LES equations – Filtered Navier-Stokes equations

- The starting point is the incompressible NSE,

$$\nabla \cdot (\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = \frac{-\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}$$

- If we apply the filtering operator directly to the primitive variables, we obtain the following filtered equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}$$

- In the previous equations, notice that,

$$\overline{\mathbf{u}\mathbf{u}} \neq \bar{\mathbf{u}}\bar{\mathbf{u}}$$

- This is an important difference between the RANS derivation and the LES derivation.

LES equations – Filtered Navier-Stokes equations

- Recall the following rules,

Averaging rules (in RANS)	Filtering rules (in LES)
$\overline{\bar{\phi}\psi} = \bar{\bar{\phi}}\bar{\psi} = \bar{\phi}\bar{\psi}$	$\overline{\phi\psi} \neq \bar{\phi}\bar{\psi}$
$\overline{\mathbf{u}\mathbf{u}} = \bar{\mathbf{u}}\bar{\mathbf{u}}$	$\overline{\mathbf{u}\mathbf{u}} \neq \bar{\mathbf{u}}\bar{\mathbf{u}}$

- By doing some algebra and some sunstitutions, we arrive to the FNS equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

LES equations – Filtered Navier-Stokes equations

- In this set of equations,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS} \qquad (1)$$

- τ^{SGS} is the sub-grid scale stress tensor and it represents the effect of small scales.
- This tensor can be written as follows,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}}$$

- If you substitute this relation into equation (1), we can recast the starting filtered equations, namely,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}}$$

LES equations – Filtered Navier-Stokes equations

- The sub-grid scale stress tensor is given by,

$$\tau^{SGS} = \bar{\mathbf{u}}\bar{\mathbf{u}} - \overline{\mathbf{u}\mathbf{u}} \qquad (1)$$

- By doing the following substitution,

Filtered quantity

Field residual

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \qquad \text{or} \qquad \overline{\mathbf{u}} = \mathbf{u} - \mathbf{u}'$$

- We obtain the following equation (expansion of eq. 1),

$$\tau^{SGS} = \left(\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}\right) + \left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right) + \overline{\mathbf{u}'\mathbf{u}'}$$

- Which can be seen as an analogous term to the Reynolds stress tensor in the RANS equations.
    - A little bit more complex thought, as it takes into account interactions between all scales, namely, resolved and unresolved.

LES equations – Filtered Navier-Stokes equations

- The sub-grid scale stress tensor τ^{SGS} represents the effect of filtered scales and small scales, and can be written as,

$$\tau^{SGS} = \underbrace{(\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}})}_{\mathbf{L}} + \underbrace{\left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right)}_{\mathbf{C}} + \underbrace{\overline{\mathbf{u}'\mathbf{u}'}}_{\mathbf{R}} = \mathbf{L} + \mathbf{C} + \mathbf{R}$$

- Where **L** is called the Leonard stresses, **C** is the called the cross-term stress, and **R** is called the sub-grid scale Reynolds stress (equivalent to the Reynolds stress tensor).
- The Leonard stresses (**L**) involves only the resolved quantities, and therefore it can be computed.
- The cross-term stresses (**C**) and SGS Reynolds stresses (**R**), involve unresolved scales and must be modeled.
- The cross-term stress represents the interaction of resolved and unresolved scales, whereas the SGS Reynolds stress represents the interaction of unresolved scales.
- At this point, the problem is how to model the cross-term stress and the sub-grid scale Reynolds stress.

LES equations – Filtered Navier-Stokes equations

- At the end of the day, these are the incompressible filtered Navier-Stokes eqauions to be used with LES models,

$$\nabla \cdot (\bar{\mathbf{u}}) = 0$$

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} + \nabla \cdot \tau^{SGS}$$

- Where,

$$\tau^{SGS} = (\overline{\bar{\mathbf{u}}\bar{\mathbf{u}}} - \bar{\mathbf{u}}\bar{\mathbf{u}}) + \left(\overline{\bar{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\bar{\mathbf{u}}}\right) + \overline{\mathbf{u}'\mathbf{u}'}$$

- At this point, we need to introduce models to approximate τ^{SGS} (similar to RANS).
- It is important to mention that the decomposition we just introduced to derive τ^{SGS} is not unique.
    - In this case we used the triple decomposition or Leonard decomposition.