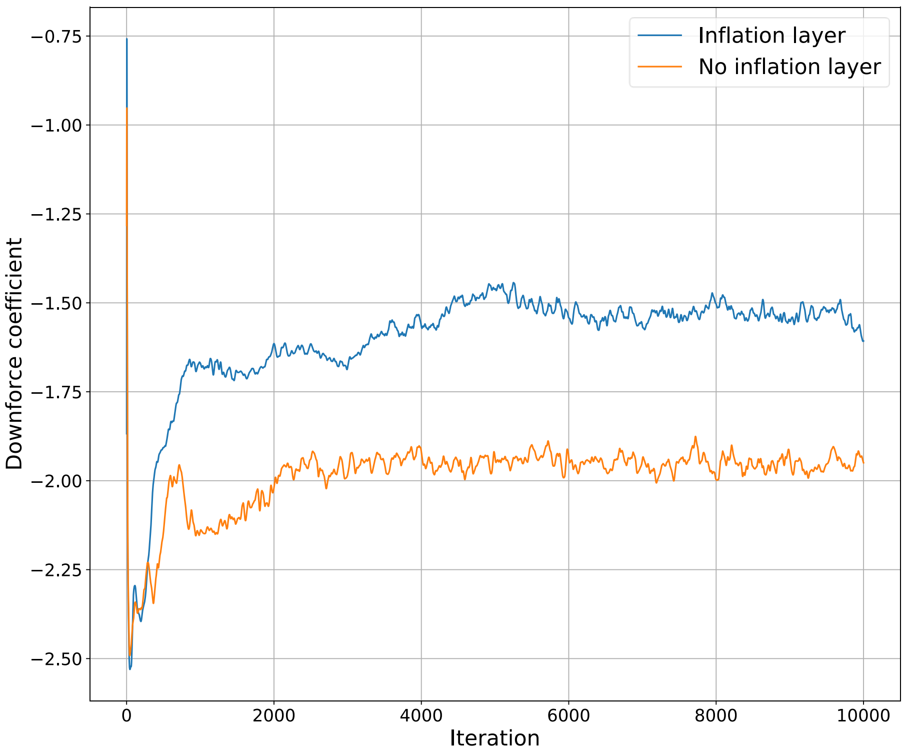


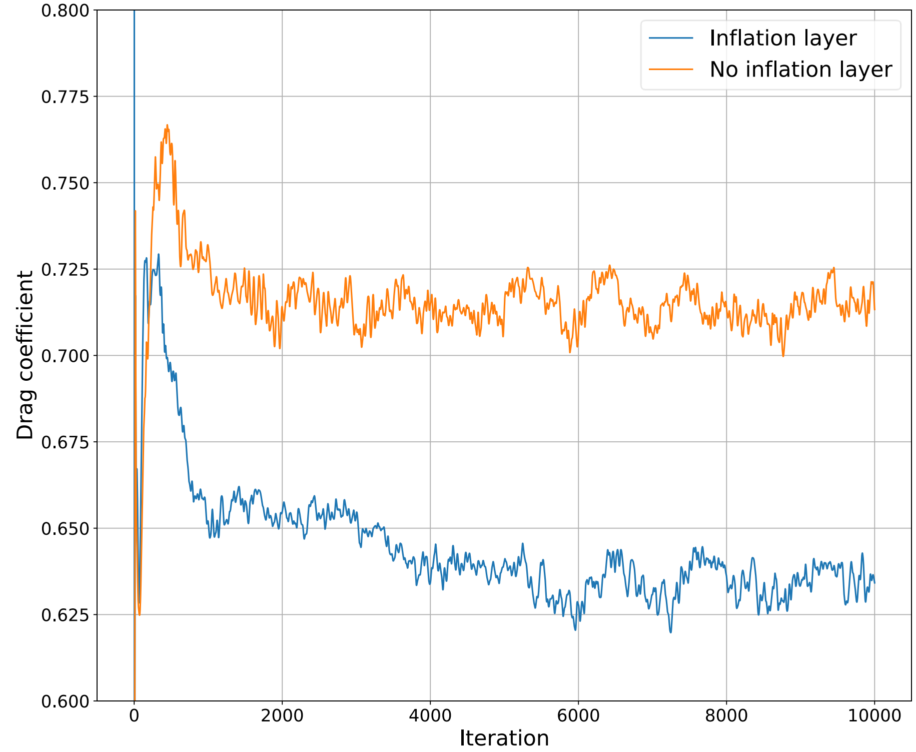
Average y^+ approximately 7

- By only adding the inflation layer to resolve the boundary layer we almost doubled the number of cells.

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count

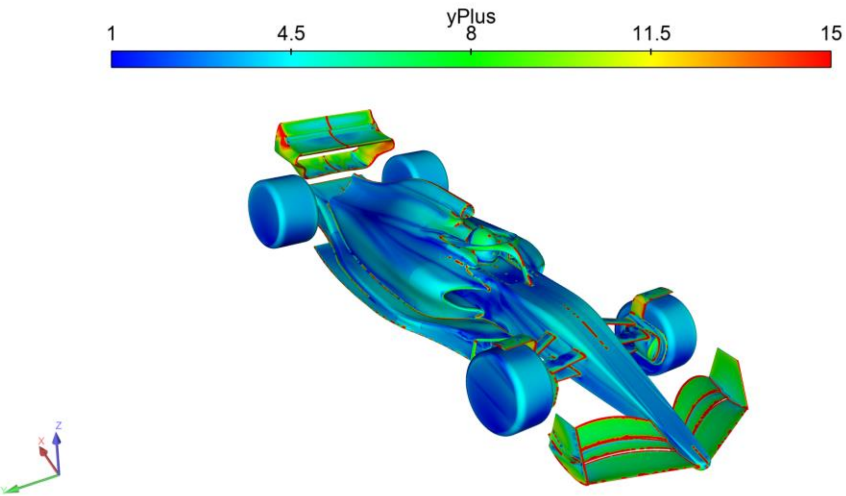




- As you can see, two different near the wall treatment give very different results.
- You should be very critical when analyzing these results.
- Not necessary the finer mesh gives the best results.

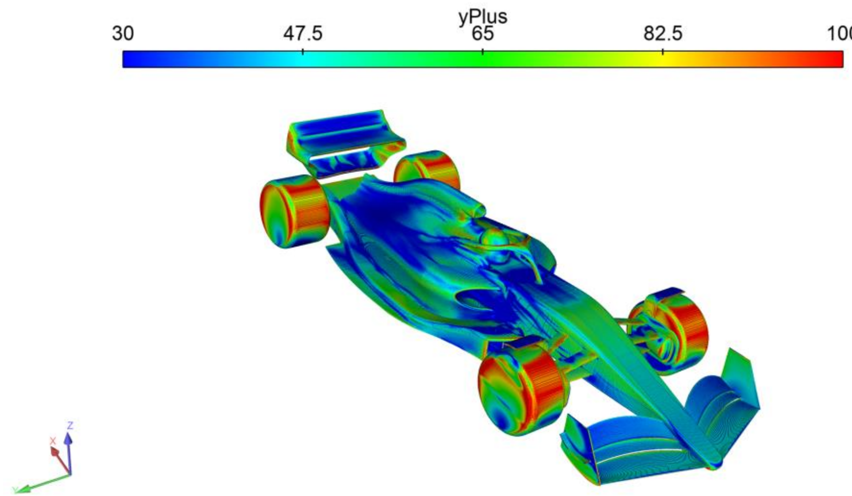
Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)

	Average y+
FWD. Wing	14
RWD. Wing	12
Body	4

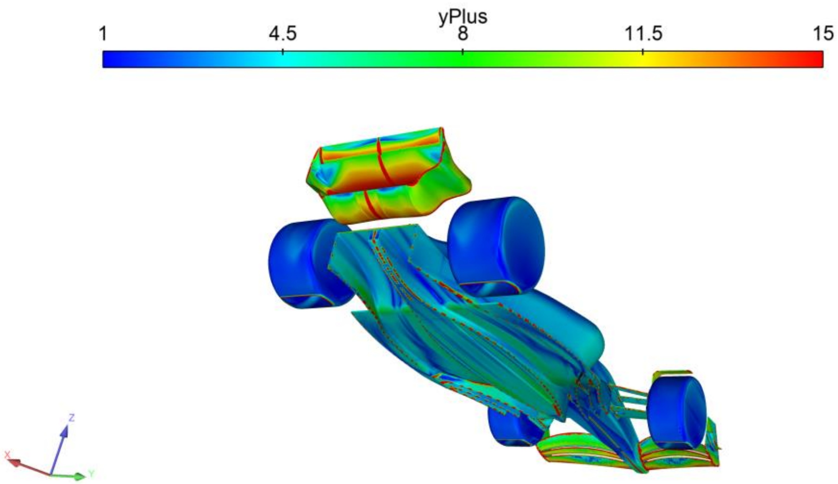


Wall modeling approach (High-RE)

	Average y+
FWD. Wing	56
RWD. Wing	62
Body	46

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)

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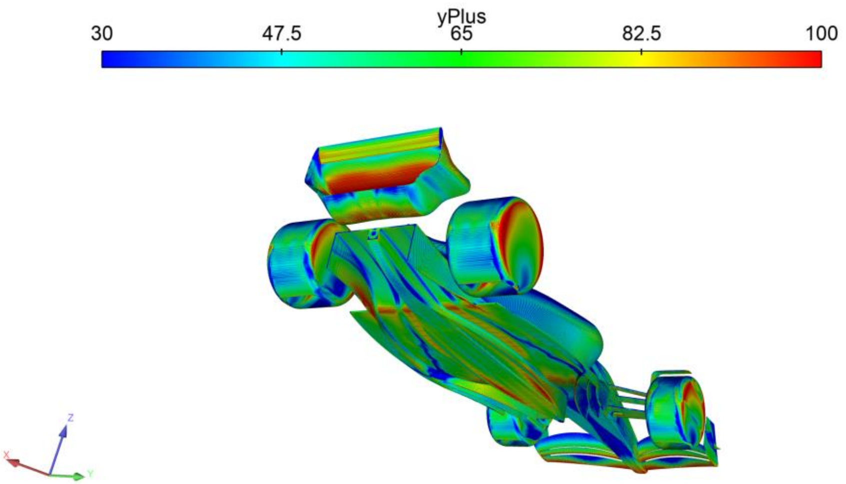

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Turbulence near the wall - Law of the wall

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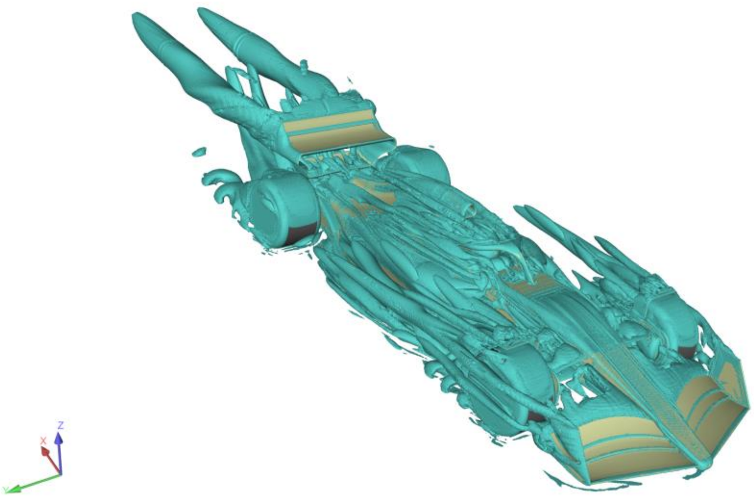
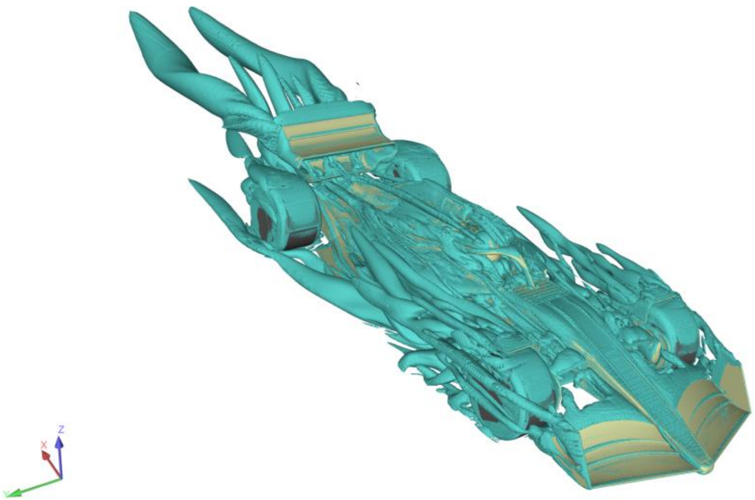


Wall resolving approach (Low-RE)

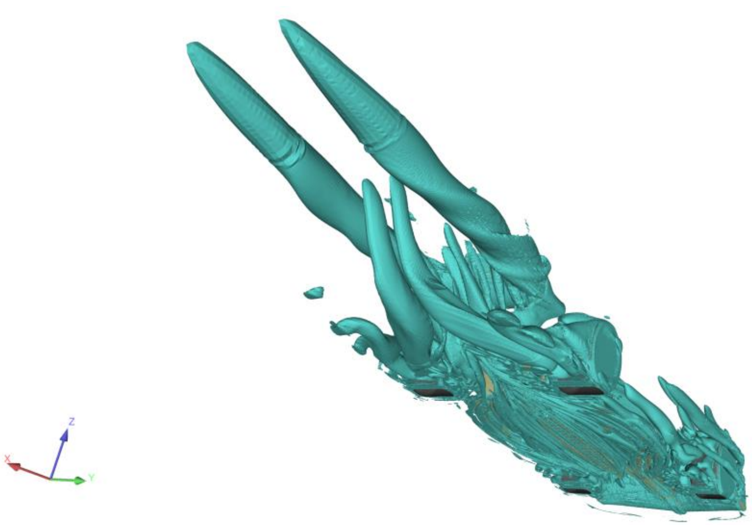


Wall modeling approach (High-RE)

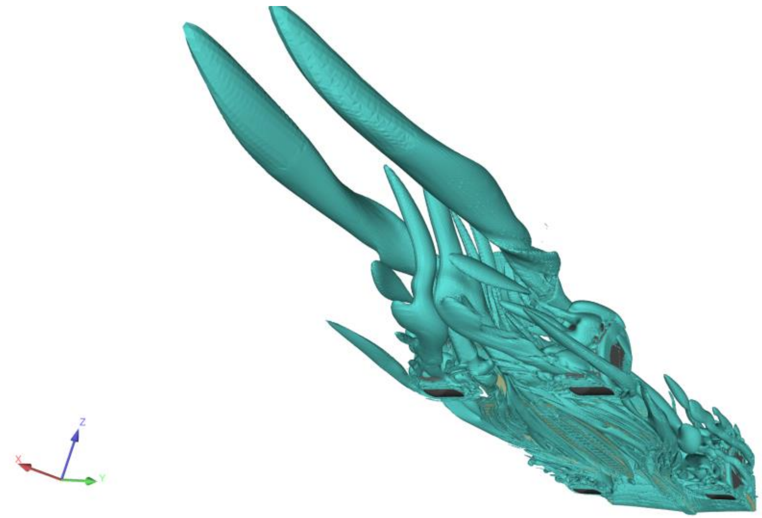
- Qualitative post-processing.
- The vortical structures are visualized using the Q-criterion.
- By the way, if you switch off the turbulence model, it is likely probably that your results will be garbage (unless you have an extremely fine mesh that resolves all scales).

Turbulence near the wall - Law of the wall

Influence of near-wall treatment in cell count



Wall resolving approach (Low-RE)



Wall modeling approach (High-RE)

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Roadmap to Lecture 3

1. ~~Turbulence modeling – Scales of turbulence From Kolmogorov scales to Taylor microscales, to integral scales~~
2. ~~Energy spectrum and energy cascade. Integral length scale and grid length scale~~
3. ~~Turbulence near the wall - Law of the wall~~
4. **A glimpse to a turbulence model**

A glimpse to a turbulence model

Turbulence modeling – Starting equations

NSE

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \dfrac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}\mathbf{u}) = -\nabla p + \nabla \cdot \tau, \\ \dfrac{\partial (\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \mathbf{u}) = \nabla \cdot q - \nabla \cdot (p \mathbf{u}) + \tau \mathbf{:} \nabla \mathbf{u}, \end{cases}$$

$$+$$

Additional closure equations for the turbulence models

- Turbulence models equations cannot be derived from fundamental principles.
- All turbulence models contain some sort of empiricism.
- Some calibration to observed physical solutions is contained in the turbulence models.
- Also, some intelligent guessing is used.
- A lot of uncertainty is involved!

A glimpse to a turbulence model

Turbulence modeling – Starting equations

- Let us write down the governing equations for an incompressible flow.
- When conducting DNS simulations (no turbulence models involved), this is our starting point,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- When using RANS turbulence models, these are the governing equations,

If we retain this term we talk about URANS equations
and if we drop it we talk about RANS equations

Reynolds stress tensor
This term requires modeling

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} - \frac{1}{\rho} \nabla \cdot \tau^R$$

A glimpse to a turbulence model

Turbulence modeling – Starting equations

- The difference between the DNS equations and RANS equations are the overbar over the primitive variables and the appearance of the Reynolds stress tensor.
- The overbar over the primitive variables in the RANS equations means that the quantities have been averaged (time average, spatial average or ensemble average).
- We will explain how to derive the RANS equations in Lecture 5.
- In the RANS equations, the Reynolds stress tensor requires modeling. Therefore, we need to define how to model this term and introduce closure equations (turbulence modeling).
- There are many turbulence models available, and none of them is universal.
- Therefore, it is essential to understand their range of applicability and limitations.

A glimpse to a turbulence model

Turbulence modeling – Starting equations

- Let us take a glimpse to a very popular turbulence model, the $k - \omega$ turbulence model [1].
- Remember, as we are introducing additional equations, we need to define boundary conditions and initial conditions for the new variables.
- These new variables, in this case, k and ω, do not have any physical meaning.
- They were introduced to model the Reynolds stress tensor (which contains the velocity fluctuations).
- In Lecture 6, we will study many turbulence models (including this one).

[1] D. Wilcox. Turbulence Modeling for CFD. DCW Industries Inc., 2010.

A glimpse to a turbulence model

$k-\omega$ Turbulence model equations

- It is called $k-\omega$ because it solves two additional equations for modeling the turbulent flow, namely,
  - The turbulent kinetic energy k .
  - The specific rate of dissipation ω.

- The closure equations of the $k-\omega$ turbulence model are,

$$\rho \frac{\partial k}{\partial t} + \rho \nabla . \left(\bar{\mathbf{u}} k \right) = \tau^R : \nabla \bar{\mathbf{u}} - \beta^* \rho k \omega + \nabla . \left[\left(\mu + \sigma^* \mu_T \right) \nabla k \right]$$

$$\rho \frac{\partial \omega}{\partial t} + \rho \nabla . \left(\bar{\mathbf{u}} \omega \right) = \alpha \frac{\omega}{k} \tau^R : \nabla \bar{\mathbf{u}} - \beta \rho \omega^2 + \nabla . \left[\left(\mu + \sigma \mu_T \right) \nabla \omega \right]$$

- These are not physical properties.
- They kind of represent the generation and destruction of turbulence.

A glimpse to a turbulence model

$k-\omega$ Turbulence model equations

- The previous equations are used to compute the turbulent viscosity μ_t.
- In the $k-\omega$ model, the turbulent viscosity is computed as follows,

$$\mu_t = \frac{\rho k}{\omega}$$

- As we have done so far, if you check the dimensional groups, you will see that this combination of variables result in viscosity units.
- The turbulent viscosity is introduced to take into account the increased mixing and shear stresses due to the turbulence.
- So at this point the question is, how do we model the Reynolds stress tensor?
- There are many methods, we will briefly outline the most widely used.

$k-\omega$ Turbulence model equations

- The most widely approach used to model the Reynolds stress tensor is to use the Boussinesq hypothesis.
- By using the Boussinesq hypothesis, we can relate the Reynolds stress tensor to the mean velocity gradient such that,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T\overline{\mathbf{D}}^R - \frac{2}{3}\rho k\mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^{\mathrm{T}}\right] - \frac{2}{3}\rho k\mathbf{I}$$

Computed from k and ω

- Each turbulence model will compute the turbulent viscosity in a different way.
- Also, recall that,

$$k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$$

A glimpse to a turbulence model

$k-\omega$ Turbulence model equations

- Let me remind you the base units of the derived quantities used in the $k-\omega$ turbulence model.

Derived quantity	Symbol	Dimensional units	SI units
Turbulent kinetic energy per unit mass	k	L^2T^{-2}	m^2/s^2
Specific dissipation rate	ω	T^{-1}	1/s
Dynamic viscosity (laminar)	μ	$ML^{-1}T^{-1}$	Kg/m-s
Dynamic viscosity (turbulent)	μ_t	$ML^{-1}T^{-1}$	Kg/m-s

- The turbulent eddy viscosity is not a fluid property, it is a property needed by the turbulence model.
- In turbulence modeling we will use many quantities that appear to be magical. To get an idea of those quantities, always check the dimensional groups.

A glimpse to a turbulence model

Overview of turbulence modeling approaches

MODEL

RANS
Reynolds-**A**veraged **N**avier-**S**tokes equations

URANS
Unsteady **R**eynolds-**A**veraged **N**avier-**S**tokes equations

SAS
Scale **A**daptive **S**imulations

DES
Detached **E**ddy **S**imulations

LES
Large **E**ddy **S**imulations

DNS
Direct **N**umerical **S**imulations

SRS
Scale-**R**esolving **S**imulations

Increasing computational cost

Increasing modelling and mathematical complexity