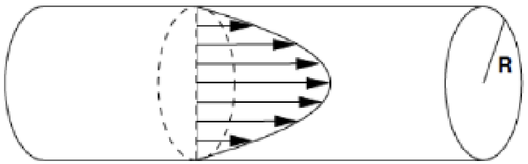
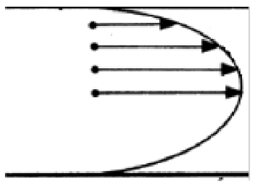
Turbulence and CFD models

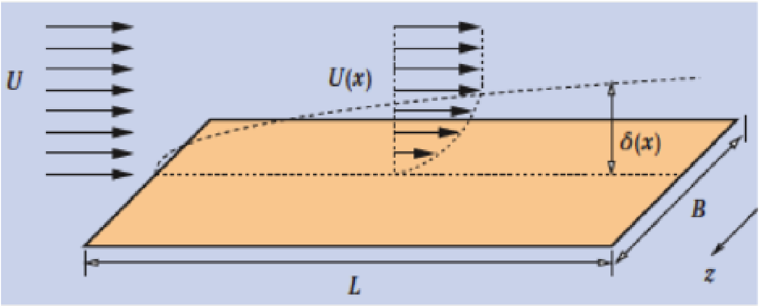
Roadmap

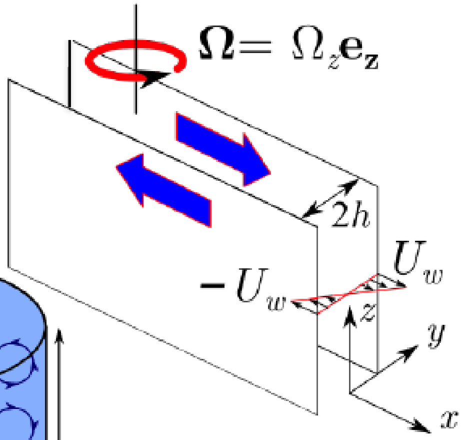
1. **Transition to turbulence in shear flows**

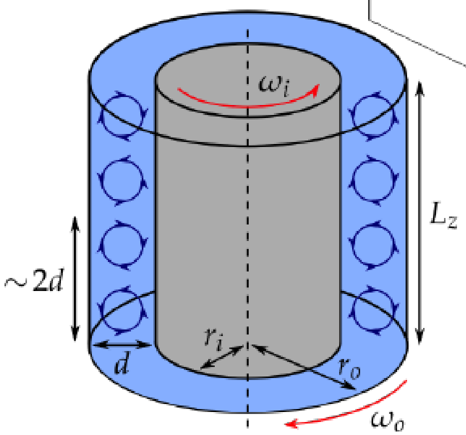
Transition to turbulence in shear flows

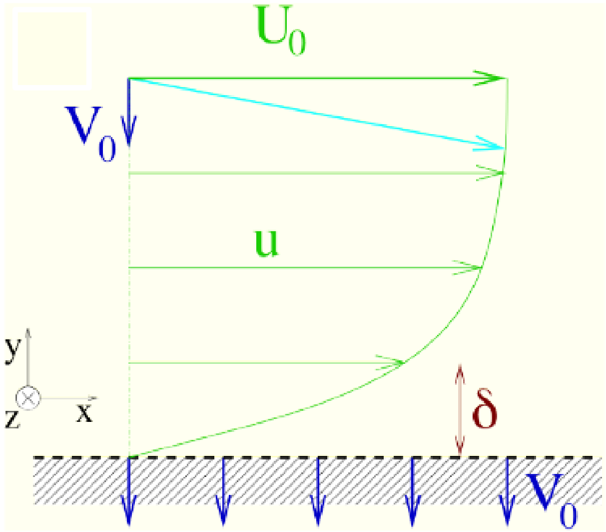












Transition to turbulence in shear flows

Motivation for transition work

Transition 1st Order Impact:

-Aerodynamic Drag and Control Authority

-Engine Performance and Operability

-Thermal Protection Requirements

-Structural Concepts and Weight

Example of Maneuvering RV:

•Heating and drag increase significantly at transition
~6X between peak turbulent and laminar heating rates

•Substructure failure due to excessive temperatures
if transition earlier than anticipated

•Added shielding mass

Motivation for transition work

Control:

Desire:

Delay transition (LFC - fuel efficiency, long range)

Encourage for enhanced mixing or separation delay

Most effective strategy:

Capitalize on the physics

Identify most unstable disturbances.

If laminar flow could be maintained on wings of transport aircraft, fuel savings of up to 25% would be obtained.

Transport aircraft drag

50% skin friction

40% of that from wings

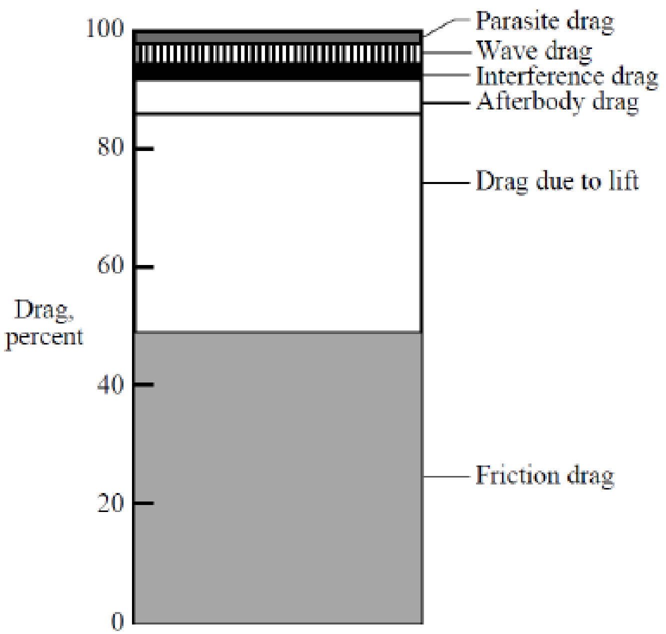
Transition to turbulence in shear flows

Motivation for transition work

Control:

Added benefits: CO_2 emission reductions and reduced operatings costs

It has been estimated (Joslin, 1998) that aircraft laminar flow control over wings, tail, nacelles, etc. can reduce DOC by a few percentage points, leading to savings of several M$/year.



Transition to turbulence in shear flows

Motivation for transition work

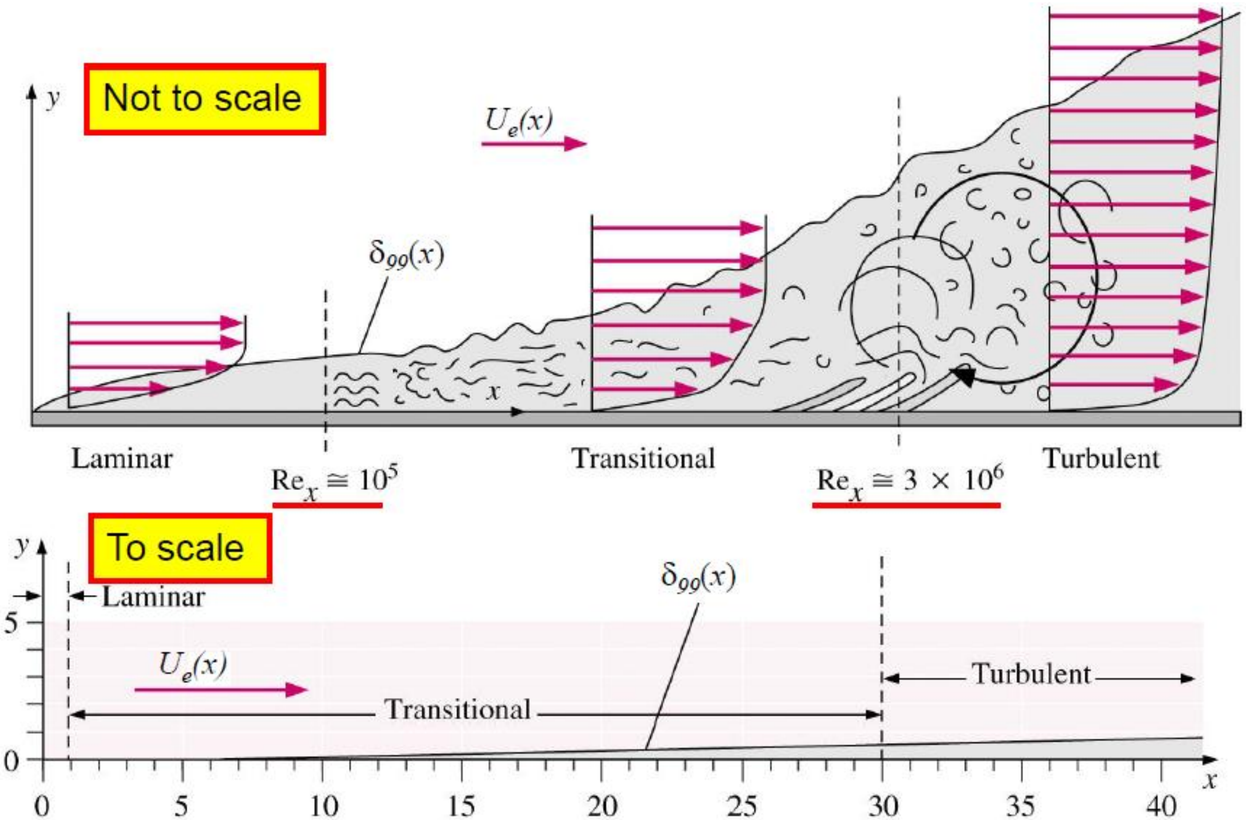
- Of interest to turbulence community, boundary-layer flows are open systems, strongly influenced by freestream and wall conditions.

- Breakdown well documented to vary considerably when operating conditions change.

- Transition process then provides vital upstream conditions from which downstream turbulent flowfield evolves. Different transition patterns give rise to different turbulence characteristics.

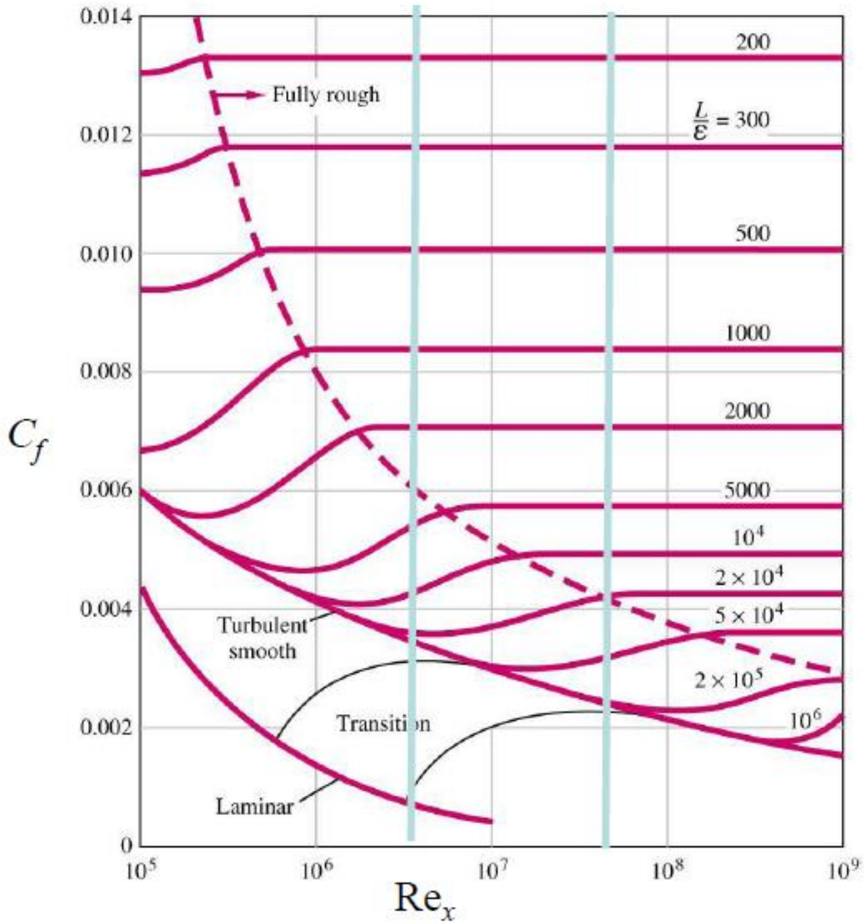
Transition to turbulence in shear flows

The usual picture



Transition to turbulence in shear flows

Effect of roughness on skin friction



$$\mathrm{Re}_x = U_\infty x / \nu$$

$$C_f = 2 \tau_w / (\rho U_\infty^2)$$

Transition to turbulence in shear flows

- When/where/why/how do instabilities start?
- Why does roughness affect skin friction?
- What kind of waves are most likely to be amplified?
- Can they be controlled (eliminated, anticipated, delayed)?
- How long does transition last?
- Once the turbulent flows sets in does it present a universal character?
- Can we control turbulence?

Transition to turbulence in shear flows

The usual methodology starts with:

- Basic State: Flow about which stability question is asked
  - Boundary layer, pipe flow, some solution of Navier-Stokes equations (**analytical**)
  - Developed in-house or commercial (**numerical**)
- Stability: Do small disturbances grow or decay in space or time?
- Procedure: Superpose small disturbances on basic state, solve

Transition to turbulence in shear flows

- Numerical accuracy of basic state must be very high, because stability and transition results very sensitive to small departures of mean flow from its “exact” shape.
- Stability of flow can depend on small variations of boundary conditions for the basic state, such as freestream velocity or wall temperature. Basic-state boundary conditions must also be very accurate.

- Example: For LFC, suction 10^{-3} to 10^{-4} U_∞
  - relative growth reduced from e^{26} to e^{5} at $F = 10 \times 10^{-6}$

($F = \omega / Re \times 10^6$ reduced frequency)

Transition to turbulence in shear flows

Environmental conditions

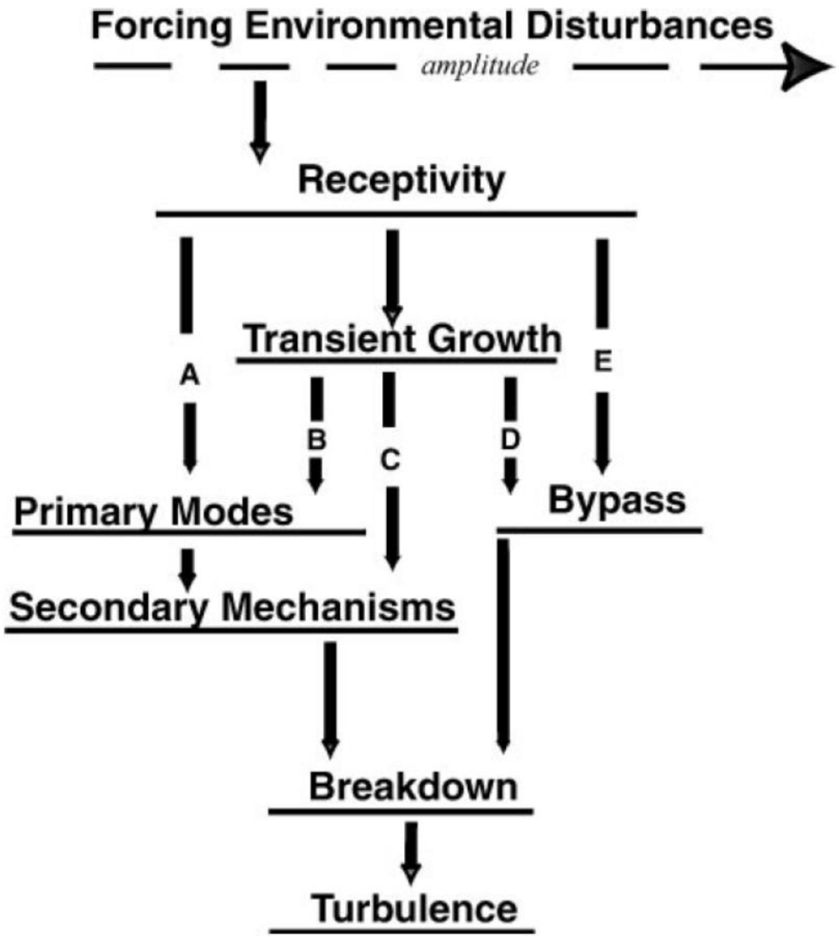
The type(s) of disturbances which grow, their self- or mutual interactions, and the amount by which perturbations are amplified (in other words, the **transition process**) depend on the forcing conditions provided by the environment.

Transition to turbulence in shear flows

Receptivity

Broadly speaking, the manner in which exogeneous perturbations [sound waves (irrotational), free-stream turbulence (rotational), leading edge curvature and/or vibrations, gusts, vortical structures, wall roughness, discontinuities in surface curvature at junction LE/flat plate, etc. …] enter the boundary layer and are filtered, eventually turning into instability waves, determines the path to turbulence, the coherent flow structures arising, the 'critical' or 'transitional' Reynolds number, the skin friction and heat transfer to/from the wall.

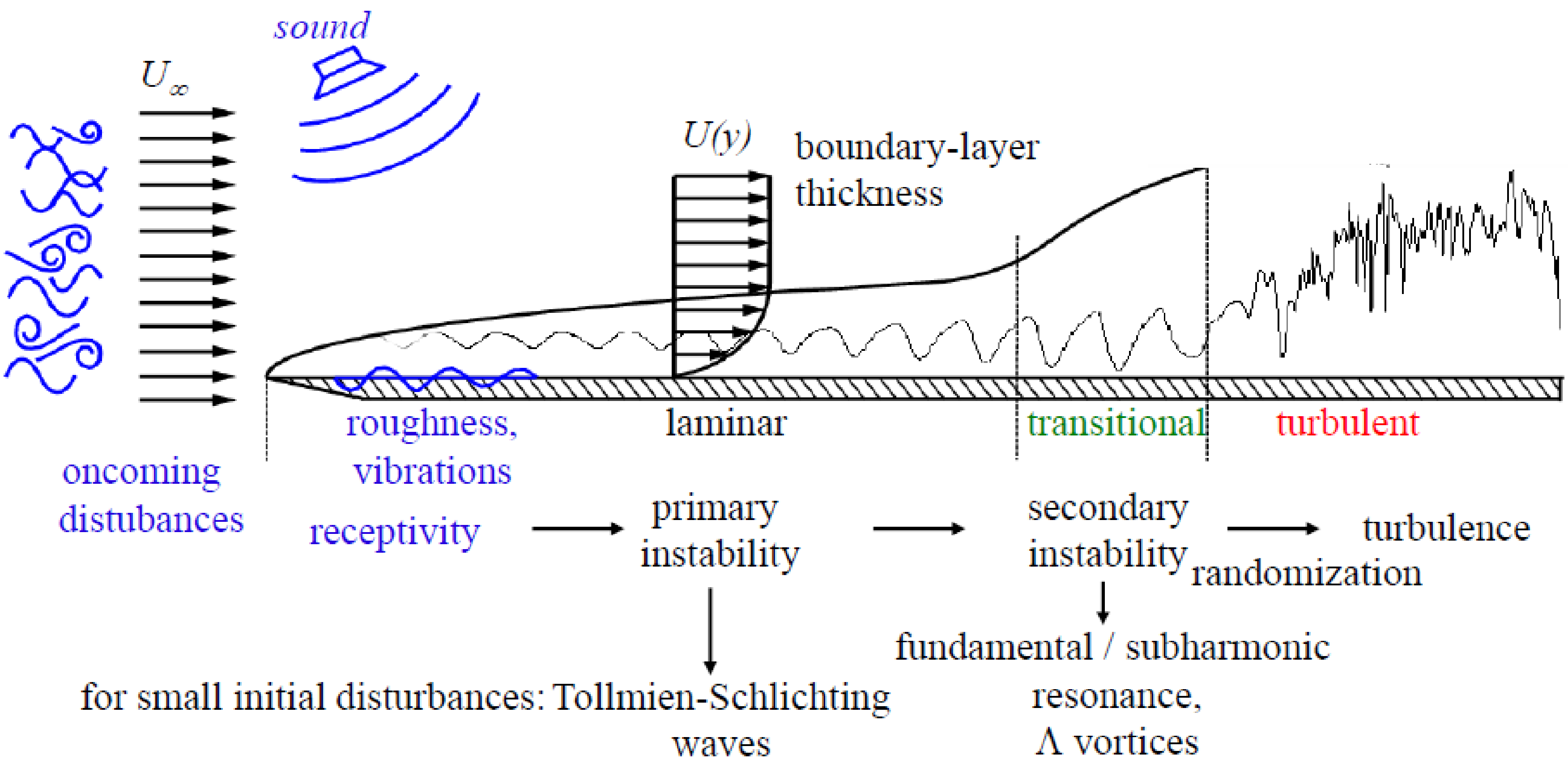
Transition to turbulence in shear flows



Morkovin, 1994

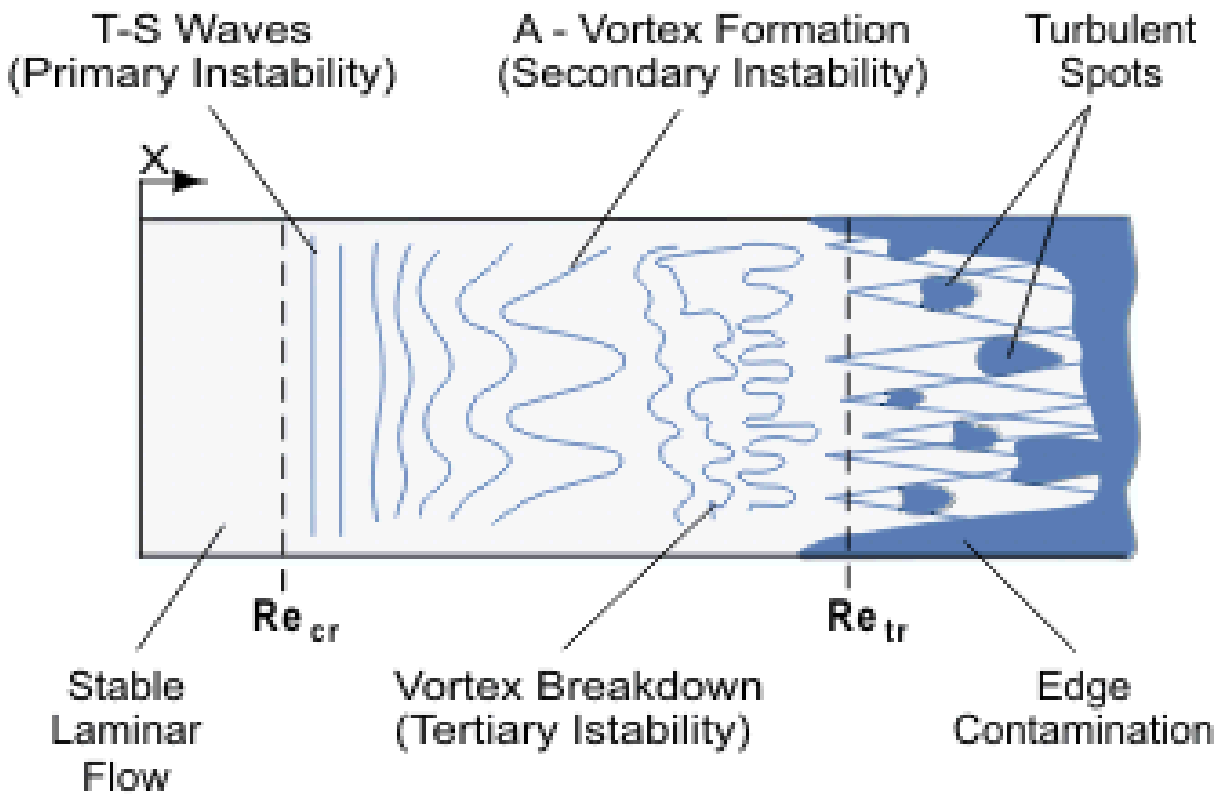
Transition to turbulence in shear flows

Standard scenario for 2D boundary layer (A)



Transition to turbulence in shear flows

Standard scenario for 2D boundary layer (A)



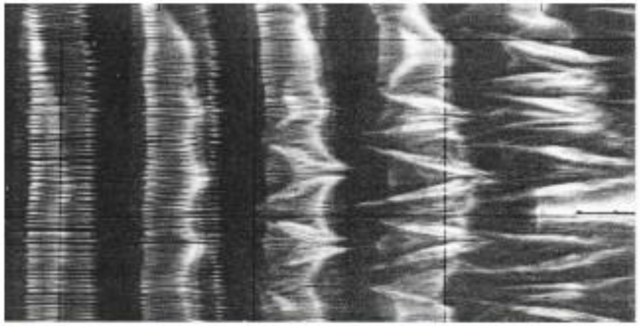
Transition to turbulence in shear flows

Walter Tollmien (1900-1968)

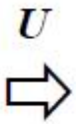
Hermann Schlichting (1907-1982)

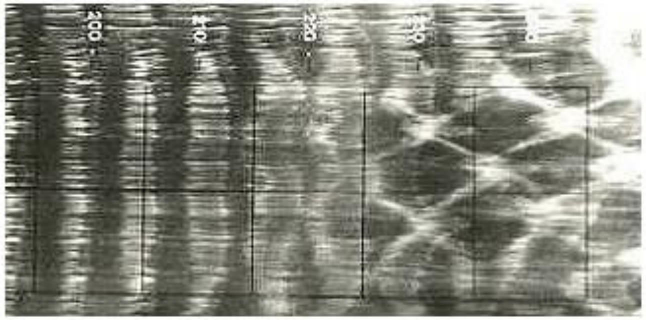
Transition to turbulence in shear flows

Experiments: smoke and laser light sheet



K-type transition

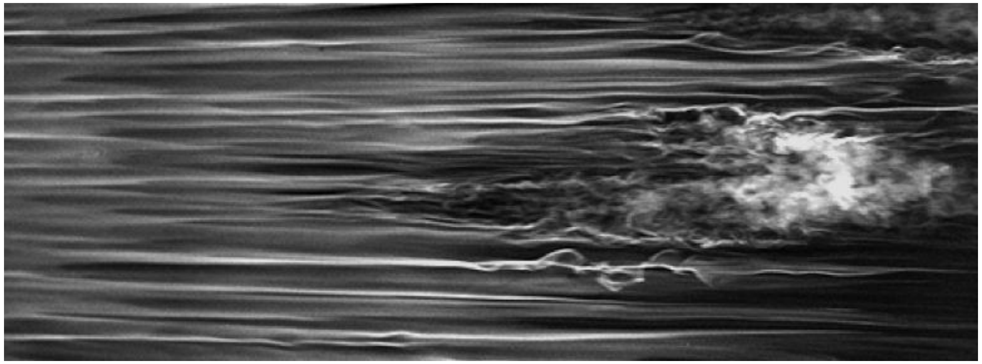




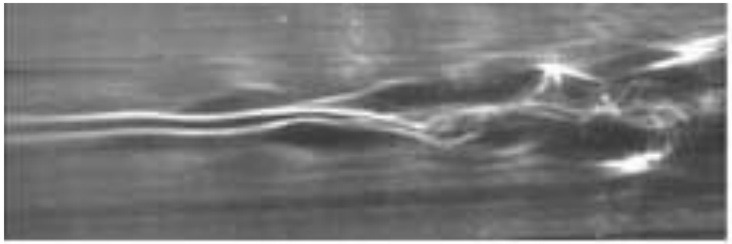
H-type transition

Streaks-induced transition

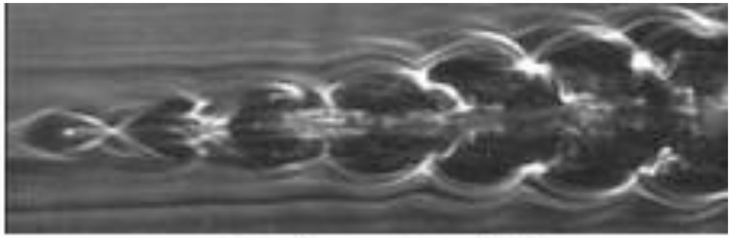
Transition to turbulence in shear flows



Turbulent spot
(Matsubara & Alfredsson 2005)

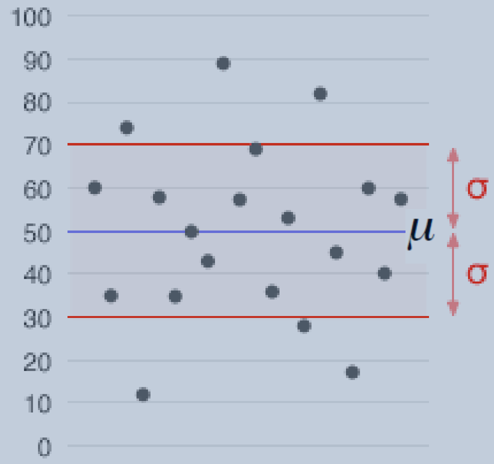


Sinuous instability



Varicose instability

Transition to turbulence in shear flows



$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}, \quad \sigma = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}}$$

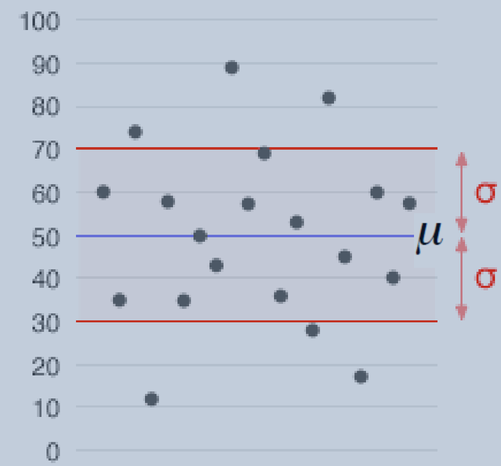
mean and rms values

Transition to turbulence in shear flows

$$u_{rms} = \sqrt{\sum_{i=1}^{n} \frac{(u_i - \overline{u})^2}{(n-1)}}$$

$$T_u = \frac{u_{rms}}{\overline{u}} \times 100,$$ turbulence intensity

- Flight conditions and few wind tunnels: $T_u <$ 0.1%
- Most wind tunnels: $T_u <$ 1%
- Turbines/compressors: $T_u >$ 10%

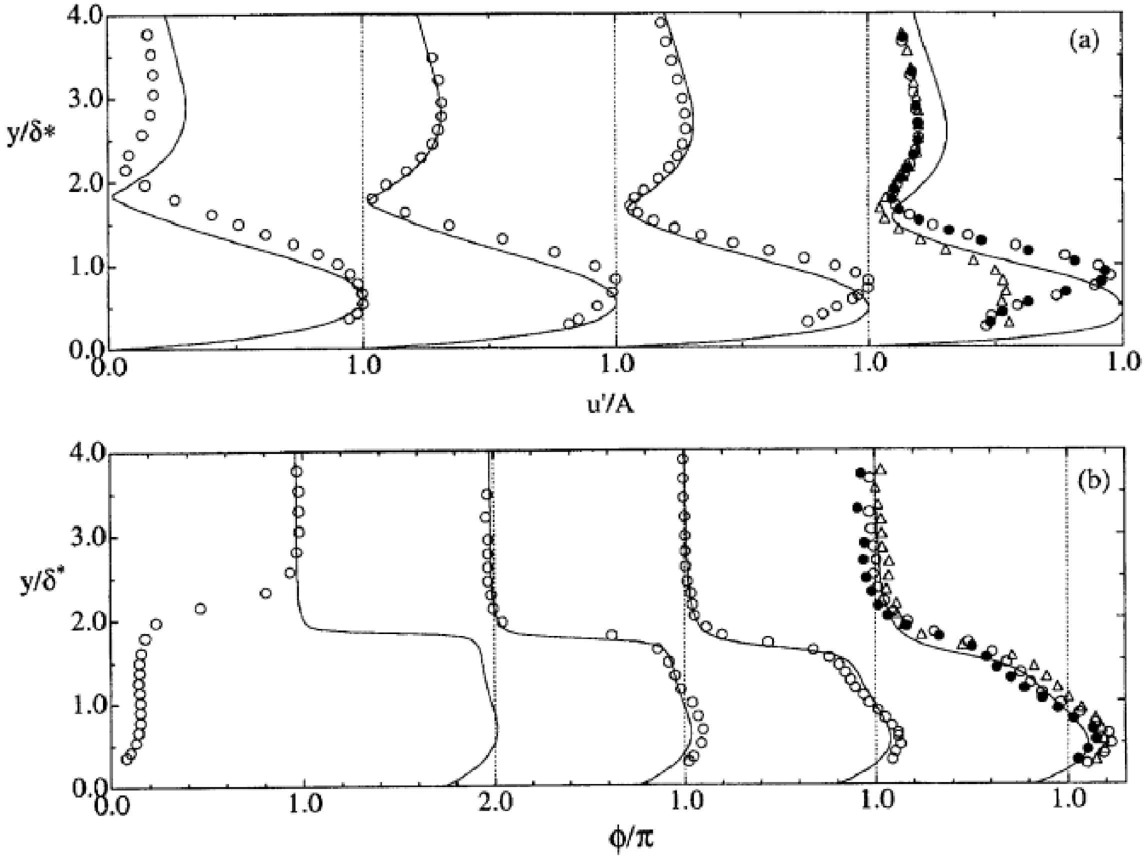


$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}, \quad \sigma = \sqrt{\frac{\sum_{i=1}^{N} (X_i - \mu)^2}{N}}$$

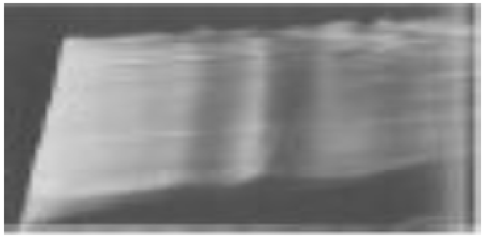
mean and rms values

Wind tunnels can give trends opposite to flight

Experiments versus theory (TS waves)



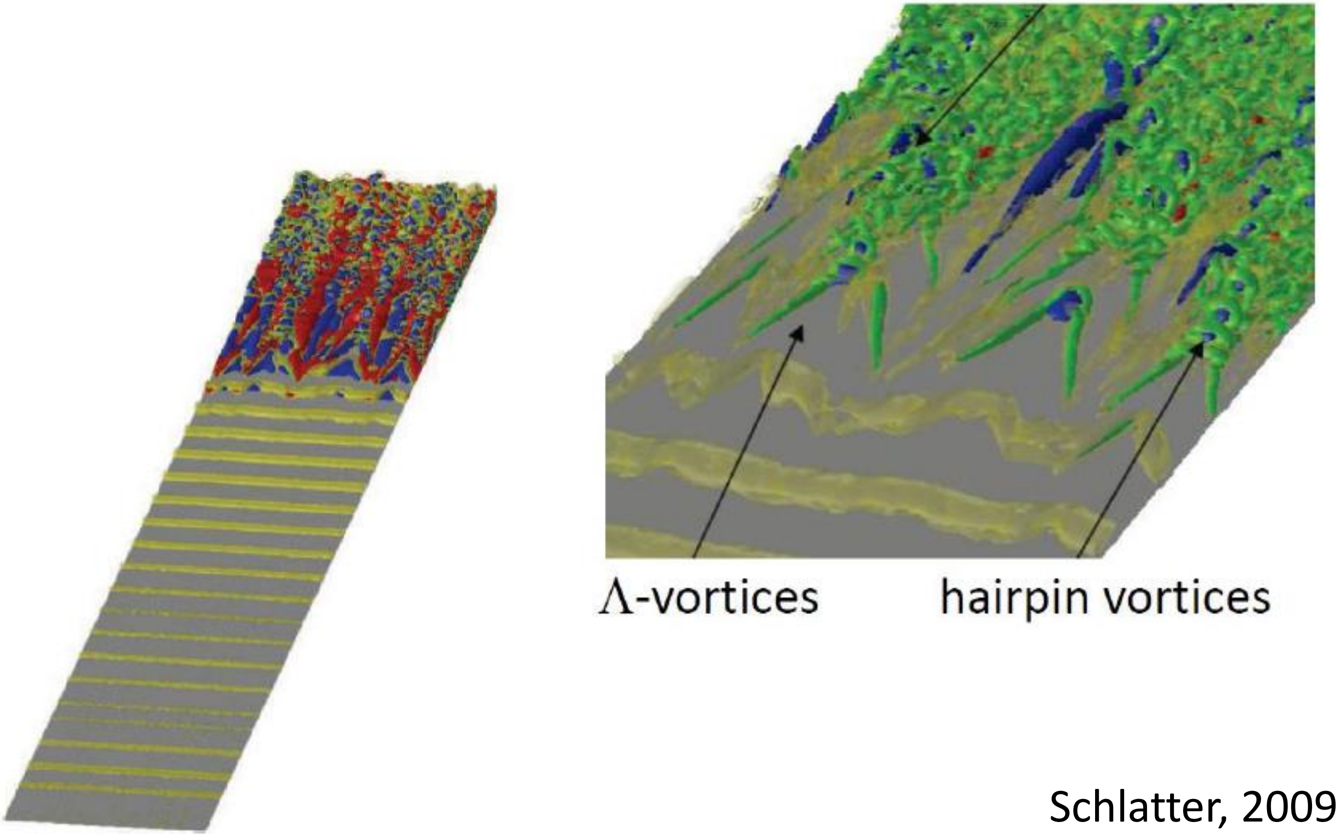
Very well-controlled experimental conditions



Bakchinov et al., 1998
(very low free stream Tu)

Transition to turbulence in shear flows

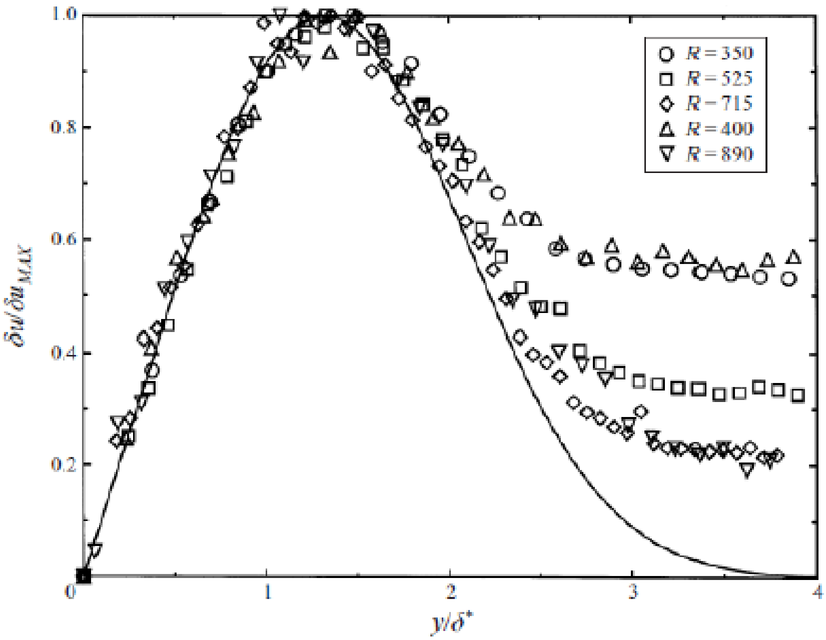
… and DNS



Schlatter, 2009

Transition to turbulence in shear flows

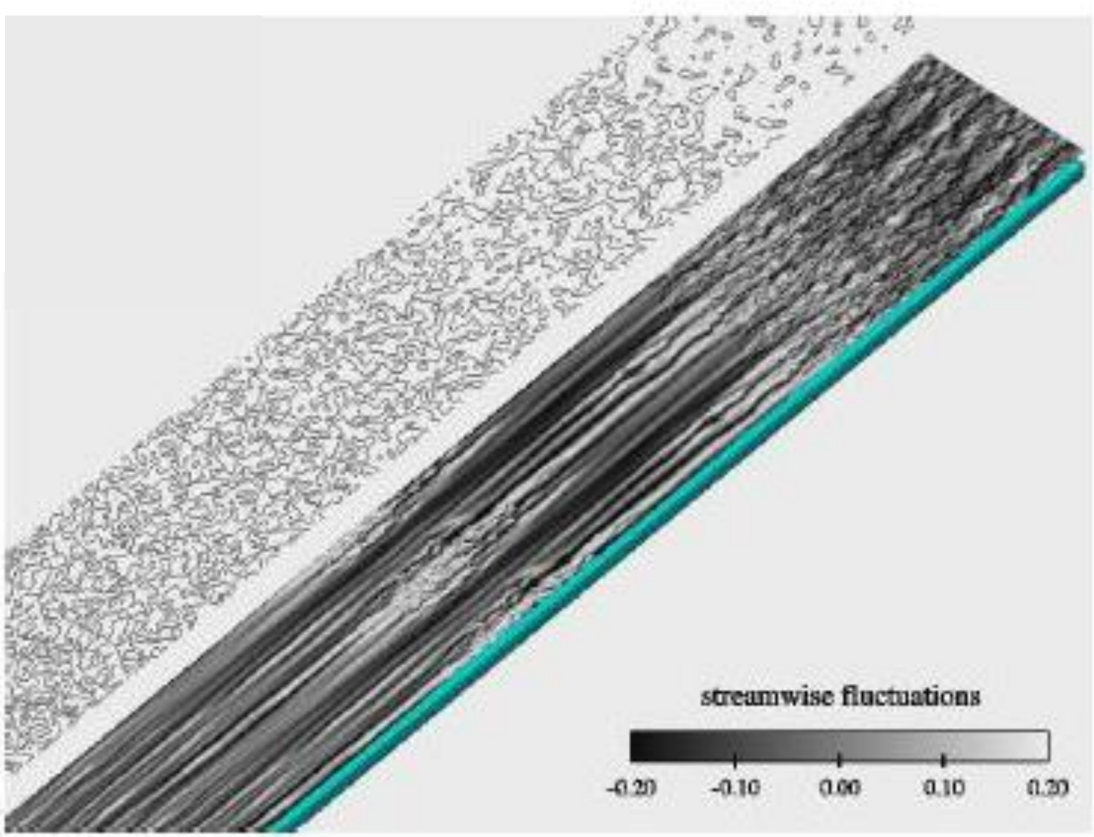
Experiments versus theory (streaks)



Luchini, 2000
(large free stream Tu)

Transition to turbulence in shear flows

… and DNS



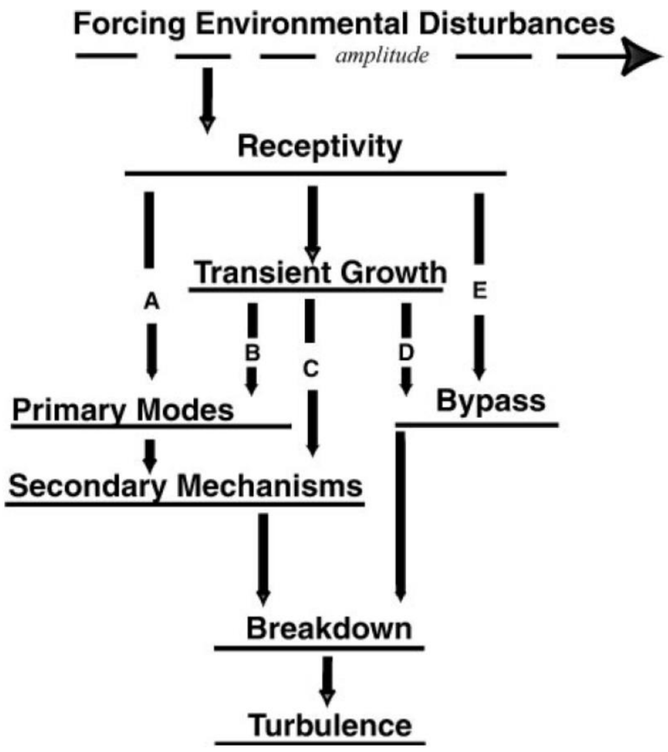
Zaki & Durbin, 2000
(large free stream Tu)

Transition to turbulence in shear flows

The initial stages of transition

Except for the cases of transition scenarios **D** or **E**, small disturbances are initially filtered and amplified; this justifies focussing on the growth of infinitesimal perturbations: the equations are thus linearized.

Nonlinear interactions acquire importance only once the amplitude of the disturbances becomes large enough.



Recap on linear matrix algebra

Generic evolution system:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}[\boldsymbol{x}(t), t; r]$$

$\boldsymbol{x}$ = state vector (N components, column vector)

$\boldsymbol{f}$ = evolution function (another N-column vector)

t = time

r = control parameter

Autonomous system: $$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}[\boldsymbol{x}(t); r]$$

Transition to turbulence in shear flows

Recap on linear matrix algebra

Statement of the problem:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}[\boldsymbol{x}(t); r]$$

Predict the characteristics of the asymptotic state ($t \to \infty$) as function of the initial conditions and the control parameter.

Note: we will see later that the behavior of the system for small times is also of importance

Transition to turbulence in shear flows

Recap on linear matrix algebra

Basic state: $\boldsymbol{x}_0$ that satisfies

$$\frac{d\boldsymbol{x}_0}{dt} = \boldsymbol{f}[\boldsymbol{x}_0; r]$$

Perturbation: $\epsilon\, \boldsymbol{x}'(t)$ (ϵ small amplitude) satisfying

$$\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{A}\, \boldsymbol{x}'(t)$$

Transition to turbulence in shear flows

Recap on linear matrix algebra

$$\boldsymbol{x} = \boldsymbol{x}_0 + \epsilon\, \boldsymbol{x}'$$

$$\frac{d\boldsymbol{x}_0}{dt} + \epsilon \frac{d\boldsymbol{x}'}{dt} = \boldsymbol{f}[\boldsymbol{x}_0 + \epsilon\, \boldsymbol{x}'] =$$

$$\boldsymbol{f}(\boldsymbol{x}_0) + \epsilon \left.\frac{\partial f}{\partial x}\right|_{\boldsymbol{x}_0} \boldsymbol{x}' + O(\epsilon^2) =$$

$$\boldsymbol{f}(\boldsymbol{x}_0) + \epsilon\, \boldsymbol{A}\, \boldsymbol{x}' + O(\epsilon^2)$$

$\boldsymbol{A}$ = Jacobian matrix of coefficients ($N\, x\, N$)

Recap on linear matrix algebra

Setting the eigenproblem

$$\boldsymbol{x}'(t) = \boldsymbol{x}'(0) + t\,\frac{d\boldsymbol{x}'}{dt}\bigg|_{t=0} + \frac{t^2}{2}\frac{d^2\boldsymbol{x}'}{dt^2}\bigg|_{t=0} + \ldots$$

$$\frac{d\boldsymbol{x}\prime}{dt} = \boldsymbol{A}\,\boldsymbol{x}', \quad \frac{d^2\boldsymbol{x}\prime}{dt^2} = \boldsymbol{A}^2\,\boldsymbol{x}', \quad \ldots \quad \frac{d^n\boldsymbol{x}\prime}{dt^n} = \boldsymbol{A}^n\,\boldsymbol{x}'$$

Recap on linear matrix algebra

Setting the eigenproblem

$$\boldsymbol{x}'(t) = \boldsymbol{x}'(0) + t\,\frac{d\boldsymbol{x}'}{dt}\bigg|_{t=0} + \frac{t^2}{2}\frac{d^2\boldsymbol{x}'}{dt^2}\bigg|_{t=0} + \dots$$

$$\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{A}\,\boldsymbol{x}', \quad \frac{d^2\boldsymbol{x}'}{dt^2} = \boldsymbol{A}^2\,\boldsymbol{x}', \quad \dots \quad \frac{d^n\boldsymbol{x}'}{dt^n} = \boldsymbol{A}^n\,\boldsymbol{x}'$$

$$\boldsymbol{x}'(t) = \boldsymbol{x}'(0) + t\,\boldsymbol{A}\,\boldsymbol{x}'(0) + \frac{t^2}{2}\boldsymbol{A}^2\,\boldsymbol{x}'(0) + \dots$$

Recap on linear matrix algebra

Setting the eigenproblem

$$\boldsymbol{x}'(t) = \boldsymbol{x}'(0) + t\,\boldsymbol{A}\,\boldsymbol{x}'(0) + \frac{t^2}{2}\boldsymbol{A}^2\,\boldsymbol{x}'(0) + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{t^n \boldsymbol{A}^n}{n!}\boldsymbol{x}'(0)$$

Definition of the analytic function of a matrix:

$$e^{\boldsymbol{A}t} = \sum_{n=0}^{\infty} \frac{(t\boldsymbol{A})^n}{n!}$$

Recap on linear matrix algebra

Setting the eigenproblem

The solution of our disturbance problem is thus:

$$\boldsymbol{x}'(t) = e^{\boldsymbol{A}t}\, \boldsymbol{x}'(0)$$

and to assess the stability of the system it is useful to decompose the matrix $\boldsymbol{A}$ in the sum of products of left and right eigenvectors

Recap on linear matrix algebra

Setting the eigenproblem

The N eigenvalues of the matrix $\boldsymbol{A}$ are the solutions λ_k of the characteristic equation

$$\det\left(\boldsymbol{A} - \lambda_k \boldsymbol{I}\right) = 0$$

The right eigenvectors $\boldsymbol{u}_k$ are non-trivial solutions, defined up to an arbitrary factor, of the system:

$$\boldsymbol{A}\,\boldsymbol{u}_k = \lambda_k\,\boldsymbol{u}_k$$

Recap on linear matrix algebra

Setting the eigenproblem

The left eigenvectors $\boldsymbol{v}_k$ are non-trivial solutions, defined up to an arbitrary factor, of the system:

$$\boldsymbol{v}_k^T\,\overline{\boldsymbol{A}} = \overline{\lambda}_k\,\boldsymbol{v}_k^T$$

Note: the left eigenvectors of $\boldsymbol{A}$ are also the right eigenvectors of the conjugate transpose of $\boldsymbol{A}$

$$\overline{\boldsymbol{A}}^T\boldsymbol{v}_k = \overline{\lambda}_k\boldsymbol{v}_k$$

Recap on linear matrix algebra

Definition of the scalar product between (in general complex) vectors:

$$(\boldsymbol{u}_k, \boldsymbol{v}_k) \equiv \overline{\boldsymbol{u}_k}^T \boldsymbol{v}_k$$

Recap on linear matrix algebra

Definition of the scalar product between (in general complex) vectors:

$$(\boldsymbol{u}_k, \boldsymbol{v}_k) \equiv \overline{\boldsymbol{u}_k}^T \boldsymbol{v}_k$$

Definition of the adjoint matrix:

$$(\boldsymbol{A}\boldsymbol{u}, \boldsymbol{v}) = \overline{\boldsymbol{A}\boldsymbol{u}}^T \boldsymbol{v} = \overline{\boldsymbol{u}}^T \overline{\boldsymbol{A}}^T \boldsymbol{v} = (\boldsymbol{u}, \overline{\boldsymbol{A}}^T \boldsymbol{v})$$

$$\boldsymbol{A}^{\dagger} \equiv \overline{\boldsymbol{A}}^T$$

Recap on linear matrix algebra

Adjoint operators/matrices are important in many areas, including

- hydrodynamic stability, receptivity, sensitivity
- optimal and robust control theory
- optimal shape design
- inverse design
- data assimilation

Recap on linear matrix algebra

If $\boldsymbol{A}^{\dagger} = \boldsymbol{A}$, the matrix $\boldsymbol{A}$ is self-adjoint

In this case the matrix is a *real*, *symmetric* matrix, its eigenvalues are real and the eigenvectors form an orthogonal basis. Furthermore, left and right eigenvectors coincide.

A non-self-adjoint matrix has, in general, complex eigenvalues, plus its conjugates.

Recap on linear matrix algebra

Property of orthogonality among eigenvectors

$$(\boldsymbol{v}_h, \boldsymbol{A}\,\boldsymbol{u}_k) = (\boldsymbol{v}_h, \lambda_k \boldsymbol{u}_k) = \lambda_k\,(\boldsymbol{v}_h, \boldsymbol{u}_k)$$

$$\parallel$$

$$(\overline{\boldsymbol{A}}^T \boldsymbol{v}_h, \boldsymbol{u}_k) = \left(\overline{\lambda}_h \boldsymbol{v}_h, \boldsymbol{u}_k\right) = \lambda_h(\boldsymbol{v}_h, \boldsymbol{u}_k)$$

Thus $(\lambda_k - \lambda_h)\,(\boldsymbol{v}_h, \boldsymbol{u}_k) = 0$, or $(\boldsymbol{v}_h, \boldsymbol{u}_k) = a\,\delta_{hk}$

a is some amplitude coefficient; if $a = 1$ left and right eigenvectors are *orthonormalized*

Recap on linear matrix algebra

Let us imagine that the N eigenvalues are distinct and the eigenvectors are linearly independent (so as to form a basis); at $t = 0$ we have

$$\boldsymbol{x}'(0) = \sum_{k=1}^{N} \boldsymbol{u}_k \, c_k$$

Recap on linear matrix algebra

Let us imagine that the N eigenvalues are distinct and the eigenvectors are linearly independent (so as to form a basis): at *t = 0* we have

$$\boldsymbol{x}'(0) = \boldsymbol{x}'_0 = \sum_{k=1}^{N} \boldsymbol{u}_k \, c_k$$

$$(\boldsymbol{v}_h, \boldsymbol{x}'_0) = \sum_{k=1}^{N} (\boldsymbol{v}_h, \boldsymbol{u}_k \, c_k) = \sum_{k=1}^{N} c_k (\boldsymbol{v}_h, \boldsymbol{u}_k) = c_h (\boldsymbol{v}_h, \boldsymbol{u}_h)$$

$$c_h = \frac{(\boldsymbol{v}_h, \boldsymbol{x}'_0)}{(\boldsymbol{v}_h, \boldsymbol{u}_h)}$$

Recap on linear matrix algebra

Let us assume that the eigenvectors are orthonormalized $\rightarrow$ $c_k = (\boldsymbol{v}_k, \boldsymbol{x}'_0)$, then

$$\boldsymbol{x}'_0 = \sum_{k=1}^{N} \boldsymbol{u}_k \, (\boldsymbol{v}_k, \boldsymbol{x}'_0) = \sum_{k=1}^{N} \boldsymbol{u}_k \, \overline{\boldsymbol{v}_k}^T \boldsymbol{x}'_0 = \boldsymbol{I} \, \boldsymbol{x}'_0$$

i.e. given that $\boldsymbol{x}'_0$ is *any* vector, the identity matrix can be retrieved from $\boldsymbol{I} = \sum_{k=1}^{N} \boldsymbol{u}_k \, \overline{\boldsymbol{v}_k}^T$

Transition to turbulence in shear flows

Recap on linear matrix algebra

The matrix $\boldsymbol{A}$ can thus be represented as

$$\boldsymbol{A} = \boldsymbol{A}\,\boldsymbol{I} = \sum_{k=1}^{N} \boldsymbol{A}\,\boldsymbol{u}_k\ \overline{\boldsymbol{v}_k}^T = \sum_{k=1}^{N} \lambda_k\,\boldsymbol{u}_k\ \overline{\boldsymbol{v}_k}^T$$

i.e. $\boldsymbol{A}$ can be written as the sum of the product of eigenvalues and eigenvectors (of course, under the assumption that eigenvalues are distinct, and eigenvectors are linearly independent)

Recap on linear matrix algebra

Let U be the matrix whose columns are the N right eigenvectors and V the matrix with the N left eigenvectors in the colums; assume eigenvalues to be distinct.

Recap on linear matrix algebra

Let $\boldsymbol{U}$ be the matrix whose columns are the N right eigenvectors and $\boldsymbol{V}$ the matrix with the N left eigenvectors in the colums; assume eigenvalues to be distinct. When e-vectors are orthonormal:

$$(\boldsymbol{v}_h, \boldsymbol{u}_k) = \delta_{hk} \rightarrow \overline{\boldsymbol{v}_h}^T \boldsymbol{u}_k = \delta_{hk} \rightarrow \overline{\boldsymbol{V}}^T \boldsymbol{U} = \boldsymbol{I}$$

$$\overline{\boldsymbol{v}_h}^T \boldsymbol{A}\, \boldsymbol{u}_k = \lambda_k \overline{\boldsymbol{v}_h}^T \boldsymbol{u}_k = \lambda_k \delta_{hk}$$

$$\rightarrow \quad \overline{\boldsymbol{V}}^T \boldsymbol{A}\, \boldsymbol{U} = \boldsymbol{\Lambda} \text{ (the diagonal matrix of e−values)}$$

Recap on linear matrix algebra

$$\overline{\boldsymbol{V}}^T \boldsymbol{A}\,\boldsymbol{U} = \boldsymbol{\Lambda} \;\rightarrow\; \boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\,\overline{\boldsymbol{V}}^T$$

for the problem $\dfrac{d\boldsymbol{x}'}{dt} = \boldsymbol{A}\,\boldsymbol{x}'(t)$ let us take

$$\boldsymbol{x}' = \boldsymbol{U}\,\boldsymbol{q}(t), \qquad \frac{d\boldsymbol{x}'}{dt} = \boldsymbol{U}\dot{\boldsymbol{q}}, \qquad \boldsymbol{U}\,\dot{\boldsymbol{q}} = \boldsymbol{A}\,\boldsymbol{U}\,\boldsymbol{q},$$

$$\dot{\boldsymbol{q}} = \boldsymbol{U}^{-1}\boldsymbol{A}\,\boldsymbol{U}\,\boldsymbol{q}, \quad \dot{\boldsymbol{q}} = \boldsymbol{\Lambda}\,\boldsymbol{q} \;\rightarrow\; \boldsymbol{q}(t) = e^{\boldsymbol{\Lambda} t}\boldsymbol{q}(0)$$

Recap on linear matrix algebra

$$\boldsymbol{q}(t) = \boldsymbol{U}^{-1}\boldsymbol{x}' = e^{\boldsymbol{\Lambda}t}\,\boldsymbol{q}(0) = e^{\boldsymbol{\Lambda}t}\,\boldsymbol{U}^{-1}\,\boldsymbol{x}'(0)$$

so that the solution of $$\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{A}\,\boldsymbol{x}'(t)$$

is $$\boxed{\boldsymbol{x}' = \boldsymbol{U}\,e^{\boldsymbol{\Lambda}t}\,\boldsymbol{U}^{-1}\,\boldsymbol{x}'(0) = \boldsymbol{U}\,e^{\boldsymbol{\Lambda}t}\,\overline{\boldsymbol{V}}^{T}\,\boldsymbol{x}'(0)}$$

$$\boldsymbol{x}' = \boldsymbol{L}\,\boldsymbol{x}'(0)$$

propagator of the initial condition

Recap on linear matrix algebra

$$\boldsymbol{A} = \sum_{k=1}^{N} \lambda_k \boldsymbol{u}_k \ \overline{\boldsymbol{v}_k}^T$$

$$\boldsymbol{A}^2 = \ \boldsymbol{A}\boldsymbol{A} = \sum_{k=1}^{N} \lambda_k \boldsymbol{u}_k \ \overline{\boldsymbol{v}_k}^T \sum_{h=1}^{N} \lambda_h \boldsymbol{u}_h \ \overline{\boldsymbol{v}_h}^T =$$

$$\sum_{k=1}^{N} \sum_{h=1}^{N} \lambda_k \lambda_h \boldsymbol{u}_k \overline{\boldsymbol{v}_k}^T \boldsymbol{u}_h \ \overline{\boldsymbol{v}_h}^T =$$

$$\sum_{k=1}^{N} \sum_{h=1}^{N} \lambda_k \lambda_h \boldsymbol{u}_k \delta_{hk} \overline{\boldsymbol{v}_h}^T =$$

$$\sum_{h=1}^{N} \lambda_h^{\boldsymbol{2}} \ \boldsymbol{u}_h \overline{\boldsymbol{v}_h}^T$$

Recap on linear matrix algebra

$$\boldsymbol{A}^n = \sum_{k=1}^{N} \lambda_k^n \, \boldsymbol{u}_k \, \overline{\boldsymbol{v}_k}^T$$

The matrix $\boldsymbol{A}^n$ has the same eigenvectors as $\boldsymbol{A}$, and eigenvalues which are λ_k^n. In general, for a linear combination g of powers of $\boldsymbol{A}$

$$g(\boldsymbol{A}) = \sum_{k=1}^{N} g(\lambda_k) \, \boldsymbol{u}_k \, \overline{\boldsymbol{v}_k}^T$$

Recap on linear matrix algebra

and in particular

$$e^{\boldsymbol{A}} = \sum_{k=1}^{N} e^{\lambda_k} \boldsymbol{u}_k \, \overline{\boldsymbol{v}_k}^T$$

The solution of our linear problem reads also:

$$\boxed{\boldsymbol{x}'(t) = e^{\boldsymbol{A}t} \, \boldsymbol{x}'(0) = \sum_{k=1}^{N} e^{\lambda_k t} \boldsymbol{u}_k \; (\boldsymbol{v}_k, \boldsymbol{x}'(0))}$$

with the left eigenvectors weighting the initial condition

Transition to turbulence in shear flows

Stability conditions

$$\boldsymbol{x}'(t) = \sum_{k=1}^{N} e^{\lambda_k t}\, \boldsymbol{u}_k\, c_k$$

the eigenvalues λ_k define the asymptotic growth/decay of the disturbance.

Should there be a *double eigenvalue,* terms of the form

$$t\, e^{\lambda_k t}$$

would appear in the expansion of the solution (resulting in a linear time growth of the disturbance even when *Re*$(\lambda_k) < 0$, for all k).

Stability conditions

An autonomous system, with evolution matrix $\boldsymbol{A}$ equipped with N distict eigenvalues is:

- **Asymptotically stable** is all eigenvalues of A have negative real part
- **Marginally stable** if one (or more) eigenvalues have real part equal to zero (and the others have negative real part)
- **Unstable** if at least one eigenvalue has real part larger than zero

Transition to turbulence in shear flows

Stability conditions

Near the marginal stability conditions, typically a single (1!) eigenvalue crosses the stability boundary, i.e. for $t \rightarrow \infty$ we have

$$\boldsymbol{x}'(t) \sim e^{\lambda_1 t}\, \boldsymbol{u}_1\, (\boldsymbol{v}_1, \boldsymbol{x}(0)) = e^{\lambda_1 t}\, \boldsymbol{u}_1\, c_1$$

all other modes ($k = 2, 3, 4, \dots N$) being damped.

Stability conditions

$$\lambda_k = \sigma_k + i\,\omega_k$$

growth rate   angular frequency

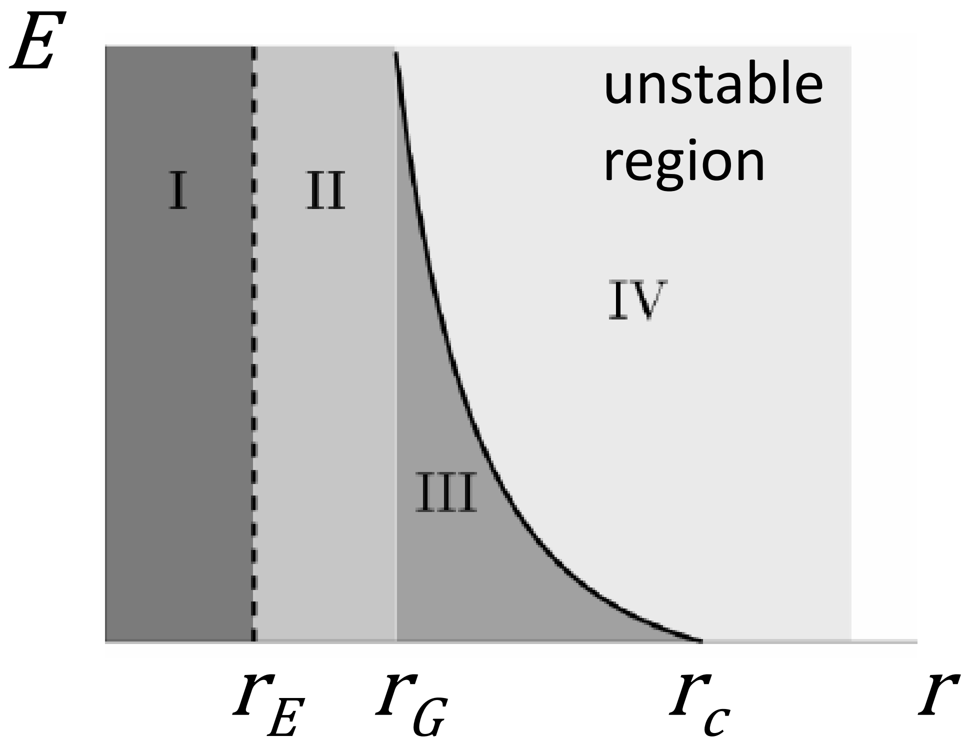
The eigenvalue problem is

$$\boldsymbol{A}\,\boldsymbol{u}_k = \lambda_k\,\boldsymbol{u}_k$$

and in hydrodynamic stability analysis the $\boldsymbol{u}_k$'s are called *normal modes*

Stability conditions

$$E(t) = (\boldsymbol{x}', \boldsymbol{x}')$$



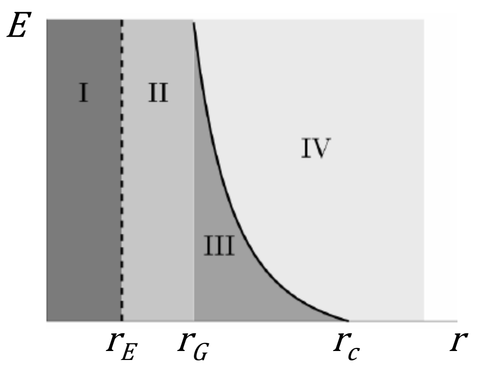
Stability conditions

Stable $\lim_{t\to\infty} \frac{E(t)}{E(0)} \to 0$

Conditionally stable (III) $\exists\ \delta > 0 :\ E(0) < \delta \Rightarrow \text{stable}$

Globally stable (II) Conditionally stable with $\delta \to \infty$

Monotonically stable (I) Globally stable and $\frac{dE}{dt} \le 0 \quad \forall t > 0$



A simple example

Consider the very simple linear system (***A*** in the chosen example is called a Jordan block):

$$\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{A}\,\boldsymbol{x}'(t) \qquad\qquad \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

A has a double eigenvalue ($\lambda_1 = \lambda_2 = 0$) to which is associated the double eigenvector $\boldsymbol{u}_1 = \boldsymbol{u}_2 = \binom{1}{0}$

Example

If $\quad \boldsymbol{x}'(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$ is the initial condition (at $t = 0$)

then the solution is

$$\boldsymbol{x}' = \begin{pmatrix} x_{10} + t\, x_{20} \\ x_{20} \end{pmatrix}$$

i.e. the disturbance vector grows linearly in time (*algebraic growth*), despite the fact that the eigenvalues have vanishing real part.

Example

Consider now the *perturbed* system with

$$\boldsymbol{A} = \begin{pmatrix} -\epsilon & 1 \\ 0 & -2\epsilon \end{pmatrix}, \qquad 0 < \epsilon \ll 1$$

eigenvalues: $\lambda_1 = -\epsilon\,,\qquad \lambda_2 = -2\epsilon$

e-vectors: $\boldsymbol{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 1 \\ -\epsilon \end{pmatrix}, \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1/\epsilon \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ -1/\epsilon \end{pmatrix}$

and the solution is: $\boldsymbol{x}' = c_1\ e^{-\epsilon t}\,\boldsymbol{u}_1\ +\ c_2\ e^{-2\epsilon t}\,\boldsymbol{u}_2$

Example

The question is: how do the two solutions (linear growth and exponential decrease) match as ϵ decreases to become $\epsilon = 0$?

Transition to turbulence in shear flows

Example

The question is: how do the two solutions (linear growth and exponential decrease) match as ϵ decreases to become $\epsilon = 0$?

To answer we must focus on the energy of the disturbance, $E(t) = (\boldsymbol{x}', \boldsymbol{x}')$. For large times (when $\epsilon t >> 1$) the exponential behavior of the previous slide holds and eventually at large times the solution goes like $\boldsymbol{x}' \sim e^{-\epsilon t}\, \boldsymbol{u}_1$. What about short times?

Transition to turbulence in shear flows

Example

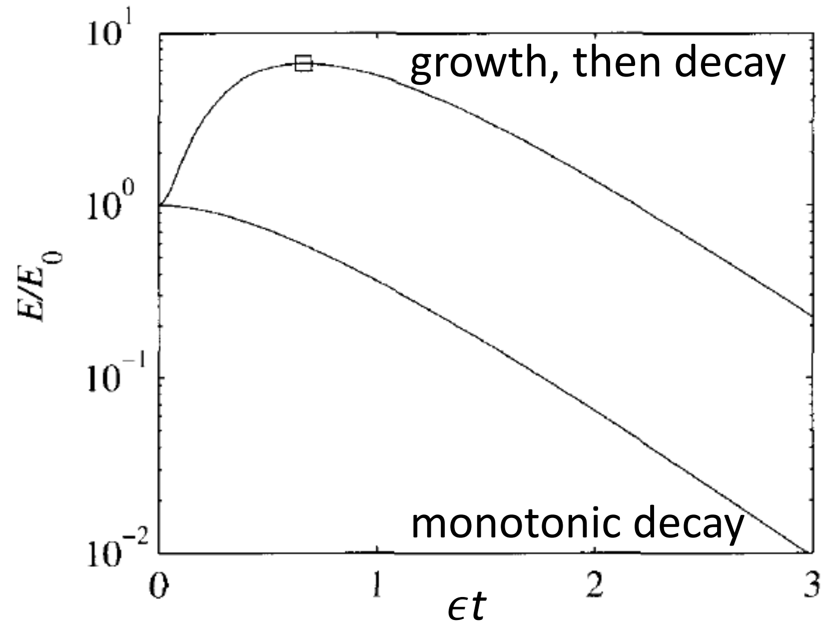
For short times (ϵt << 1) a Taylor series of the energy gives:

$$E(t) = (c_1 + c_2)^2 + \epsilon^2 c_2^2 - [2c_1^2 + 4(1+\epsilon^2)c_2^2 + 6c_1 c_2](\epsilon t) + O(\epsilon^2 t^2)$$

and a linear growth in time is possible if the factor of (ϵt) is negative. This growth is related to the fact that the two eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are not orthogonal to one another (in fact, they are almost *parallel* !)

Transition to turbulence in shear flows

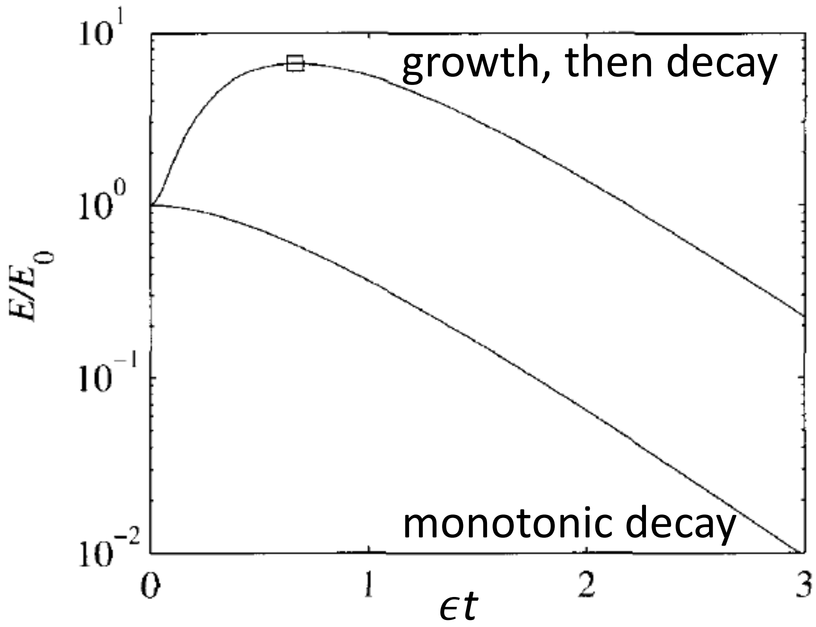
Example



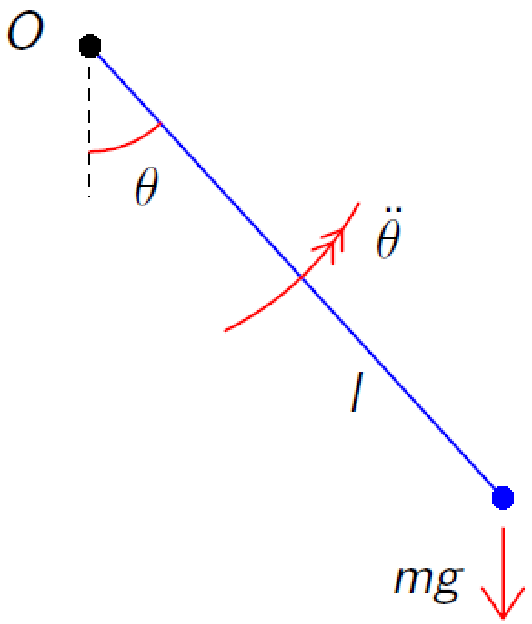
Example

The *optimal* initial condition

It is easy to see that the initial condition which yields the largest gain, ratio of final to initial energy, for t large enough, is the first left eigenvector.



Another simple example



The frictionless simple pendulum: a mass m attached to a light rod of length l that pivots freely about O.

Newton's laws tell us

$$\ddot{\theta} + \omega^2 \sin\theta = 0 \qquad (1)$$

where $\omega^2 = g/l$.

We assume there is a steady solution θ_0, and we want to find its stability. Let

$$\theta = \theta_0 + \epsilon\theta_1(t) \qquad (2)$$

where $\epsilon \ll 1$.

Another simple example

Substitute (2) into (1):

$$\begin{aligned} 0 &= \epsilon\ddot{\theta}_1 + \omega^2 \sin(\theta_0 + \epsilon\theta_1) \\ &= \epsilon\ddot{\theta}_1 + \omega^2 \sin\theta_0 \cos(\epsilon\theta_1) + \omega^2 \cos\theta_0 \sin(\epsilon\theta_1). \end{aligned} \qquad (3)$$

Substitute the Taylor expansions

$$\cos(\epsilon\theta_1) \sim 1 - \frac{(\epsilon\theta_1)^2}{2} + \ldots, \quad \sin(\epsilon\theta_1) \sim \epsilon\theta_1 - \frac{(\epsilon\theta_1)^3}{6} + \ldots$$

into (3) and equate coefficients of $O(1)$ and $O(\epsilon)$:

$$\sin\theta_0 = 0 \qquad (4)$$

$$\ddot{\theta}_1 + (\omega^2 \cos\theta_0)\theta_1 = 0 \qquad (5)$$

Another simple example

- The nonlinear equation for steady states (4) has solutions $\theta_0 = 0$ and $\theta_0 = \pi$.
- The disturbance equation (5) is linear and its coefficients depend on the steady solution θ_0.
- Substitute $\theta_0 = 0$ into (5):

$$\ddot{\theta}_1 + \omega^2 \theta_1 = 0 \quad \Rightarrow \quad \theta_1 = A \cos \omega t + B \sin \omega t, \qquad (6)$$

θ_1 remains bounded as $t \to \infty$, therefore $\theta_0 = 0$ is stable.

Another simple example

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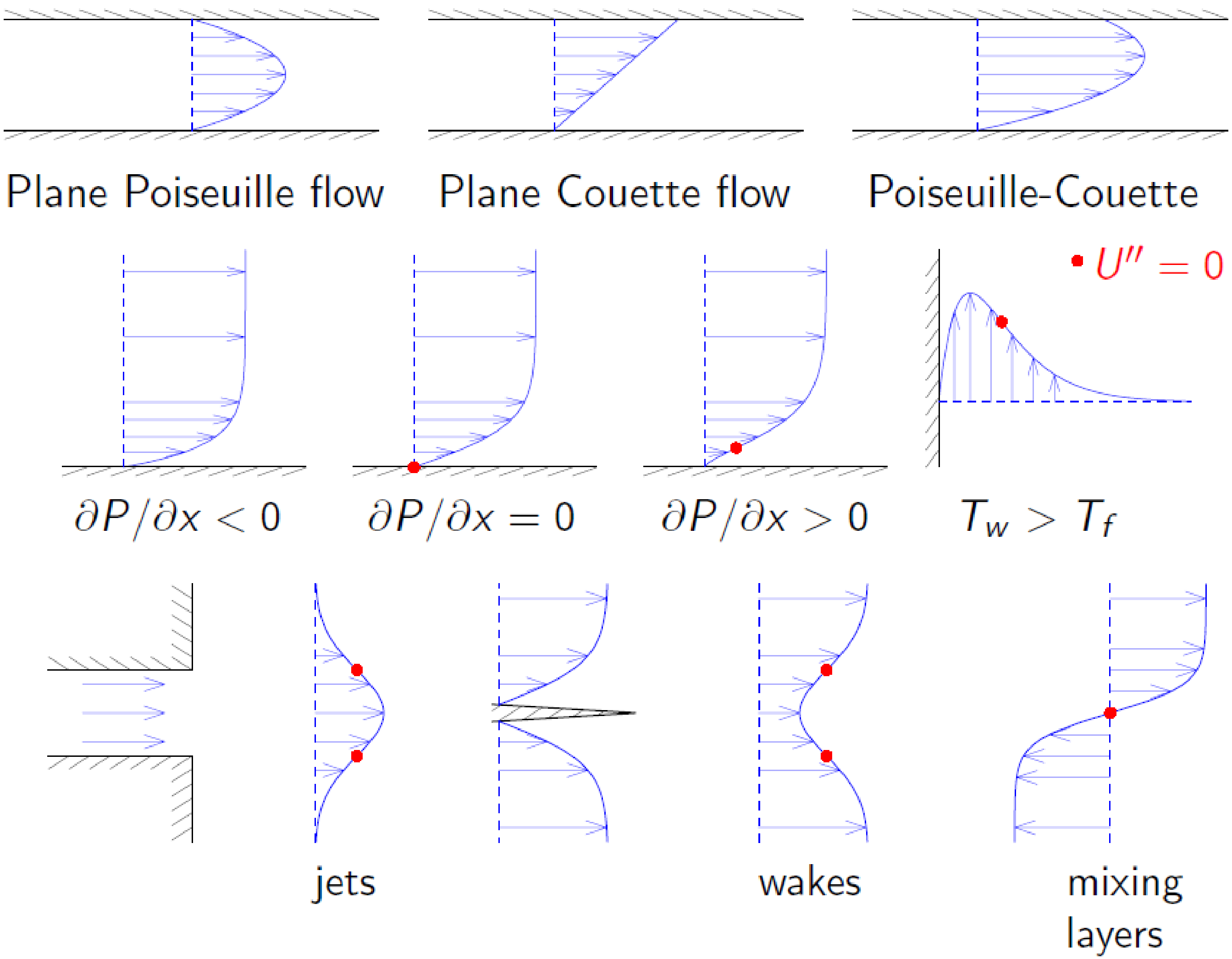
  θ_1 remains bounded as $t \to \infty$, therefore $\theta_0 = 0$ is stable.
- Substitute $\theta_0 = \pi$ into (5):

$$\ddot{\theta}_1 - \omega^2\theta_1 = 0 \quad \Rightarrow \quad \theta_1 = A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t}, \qquad (7)$$

  $\theta_1 \to \infty$ as $t \to \infty$, therefore $\theta_0 = \pi$ is unstable.

Transition to turbulence in shear flows

Shear flow problems



Shear flow problems

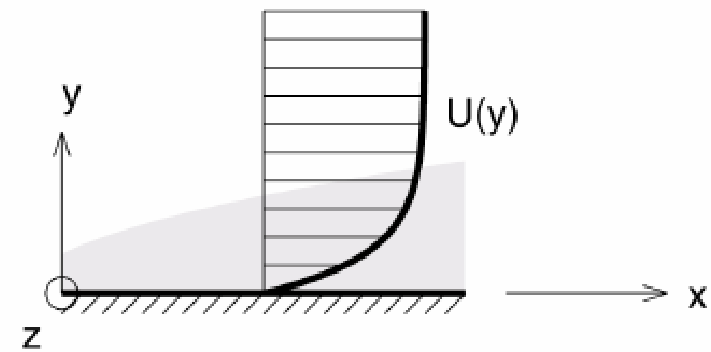
For the simple problems above, the flow is parallel or quasi-parallel and it is a good approximation to consider the velocity profile as

$$\boldsymbol{u}_0 = U(y)\,\boldsymbol{i}$$

Transition to turbulence in shear flows

Hydrodynamic stability of // shear flows

$$\begin{cases} \nabla \cdot \boldsymbol{u} = 0 \\ \dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \dfrac{1}{Re} \nabla^2 \boldsymbol{u} \end{cases}$$



$\boldsymbol{u}(\boldsymbol{x}, 0)$ assigned

$\boldsymbol{u}(\boldsymbol{x}, t) = 0$ on solid boundaries

$$\begin{cases} \boldsymbol{u} = \boldsymbol{u}_0 + \epsilon\, \boldsymbol{u}_1 + \epsilon^2 \boldsymbol{u}_2 + \dots \\ p = p_0 + \epsilon\, p_1 + \epsilon^2 p_2 + \dots \end{cases} \qquad \epsilon \ll 1$$

Transition to turbulence in shear flows

Hydrodynamic stability

$$O(1) \quad \begin{cases} \nabla \cdot \boldsymbol{u}_0 = 0 \\ (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_0 = -\nabla p_0 + \frac{1}{Re}\nabla^2 \boldsymbol{u}_0 \end{cases}$$

$$O(\epsilon) \quad \begin{cases} \nabla \cdot \boldsymbol{u}_1 = 0 \\ \frac{\partial \boldsymbol{u}_1}{\partial t} + (\boldsymbol{u}_1 \cdot \nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{u}_1 = -\nabla p_1 + \frac{1}{Re}\nabla^2 \boldsymbol{u}_1 \end{cases}$$

Hydrodynamic stability

We shall consider two-dimensional disturbances, and work in cartesian coordinates:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)
\end{aligned}$$

Hydrodynamic stability

The stability of velocity profile $U(y)$ is found by substituting

$$
\begin{aligned}
u &= U(y) + \epsilon \ \hat{u}(x,y,t) \\
v &= \epsilon \ \hat{v}(x,y,t) \\
p &= P(x) + \epsilon \ \hat{p}(x,y,t),
\end{aligned}
$$

into the equations and collecting terms of $O(\epsilon)$:

$$
\begin{aligned}
\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} &= 0 \\
\frac{\partial \hat{u}}{\partial t} + U\frac{\partial \hat{u}}{\partial x} + U'\hat{v} &= -\frac{\partial \hat{p}}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2}\right) \\
\frac{\partial \hat{v}}{\partial t} + U\frac{\partial \hat{v}}{\partial x} &= -\frac{\partial \hat{p}}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{\partial^2 \hat{v}}{\partial y^2}\right)
\end{aligned}
$$

Hydrodynamic stability

The solution of the linear $O(\epsilon)$ problem relies crucially on the streamlines being parallel, i.e. U is independent of x. This makes the solution separable, e.g. $\hat{u} = X(x)u(y)T(t)$.

More particularly, it can be shown that the solutions are the sum of exponential functions of the invariant directions, x and t (Fourier modes). A **normal mode** has the form:

$$(\hat{u}, \hat{v}, \hat{p}) = (u(y), v(y), p(y))\mathrm{e}^{\mathrm{i}(\alpha x - \omega t)} \quad ;$$

this is a *x-travelling wave*, with α the streamwise wavenumber and ω the circular frequency.

Hydrodynamic stability

$$\begin{aligned} \mathrm{i}\alpha u + v' &= 0 \\ -\mathrm{i}\omega u + \mathrm{i}\alpha U u + U' v &= -\mathrm{i}\alpha p + \frac{1}{Re}\left(u'' - \alpha^2 u\right) \\ -\mathrm{i}\omega v + \mathrm{i}\alpha U v &= -p' + \frac{1}{Re}\left(v'' - \alpha^2 v\right), \end{aligned}$$

where $' = \mathrm{d}/\mathrm{d}y$

Eliminating u and p gives the Orr-Sommerfeld equation:

$$(U-c)(v''-\alpha^2 v) - U'' v = \frac{1}{\mathrm{i}\alpha Re}\left(v'''' - 2\alpha^2 v'' + \alpha^4 v\right),$$

where $c = \omega/\alpha$.

Hydrodynamic stability

Boundary conditions

- No flow through a solid boundary: $v = 0$.
- No slip at a solid boundary: $u = 0 \Rightarrow v' = 0$
- If the flow is unbounded as $y \to \infty$ then $v \to 0$ and $v' \to 0$ as $y \to \infty$.
- The four boundary conditions may be summarised as

$$v(y_1) = 0, \quad v'(y_1) = 0, \quad v(y_2) = 0, \quad v'(y_2) = 0,$$

where y_1 and/or y_2 could be finite or infinite depending on whether we are considering channel flow, a boundary layer or an unbounded shear layer.

Hydrodynamic stability

- Nontrivial solutions satisfying these homogeneous boundary conditions are only possible for certain α and ω.
- These values satisfy a relation of the form $\Delta(\alpha, \omega) = 0$ called the dispersion relation.
- Roots of $\Delta(\alpha, \omega) = 0$ are called eigenvalues.
- Suppose that a Fourier mode with a real α has been chosen giving a complex eigenvalue, $\omega = \omega_r + \mathrm{i}\omega_i$:

$$\mathrm{e}^{\mathrm{i}(\alpha x - \omega t)} = \mathrm{e}^{\mathrm{i}(\alpha x - \omega_r t - \mathrm{i}\omega_i t)} = \mathrm{e}^{\omega_i t}\mathrm{e}^{\mathrm{i}(\alpha x - \omega_r t)}.$$

Hydrodynamic stability

- Therefore, if for any real α

$$\omega_i > 0 \Rightarrow \text{exponential growth in time} \Rightarrow \text{instability.}$$

- If for all real α

$$\omega_i < 0 \Rightarrow \text{exponential decay in time} \Rightarrow \text{stability.}$$

- Obtaining growth/decay in time is called temporal stability theory.
- Spatial stability theory (real ω, complex α) and spatio-temporal stability theory (complex ω, complex α), i.e. convective/absolute instabilities.

Stability conditions

- In the inviscid limit the Orr-Sommerfeld equation reduces to the Rayleigh equation

$$(U - c)(v'' - \alpha^2 v) - U''v = 0.$$

- The non-slip boundary conditions are dropped for inviscid flow, leaving

$$v(y_1) = 0, \quad v(y_2) = 0$$

(the Rayleigh equation is only 2nd order, while the Orr-Sommerfeld equation is 4th order).

The inviscid problem

Rayleigh inflection point theorem

$$\int_{y_1}^{y_2} \bar{v}v'' - \left(\frac{U''}{U-c} + \alpha^2\right)|v|^2 \,\mathrm{d}y = 0.$$

Integrate the first term by parts:

$$\begin{aligned}
[\bar{v}v']_{y_1}^{y_2} + \int_{y_1}^{y_2} -\bar{v}'v' - \left(\frac{U''}{U-c} + \alpha^2\right)|v|^2 \,\mathrm{d}y &= 0 \\
\Rightarrow \quad \int_{y_1}^{y_2} |v'|^2 + \alpha^2|v|^2 \,\mathrm{d}y + \int_{y_1}^{y_2} \frac{U''|v|^2}{U-c} \,\mathrm{d}y &= 0
\end{aligned}$$

The inviscid problem

Rayleigh inflection point theorem

The imaginary part of the equation above (α real) is

$$c_i \int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2}\,\mathrm{d}y = 0$$

ad this relation is satisfied for $c_i \neq 0$ only when the integral vanishes, which occurs only if $U'' \equiv 0$ or U'' changes sign at least once in $y_1 < y < y_2$

The inviscid problem

Rayleigh inflection point theorem:

a necessary, but not sufficient, condition for instability is that the velocity profile have an inflection point

The inviscid problem

Fjørtoft's theorem

Let there be an inflection point at $y = y_I$ and let $U_I = U(y_I)$.

If $c_i \neq 0$ then

$$(c_r - U_I)\int_{y_1}^{y_2} \frac{U''|v|^2}{|U-c|^2}\,\mathrm{d}y = 0.$$

The real part of the expression derived previously is

$$\int_{y_1}^{y_2} \frac{U''(U-c_r)|v|^2}{|U-c|^2}\,\mathrm{d}y = -\int_{y_1}^{y_2} |v'|^2 + \alpha^2|v|^2\,\mathrm{d}y$$

The inviscid problem

Fjørtoft's theorem

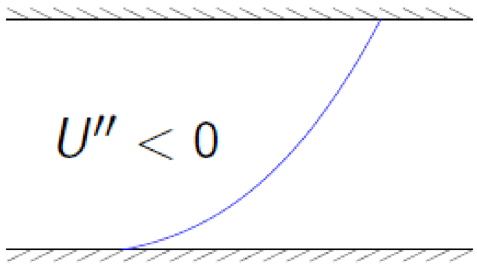
Adding up leads to:

$$\int_{y_1}^{y_2} \frac{U''(U - U_I)|v|^2}{|U - c|^2}\,\mathrm{d}y < 0$$

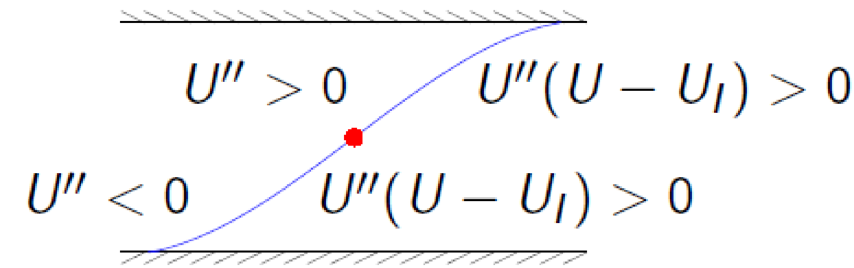
a necessary, but not sufficient, condition for instability is that $U''(U - U_I) < 0$, ***somewhere in the flow***

The inviscid problem

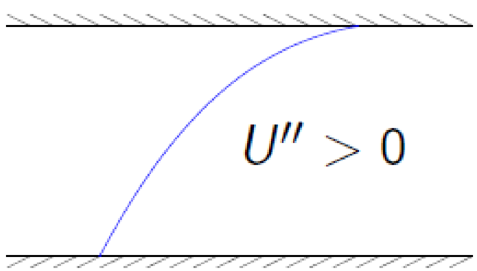
Fjørtoft's theorem



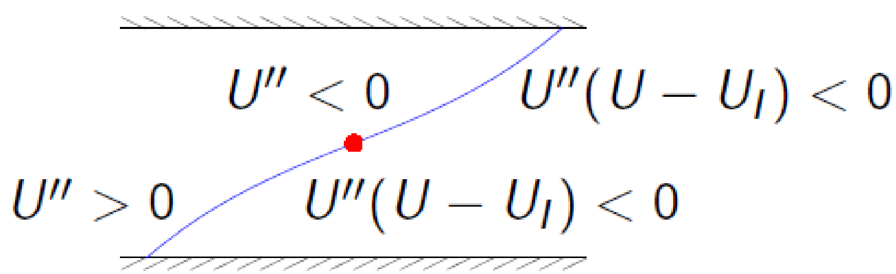
Stable by Rayleigh



Stable by Fjørtoft



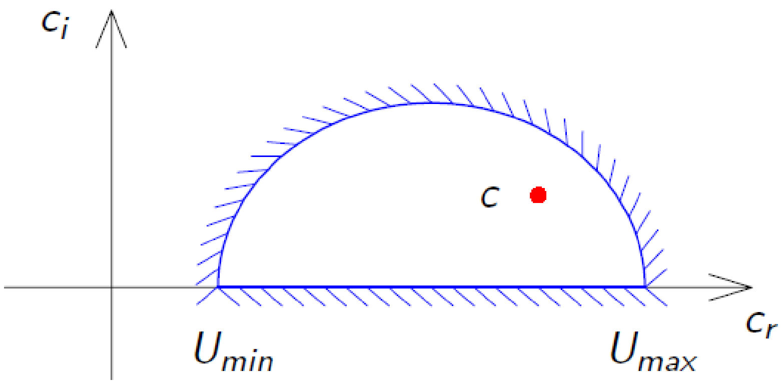
Stable by Rayleigh



Could be unstable

The inviscid problem

Howard's semi-circle theorem



the complex phase velocity lies inside, or on, the semi-circle centred on $\dfrac{U_{max} + U_{min}}{2}$ of radius $\dfrac{U_{max} - U_{min}}{2}$

The viscous problem

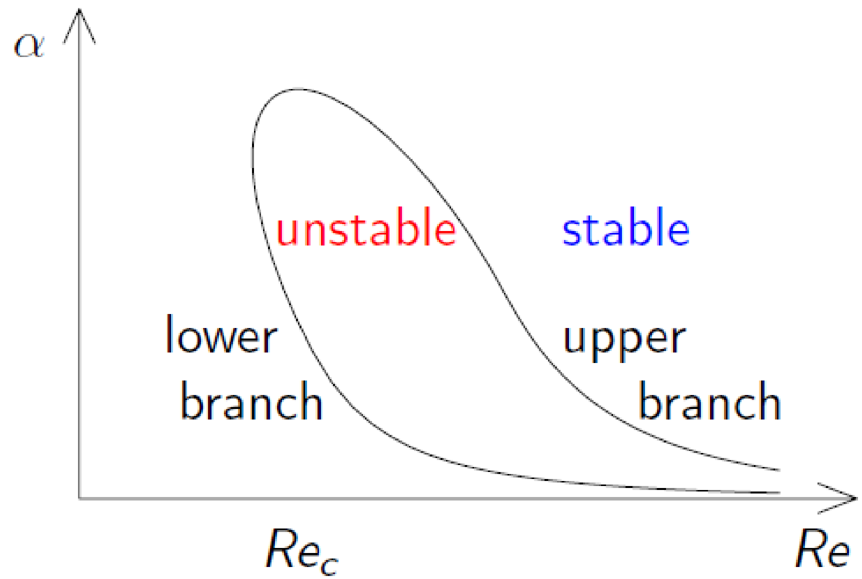
- Viscous disturbances are governed by the Orr-Sommerfeld equation:

$$(U-c)(v''-\alpha^2 v)-U''v=\frac{1}{\mathrm{i}\alpha Re}(v''''-2\alpha^2 v''+\alpha^4 v)$$

- Viscosity regularizes the inviscid solution at $U=c$.
- Viscosity dissipates kinetic energy, so it is stabilizing?
- Bizarrely, some stable inviscid flows are destabilized by viscosity!

The viscous problem

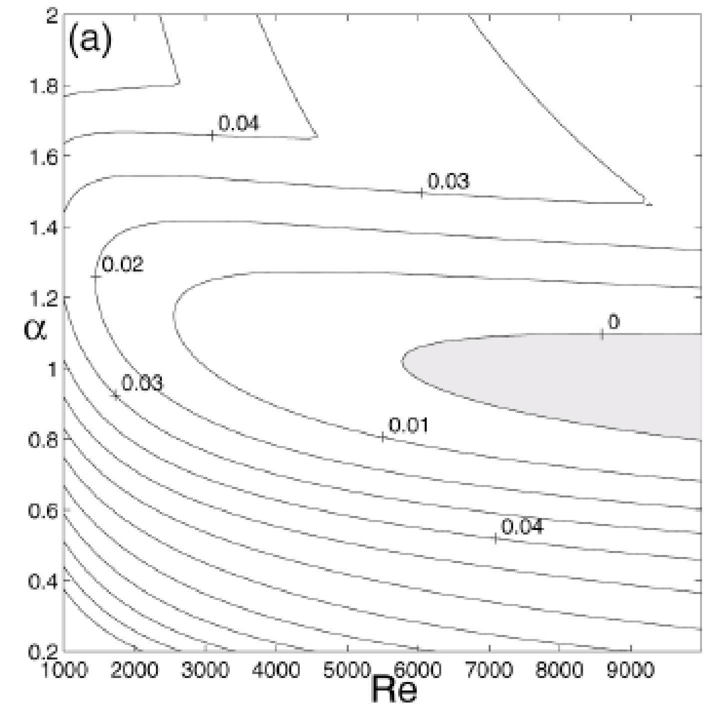
- Tollmien (1929) found asymptotic solutions to the Orr-Sommerfeld equation for profiles with no inflection point predicting instability:

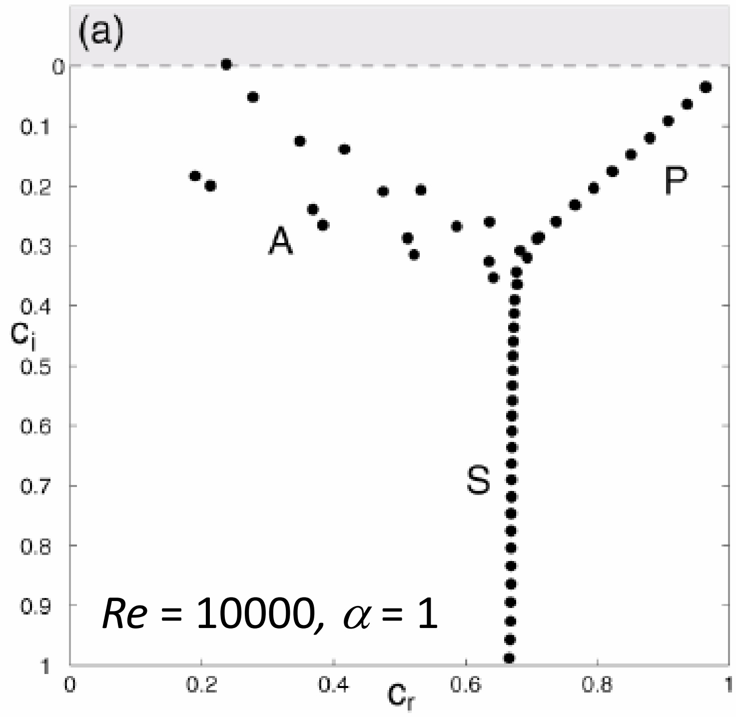


- Flow is stable for $Re < Re_c$.
- The neutral curve has upper and lower branches.
- Questionable assumptions made, results not widely accepted...
- ...until Schubauer & Skramstad (1947) verified this behaviour in wind tunnel experiments on boundary layers in the US.

Transition to turbulence in shear flows

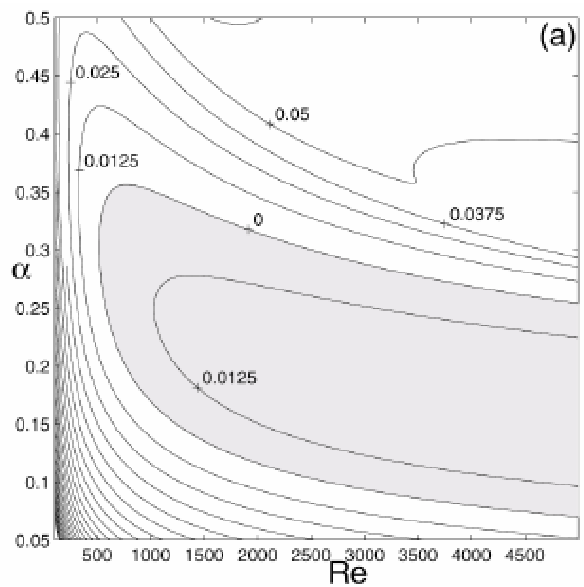
The viscous problem: PPF

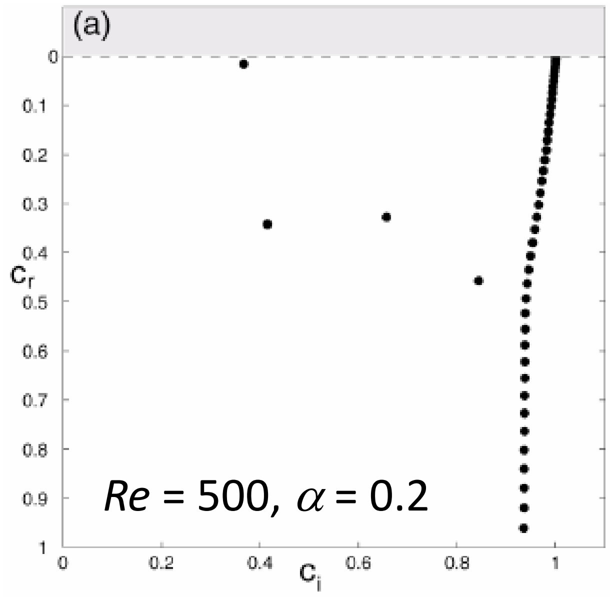


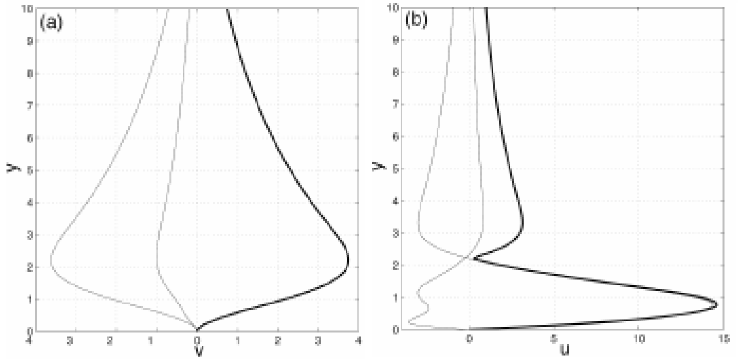


Transition to turbulence in shear flows

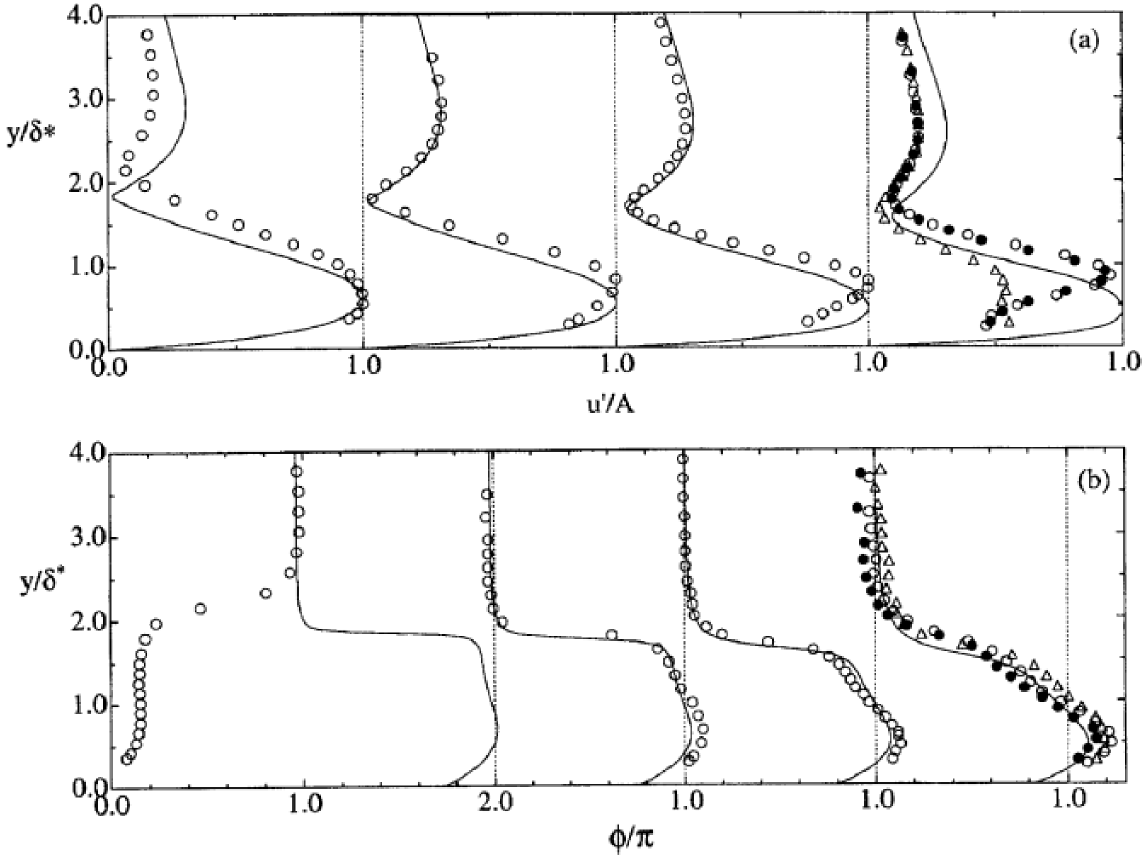
The viscous problem: Blasius



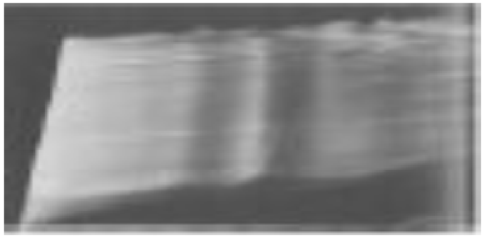




The viscous problem: Blasius



Very well-controlled experimental conditions



Bakchinov et al., 1998
(very low free stream Tu)

The viscous problem

Flow	α_{crit}	Re_{crit}	$c_{r_{crit}}$
Plane Poiseulle	1.02	5772	0.264
Blasius boundary layer flow	0.303	519.4	0.397

Typical wind tunnel experiments say that transition occurs around Re = 2000 in PPF and around Re = 400 in the Blasius boundary layer …

Could this be a 3D effect? Or something else?

The viscous problem: 3D disturbances

Consider 3D disturbances and replace

$$(\hat{u}, \hat{v}, \hat{p}) = (u(y), v(y), p(y))\mathrm{e}^{\mathrm{i}(\alpha x - \omega t)}$$

by

$$(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = (u(y), v(y), w(y), p(y))\mathrm{e}^{\mathrm{i}(\alpha x + \beta z - \omega t)}$$

to end up with the OS and Squire equations

The viscous problem: 3D disturbances

$$\left[(-i\omega + i\alpha U)(D^2 - k^2) - i\alpha U'' - \frac{1}{Re}(D^2 - k^2)^2\right] v \quad = \quad 0$$

$$\left[(-i\omega + i\alpha U) - \frac{1}{Re}(D^2 - k^2)\right] \eta \quad = \quad -i\beta U' v$$

$\eta = i\,\beta\,u\, - i\,\alpha\,w$ mode shape of the normal vorticity

$v = v' = \eta = 0$ at a solid wall and in the far field

$k^2 = \alpha^2 + \beta^2$ $D^i = \partial^i / dy^i$

The 3D viscous problem

In discrete form the *temporal problem* is a generalized eigenvalue problem of the form:

$$\begin{pmatrix} \boldsymbol{A_1} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{A_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\eta} \end{pmatrix} = \omega \begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\eta} \end{pmatrix}$$

$$\boxed{\boldsymbol{A}\,\boldsymbol{x}' = \omega\,\boldsymbol{B}\,\boldsymbol{x}'}$$

The 3D viscous problem

There are two families of solutions of the Orr-Sommerfeld and Squire problems

OS modes: $\{v_n, \eta_n^p, \omega_n\}_{n=1}^{N}$

Squire modes: $\{v = 0, \eta_m, \omega_m\}_{m=1}^{M}$

The 3D viscous problem: Squire theorem

It is easy to show that

(i) Squire modes are always damped and
(ii) for each 3D OS mode there is a 2D OS mode of lower Reynolds number

This means that the search of the *critical Reynolds number* (smaller value of Re at which ω_i becomes positive for the first time) can be carried out looking at 2D OS modes only.

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So, if 3D disturbances are not the answer, what happens?

The forced problem

Let us imagine that an oscillating source term forces our system of equations (*before* Laplace-Fourier transforming them):

$$\frac{\partial f}{\partial t} = L\, f + s e^{\sigma_s t}$$

f is some state function, L is a linear evolution operator, s is the spatial distribution of the forcing signal and σ_s its growth rate.

The forced problem

The adjoint system is

$$-\frac{\partial g}{\partial t} = L^{+}\, g$$

g is the adjoint function, L^{+} is the adjoint operator. This equation runs **backward in time**, i.e. it is integrated from $t = T$ to $t = 0$.

It arises easily from the definition of the inner product:

$$[a, b] = \int_0^T \int_V \bar{a}\, b \;\, dV\, dt$$

The forced problem

$$\left[g, \frac{\partial f}{\partial t}\right] = [g, Lf] + [g, se^{\sigma_s t}]$$

$$-\left[\frac{\partial g}{\partial t}, f\right] + \int_V \bar{g}\, f\,\Big|_0^T\, dV = [L^+ g, f] + [g, se^{\sigma_s t}]$$

Transition to turbulence in shear flows

The forced problem

$$-\left[\frac{\partial g}{\partial t}, f\right] + \int_V \bar{g}\, f\big|_0^T \, dV = \left[L^+ g, f\right] + \left[g, s e^{\sigma_s t}\right]$$

the solution of the direct problem at the final time is:

$$\int_V \bar{g}\, f\big|_{t=T} \, dV = \int_V \bar{g}\, f\big|_{t=0} \, dV + \left[g, s e^{\sigma_s t}\right]$$

The forced problem

As terminal condition of the adjoint problem choose

$$g(T) = \delta(x - x')$$

so that the direct solution at T is

$$f(x',T) = \int_V \bar{g}\, f|_{t=0}\, dV + [g, se^{\sigma_s t}]$$

i.e. the adjoint solution weights both the initial condition and the source term in determining the solution at the final time T.

The forced problem

If we solve the linear stability problem for a plane shear flow it is very easy to show that the eigenfunctions of the adjoint operator act as receptivity functions, both for the temporally evolving case and for the eigenproblem. In particular, adjoint e-functions provide *inflow/wall/source* receptivity coefficients.

Hill, 1995
Luchini & Bottaro, 1998

Sensitivity analysis

Let us go back to the discrete world, and imagine that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are perturbed, for example by disturbances in the boundary conditions or by a noisy base flow. This will produce perturbations in both the eigenvalues and the eigenvectors.

$$\boldsymbol{A}\,\boldsymbol{x}' = \omega\,\boldsymbol{B}\,\boldsymbol{x}'$$

$$(\boldsymbol{A} + \delta\boldsymbol{A})(\boldsymbol{x}' + \delta\boldsymbol{x}') = (\omega + \delta\omega)(\boldsymbol{B} + \delta\boldsymbol{B})(\boldsymbol{x}' + \delta\boldsymbol{x}')$$

and to first order (for small variations):

$$\boldsymbol{A}\,\delta\boldsymbol{x}' + \delta\boldsymbol{A}\,\boldsymbol{x}' = \omega\,\boldsymbol{B}\,\delta\boldsymbol{x}' + \omega\,\delta\boldsymbol{B}\,\boldsymbol{x}' + \delta\omega\,\boldsymbol{B}\,\boldsymbol{x}'$$

Sensitivity analysis

The left eigenproblem is:

$$\boldsymbol{y}^T \ \overline{\boldsymbol{A}} = \overline{\omega}\ \boldsymbol{y}^T \ \overline{\boldsymbol{B}}$$
$$\overline{\boldsymbol{y}^T}\ \boldsymbol{A} = \omega\ \overline{\boldsymbol{y}^T}\ \boldsymbol{B}$$

Left multiply $\boldsymbol{A}\,\delta\boldsymbol{x}' + \delta\boldsymbol{A}\,\boldsymbol{x}' = \omega\,\boldsymbol{B}\,\delta\boldsymbol{x}' + \omega\,\delta\boldsymbol{B}\,\boldsymbol{x}' + \delta\omega\,\boldsymbol{B}\,\boldsymbol{x}'$

by $\overline{\boldsymbol{y}^T}$ to obtain the eigenvalue drift:

$$\boxed{\delta\omega = \frac{(\boldsymbol{y}, \delta\boldsymbol{A}\,\boldsymbol{x}')}{(\boldsymbol{y}, \boldsymbol{B}\,\boldsymbol{x}')} - \omega\,\frac{(\boldsymbol{y}, \delta\boldsymbol{B}\,\boldsymbol{x}')}{(\boldsymbol{y}, \boldsymbol{B}\,\boldsymbol{x}')}}$$

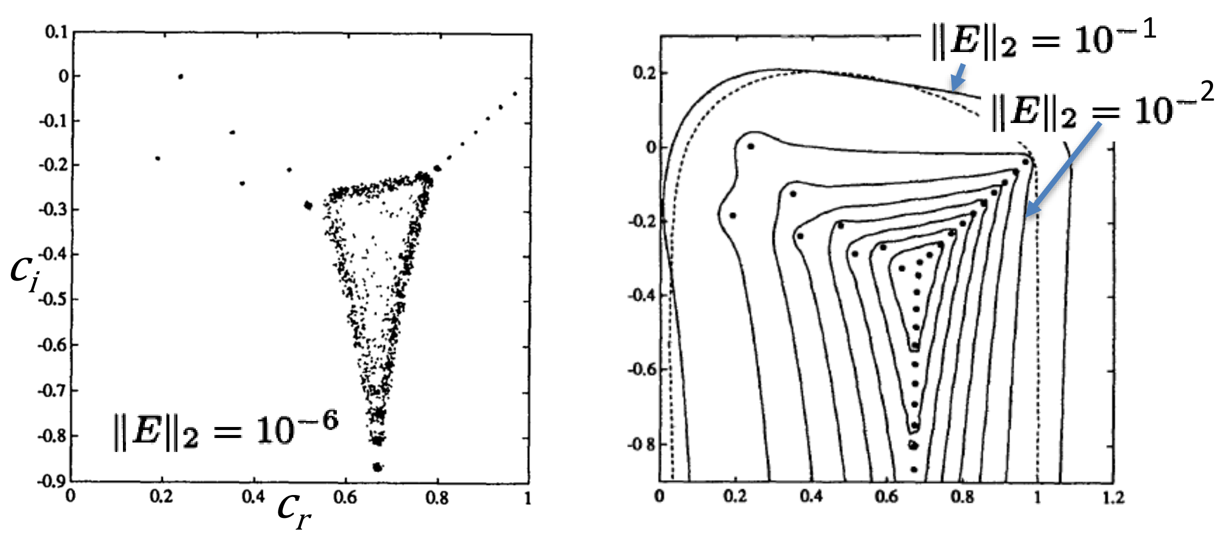
Transition to turbulence in shear flows

ϵ-pseudospectrum

Small variations in $\boldsymbol{A}$ and $\boldsymbol{B}$ with respect to their ideal behavior, linked to noise or imperfect knowledge of base flow and/or boundary conditions, can destabilize (and modify the frequency) of a nominally stable flow. The ϵ-pseudospectrum of a matrix $\boldsymbol{C}$ is the set of all eigenvalues which are ϵ-close to $\boldsymbol{C}$:

$$\Lambda_\epsilon(\boldsymbol{C}) = \{\lambda \in \mathbb{C} \mid \exists\, \boldsymbol{x} \in \mathbb{C}^n \backslash \{0\}, \exists\, \boldsymbol{E} \in \mathbb{C}^{n\times n} : (\boldsymbol{C}+\boldsymbol{E})\boldsymbol{x} = \lambda \boldsymbol{x}, ||\boldsymbol{E}||_2 \leq \epsilon\}$$

PPF, *Re* = 10 000, α = 1



ϵ-pseudospectrum

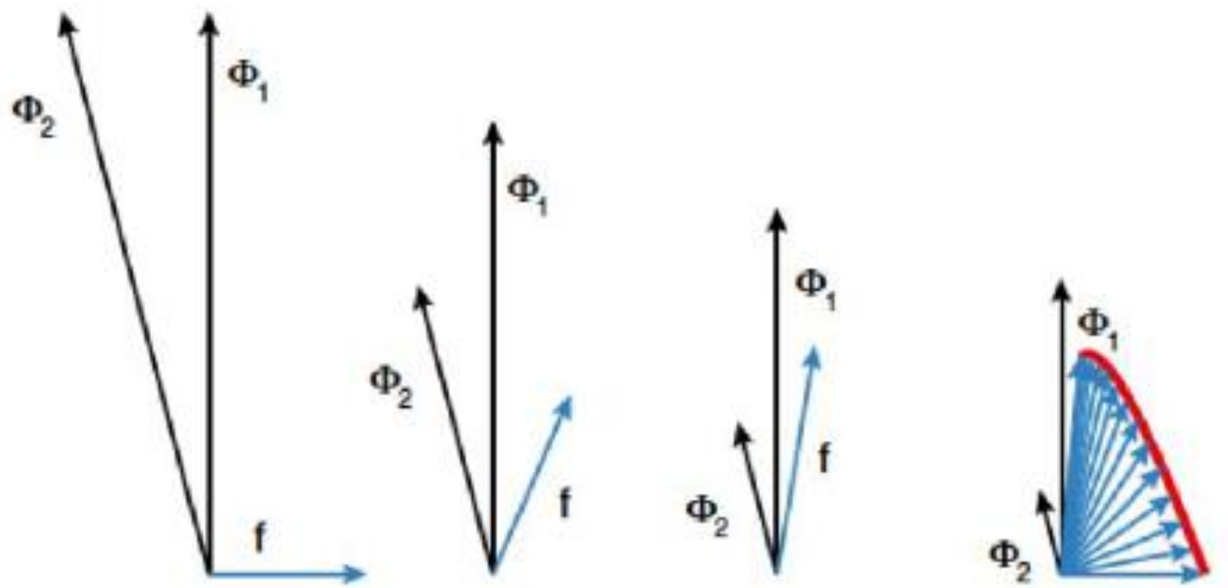
The ϵ−pseudospectrum is particularly useful to understand *non-normal matrices* and their eigenvectors, i.e. matrices which do not commute with their conjugate traspose, and for which the eigenvectors are not orthogonal to one another

$$\boldsymbol{C}\,\overline{\boldsymbol{C}}^T \neq \overline{\boldsymbol{C}}^T\,\boldsymbol{C}$$

Clearly non-normal matrices are not self-adjoint.

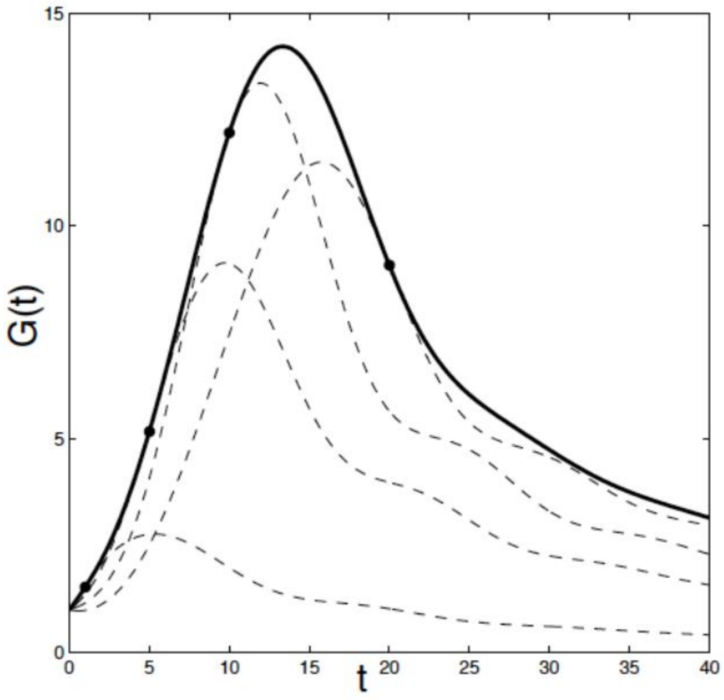
$\boxed{\text{The OS operator/matrix is strongly non-normal}}$

Transient growth



Damped e-vectors in time can produce a disturbance
f = $\Phi_1 - \Phi_2$ whose amplitude is initially/transiently amplified

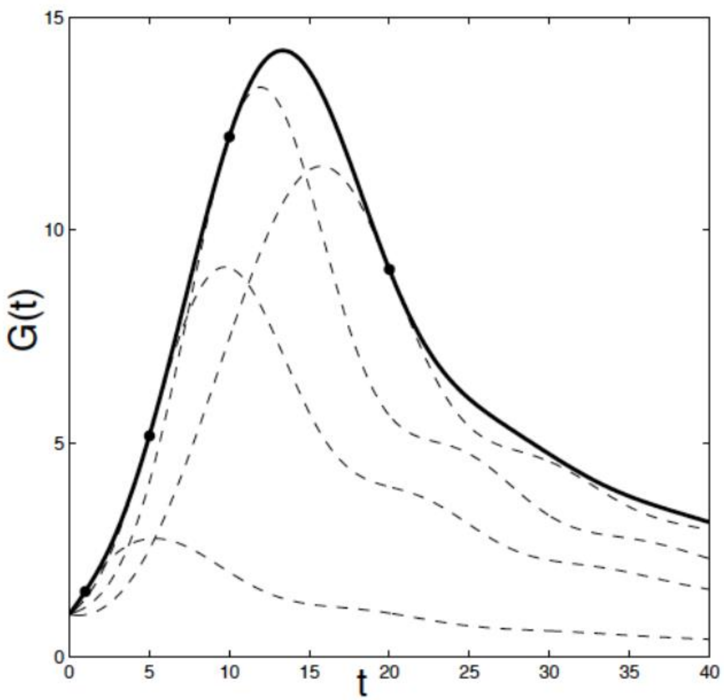
Transient growth



PPF, *Re* = 1000, α = 1

Transition to turbulence in shear flows

Transient growth



Butler & Farrell, 1992

TABLE III. Optimal perturbations in Poiseuille flow at $R=5000$.

	τ	α	β	E_τ/E_0
Antisymmetric global optimal	379	0	2.044	4897
Gustavsson—antisymmetric peak	420	0	1.98	4448
Symmetric global optimal	270	0	2.644	2819
Gustavsson—symmetric peak	286	0	2.60	2708
Best optimal at $\tau=20$	20	0.93	3.1	512
Best optimal at $\tau=5$	5	3.6	7.3	49.1
Best 2-D optimal	14.1	1.48	0	45.7

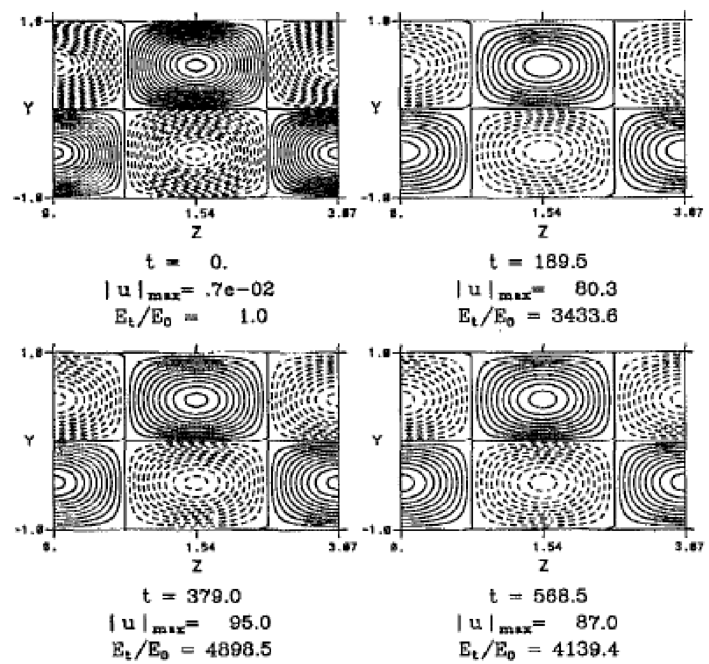


FIG. 14. Development of the perturbation streamwise velocity u for the global optimal in Poiseuille flow with $R=5000$, located at $\alpha=0$, $\beta=2.044$, and $\tau=379$. Values are normalized by the maximum value of v at time $t=0$.

Transient growth

The optimal transient growth (i.e. that producing the largest energy gain) transforms streamwise elongated vortices (present at t = 0 or x = 0) into streamwise elongated streaks at the final time/position.

The mechanism is inviscid.

Let us take the OS/Squire system, in symbolic form $\boldsymbol{A}\,\boldsymbol{x}' = \omega\,\boldsymbol{B}\,\boldsymbol{x}'$ and let's go one step backwards, i.e. *before* the Laplace transform: $\boldsymbol{A}\,\boldsymbol{x}' = -i\,\boldsymbol{B}\dfrac{d\boldsymbol{x}'}{dt}, \quad \boldsymbol{x}'(t=0) = \boldsymbol{x}'_0$

Transient growth

This equation also reads: $\frac{d\boldsymbol{x}'}{dt} = \boldsymbol{C}\,\boldsymbol{x}'$, with $\boldsymbol{C} = i\,\boldsymbol{B}^{-1}\boldsymbol{A}$

We have already seen that in this case we can decompose the solution in the sum of eigenvectors, i.e. $\boldsymbol{x}' = \boldsymbol{U}\ e^{\boldsymbol{\Lambda} t}\,\overline{\boldsymbol{V}}^T \boldsymbol{x}_0' = \boldsymbol{L}\,\boldsymbol{x}_0'$

propagator of the IC

The energy of the disturbance is $E(t) = (\boldsymbol{x}', \boldsymbol{x}')$ and the **gain** of the disturbance at a generic time T is

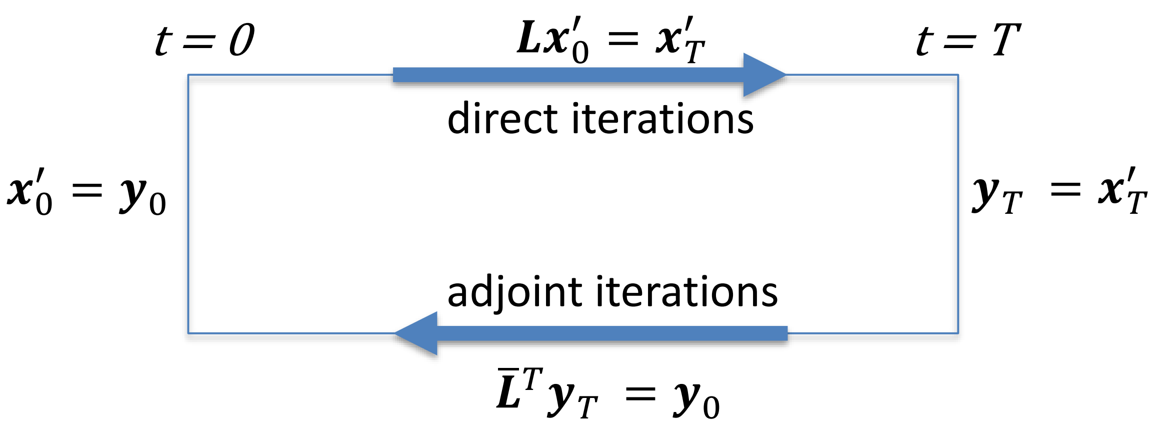
$$G(T) = \frac{E(T)}{E(0)} = \frac{(\boldsymbol{L}\boldsymbol{x}_0', \boldsymbol{L}\boldsymbol{x}_0')}{(\boldsymbol{x}_0', \boldsymbol{x}_0')} = \frac{(\overline{\boldsymbol{L}}^T \boldsymbol{L}\boldsymbol{x}_0', \boldsymbol{x}_0')}{(\boldsymbol{x}_0', \boldsymbol{x}_0')}$$

Transient growth

This is called Rayleigh quotient

$$G(T) = \frac{\left(\bar{\boldsymbol{L}}^T \boldsymbol{L}\boldsymbol{x}_0', \boldsymbol{x}_0'\right)}{\left(\boldsymbol{x}_0', \boldsymbol{x}_0'\right)}$$

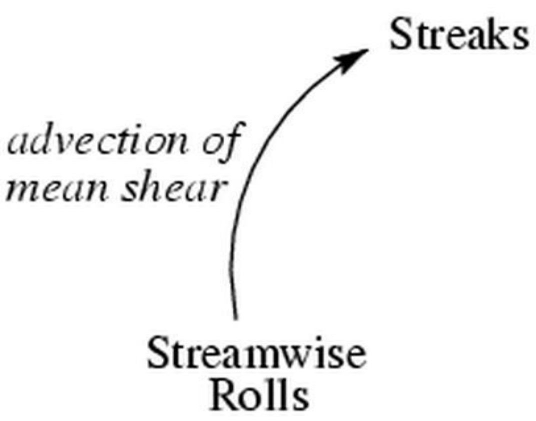
and the initial condition $\boldsymbol{x}_0'$ which yields the largest gain is easily identified by power iterations (*adjoint looping*)



Transition to turbulence in shear flows

Case closed?

Despite the fact that non-normality (and – as a direct consequence – transient growth) is an important concept, it is not sufficient to explain the breakdown to turbulence observed in experiments.



Transition to turbulence in shear flows

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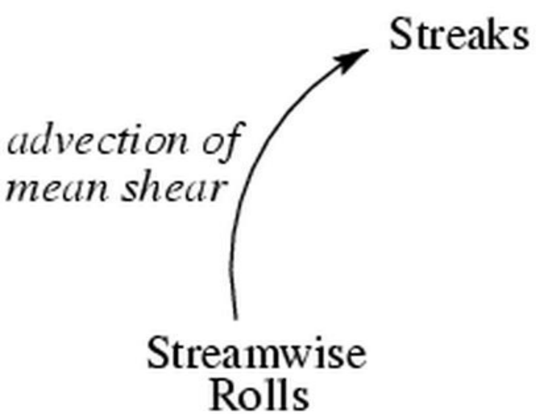
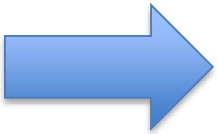
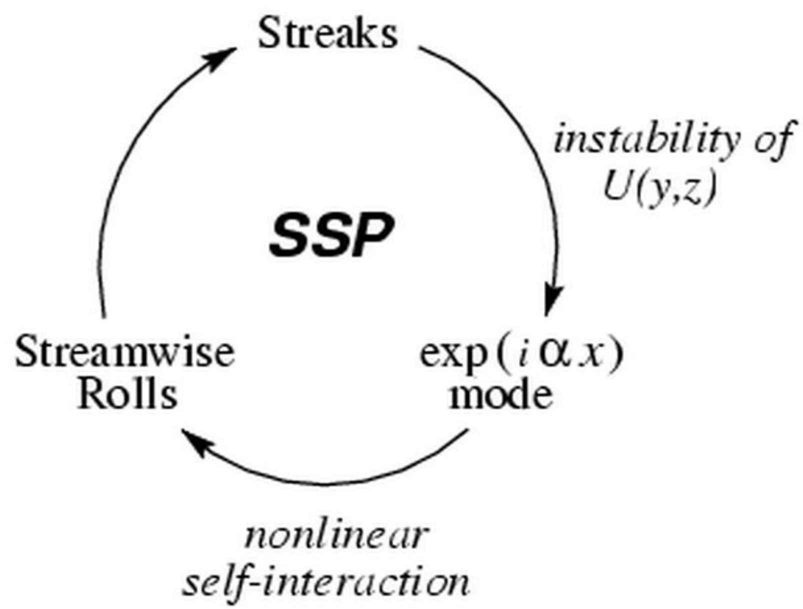
Despite the fact that non-normality (and – as a direct consequence – transient growth) is an important concept, it is not sufficient to explain the breakdown to turbulence observed in experiments.







Fabian Waleffe, *Physics of Fluids*, **9**, 1997

Transition to turbulence in shear flows

Non-linearities matter!

- waves are coupled
- growth is due to linear mechanism
- nonlinear terms redistribute kinetic energy among modes

Weakly non-linear approach

Consider a simple 1D model system:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = -u\frac{\partial u}{\partial x} \equiv -\frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

$$\boxed{u = \sum_{k=-\infty}^{\infty} a_k(t)e^{ik\alpha x}}$$

Weakly non-linear approach

Consider a simple 1D model system:

$$\sum_{k=-\infty}^{\infty}\left[\frac{da_k}{dt}+ik\alpha U a_k+\nu k^2\alpha^2 a_k\right]\ e^{ik\alpha x} =$$

$$-\ \frac{1}{2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} i(m+n)\alpha\left[a_m(t)\ a_n(t)\right]\ e^{i(m+n)\alpha x} =$$

$$-\ \frac{1}{2}\sum_{k=-\infty}^{\infty} i\alpha k \sum_{m+n=k}\left[a_m(t)\ a_n(t)\right]\ e^{ik\alpha x}$$

Weakly non-linear approach

Thus, for each mode k

$$\frac{da_k}{dt} + ik\alpha U a_k + \nu k^2 \alpha^2 a_k = -\frac{1}{2} ik\alpha \sum_{m+n=k} a_m \, a_n$$

and if we only considered three modes, $k = -1, 0, 1$, we would have:

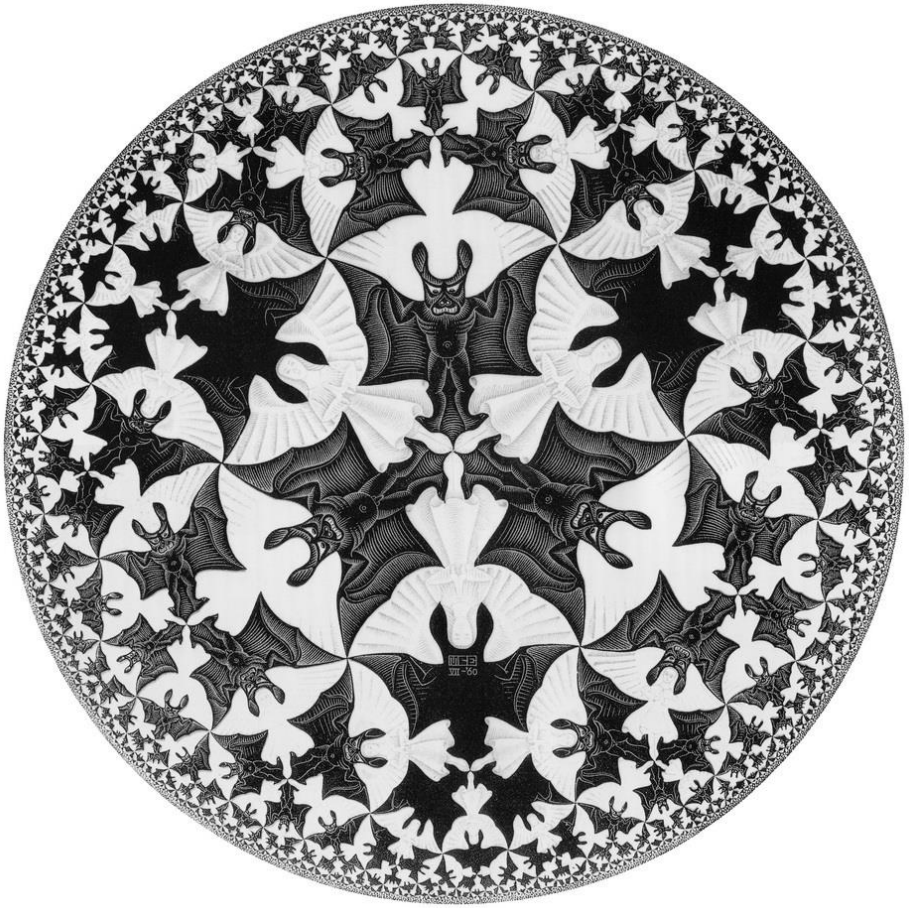
$$\begin{cases} \dfrac{da_0}{dt} = 0, \quad a_0 \text{ is the } \textit{mean flow correction} \\ \dfrac{da_1}{dt} + i\alpha U a_1 + \nu\alpha^2 a_1 = -i\alpha a_0 a_1 \\ \dfrac{da_{-1}}{dt} - i\alpha U a_{-1} + \nu\alpha^2 a_{-1} = i\alpha a_0 a_{-1} \end{cases}$$

Transition to turbulence in shear flows

Fully non-linear analysis

→ **Prof. Joel Guerrero!**

Transition to turbulence in shear flows



M.C. Escher, *Angels and Demons*
https://www.youtube.com/watch?v=YWVFIz4f5qk