Variational Methods in Fracture Mechanics

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Formulation of equilibrium problems in Mechanics based on global energy minimization

$$E(v) = \int_\Omega w(\nabla v(x))\, dx \; - \; \ell(v)$$

$$E(u) = \min\left\{ E(v) \quad \middle| \quad v \in H,\; v(x) = \bar{u}(x)\; \forall x \in \Gamma \right\}$$

Unfortunately, the approach based on global minimization does not go far beyond linear elasticity. Even for finite elasticity, there are still many open problems [1].

[1] JM Ball, *Some open problems in elasticity*, CISM Course Poly-, Quasi-, and Rank-One Convexity in Applied Mechanics (2007), CISM Courses and Lectures, to appear

For inelastic bodies, global minimization makes no sense because the solutions are not path-independent, in the sense that different loading processes with the same final value do not determine, in general, the same solution.

The incremental problem in small-strain plasticity

$$\nabla v = \nabla v^e + \nabla v^p$$

$$\boldsymbol{E = E^e + E^p}$$

$$E^e(v) = \frac{1}{2}\int_\Omega \mathbb{C}\nabla v^e(x)\cdot\nabla v^e(x)\,dx - \ell(v)$$

$$E^p(\boxed{\lambda \mapsto u_\lambda}) = \int_0^1\int_\Omega \mathbb{C}\nabla u^e_\lambda(x)\cdot\nabla \dot{u}^p_\lambda(x)\,dx\,d\lambda$$

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$t \mapsto \ell_t$	loading process
$t \mapsto u_t$	solution of incremental problem

Energy increment

$$E_{t+\varepsilon} - E_t = \varepsilon\,\delta E_t + \frac{1}{2}\varepsilon^2\delta^2 E_t + o(\varepsilon^2)$$

Energy increment

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Displacement and load increment

$$v_{t+s} - v_t = \varepsilon \dot{v}_t + \frac{1}{2}\varepsilon^2 \ddot{v}_t + o(\varepsilon^2)$$

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First-order energy increment

$$\delta E_t = \int_\Omega \mathbb{C}\nabla u_t^e(x)\cdot\nabla\dot{u}_t^e(x)dx - \ell_t(\dot{u}_t) - \dot{\ell}_t(u_t) + \int_\Omega \mathbb{C}\nabla u_t^e(x)\cdot\nabla\dot{u}_t^p(x)dx$$

$$= \int_\Omega \mathbb{C}\nabla u_t^e(x)\cdot\nabla\dot{u}_t dx - \ell_t(\dot{u}_t) - \dot{\ell}_t(u_t)$$

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$$= \cancel{\int_\Omega \mathbb{C}\nabla u_t^e(x)\cdot\nabla \dot{u}_t dx - \ell_t(\dot{u}_t)} - \dot{\ell}_t(\cancel{u_t})$$

there is nothing to minimize in δE

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Second-order energy increment

$$\delta^2 E_t \equiv \int_\Omega \left(\mathbb{C}\nabla \dot{u}_t^e(x) \cdot \nabla \dot{u}_t(x) + \mathbb{C}\nabla u_t^e(x) \cdot \nabla \ddot{u}_t(x) \right) dx \\ - \ell_t(\ddot{u}_t) - 2\dot{\ell}_t(\dot{u}_t) - \ddot{\ell}_t(u_t)$$

there is nothing to minimize in δE

Second-order energy increment

$$\delta^2 E_t \equiv \int_\Omega \Big(\mathbb{C}\nabla \dot{u}^e_t(x) \cdot \nabla \dot{u}_t(x) + \cancel{\mathbb{C}\nabla u^e_t(x) \cdot \nabla \ddot{u}_t(x)} \Big) dx$$

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one is reduced to minimize

$$\boxed{I(\dot{u}_t) = \frac{1}{2}\int_\Omega \mathbb{C}\nabla \dot{u}_t^e(x) \cdot \nabla \dot{u}_t(x) dx - \dot{\ell}_t(\dot{u}_t)}$$

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This is Hill’s *principle of maximum plastic work*

R Hill, *The Mathematical Theory of Plasticity*, Oxford University Press (1950). Reprinted in Oxford Classic Series, Clarendon Press, Oxford (1998)

What is remarkable is that the quantity to be minimized is the energy increment, which includes both elastic energy and dissipation

B Fedelich, A Ehrlacher, *Sur un principe de minimum concernant des matériaux à comportement indépendant du temps physique*, C.R. Acad. Sci. Paris 308/II, 1391-1394, 1989

A Mielke, *Energetic formulation of multiplicative elasto-plasticity using dissipation distances*, Cont. Mech. Thermodyn. 15, 351-382, 2003

The evolutionary problem in fracture mechanics

$$E(v) = \int_\Omega w(\nabla v(x))\, dx \; - \; \ell(v) + \gamma \text{ area}\, S(v)$$

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condition of irreversibility of fracture

$$\mu < \lambda \quad \Longrightarrow \quad S(u_\mu) \subseteq S(u_\lambda)$$

G Francfort, J-J Marigo, *Revisiting brittle fracture as an energy minimization problem*,
J. Mech. Phys. Solids 46, 1319-1342, 1998

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This condition transforms global minimization into evolutionary global minimization

An inconvenience of evolutionary global minimization

one-dimensional problem

$$E(v) = \frac{1}{2} \int_0^l k v'^2(x)\, dx + \gamma \#_v$$

no loads

boundary conditions $v(0) = 0$, $v(l) = \beta l$

$$\int_0^l v'(x)\, dx + \sum_{x \in S(v)} [v](x) = \beta l$$

An inconvenience of evolutionary global minimization

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$$\int_0^l v'(x)\, dx + \sum_{x \in S(v)} [v](x) = \beta l$$

necessary condition for a minimum: $v'(x) = const$

$$E(v) = \frac{1}{2} kl \left(\beta - l^{-1} \sum_{x \in S(v)} [v](x)\right)^2 + \gamma \,\#_v$$

Solution of the minimum problem for

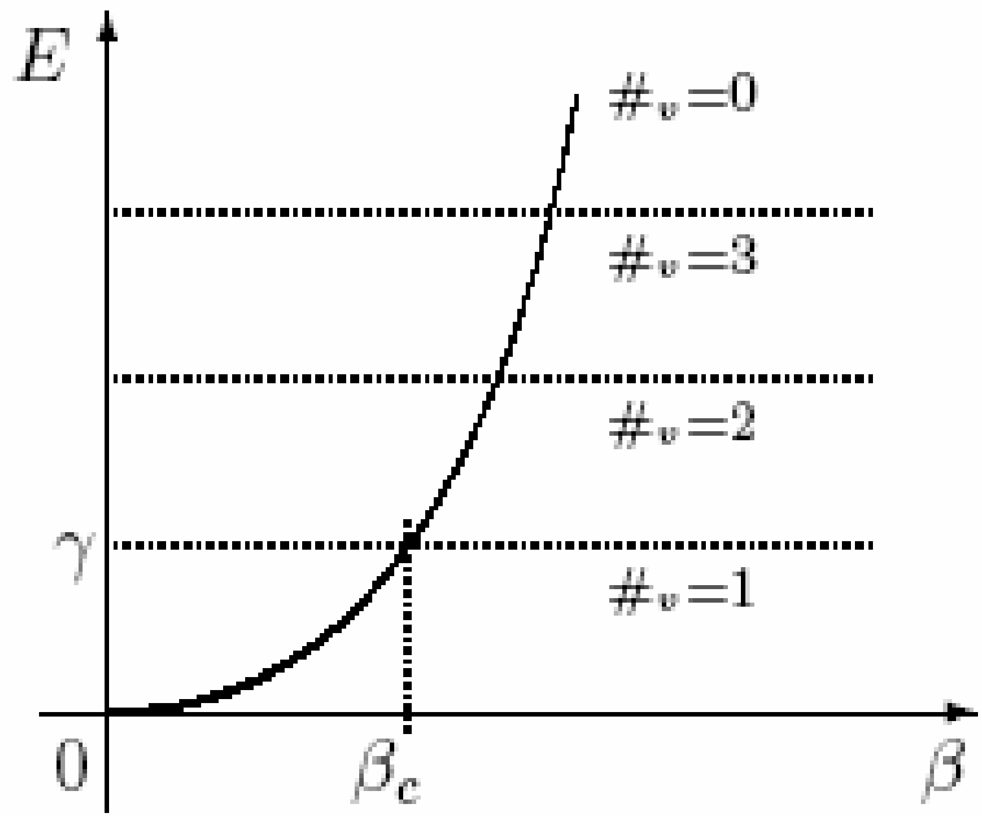
$$E(v) = \tfrac{1}{2}\, kl \left(\beta - l^{-1} \sum_{x \in S(v)} [v](x)\right)^2 + \gamma\, \#_v$$

Local minimizers

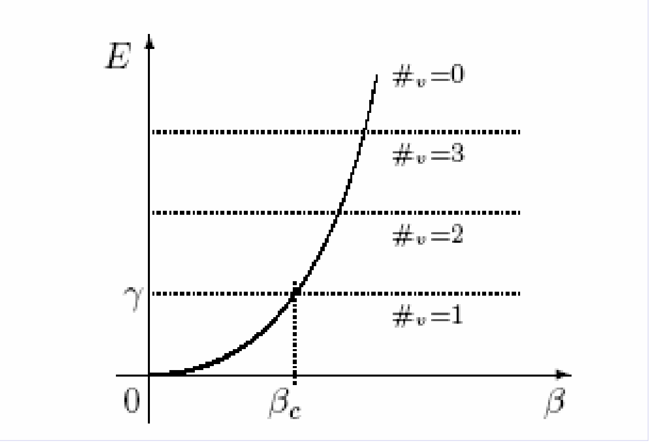
for $\#v = 0$: $u(x) = \beta x$, $E(u) = \tfrac{1}{2} kl\beta^2$,

for $\#v > 0$: $u'(x) = 0$, $E(u) = \gamma \#_u$.

$$\text{for } \#v = 0: \quad u(x) = \beta x \,, \quad E(u) = \tfrac{1}{2} kl\beta^2,$$
$$\text{for } \#v > 0: \quad u'(x) = 0 \,, \quad E(u) = \gamma \#u \,.$$

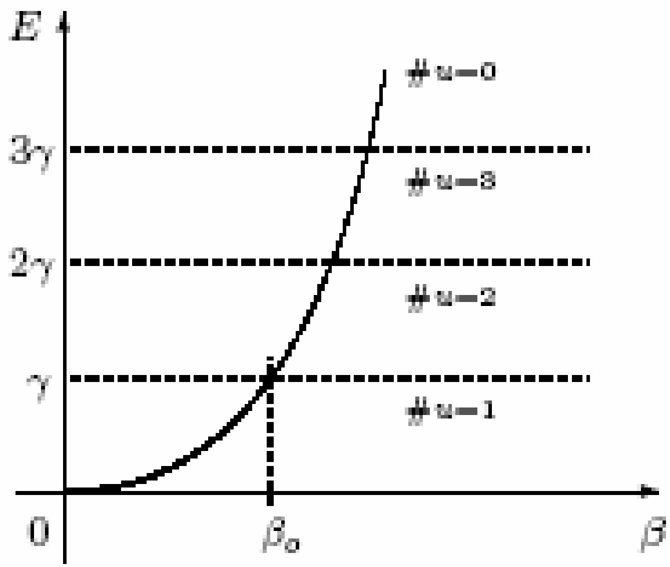


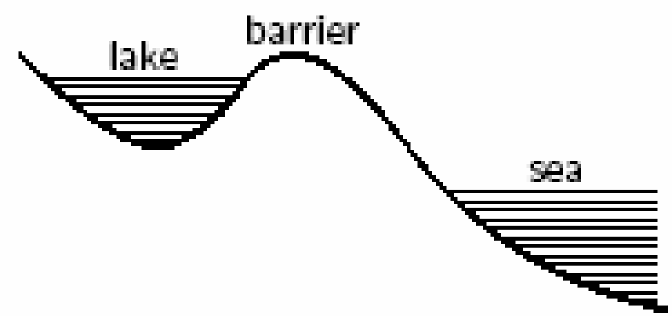
it is wrong to conclude that the bar breaks at $\beta = \beta_c$. Indeed, $\beta = \beta_c$ is only a situation in which two different configurations have the same energy, which is a global minimum. But these configurations need not be accessible from each other.



a transition from $\#_v = 0$ to $\#_v = 1$ requires the finite energy γ, while a continuation along the branch $\#_v = 0$ only requires an infinitesimal energy [3]. The conclusion is that the bar never breaks. This confirms the well known fact that Griffith's theory is unable to predict the fracture onset

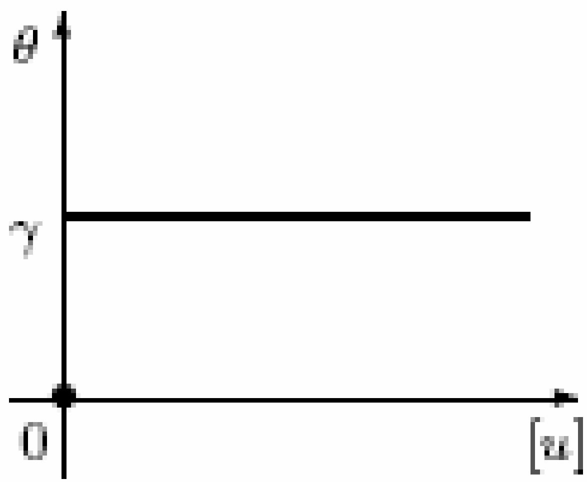
This shows that an analysis based on global minimization is unrealistic, and makes it evident that the solutions of the evolutionary problem are, in general, only local minimizers



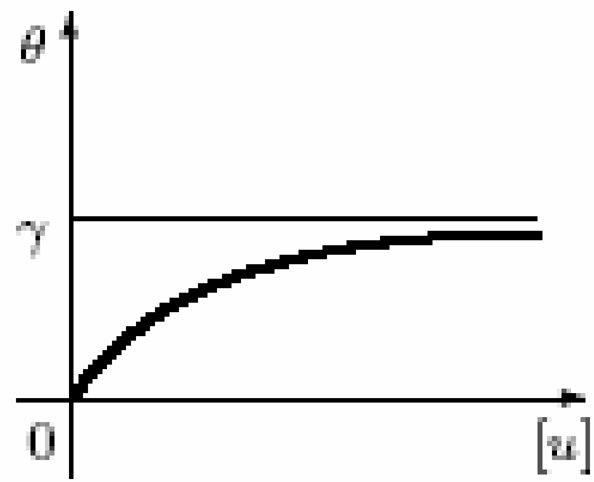


Barenblatt's regularization

$$E(u) = \frac{1}{2}\int_0^l k u'^2(x)\,dx + \sum_{i=1}^{\#u} \theta([u](x_i))$$

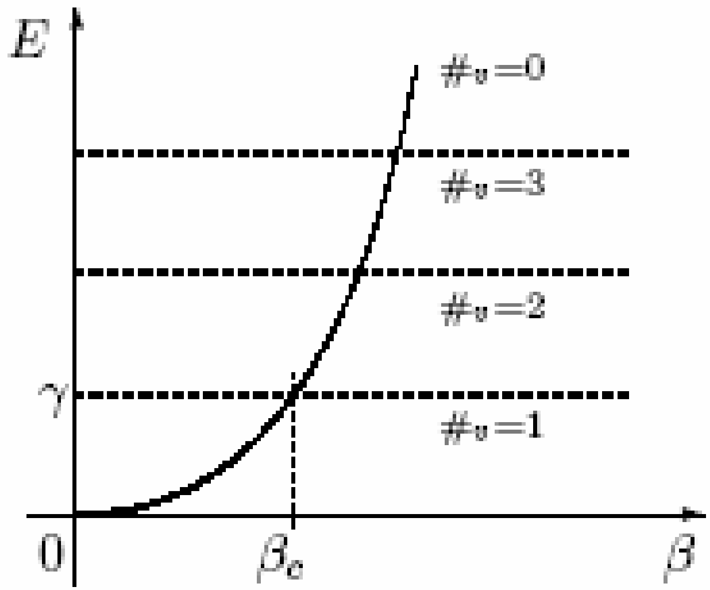


Griffith



Barenblatt

Griffith's and Barenblatt's solutions



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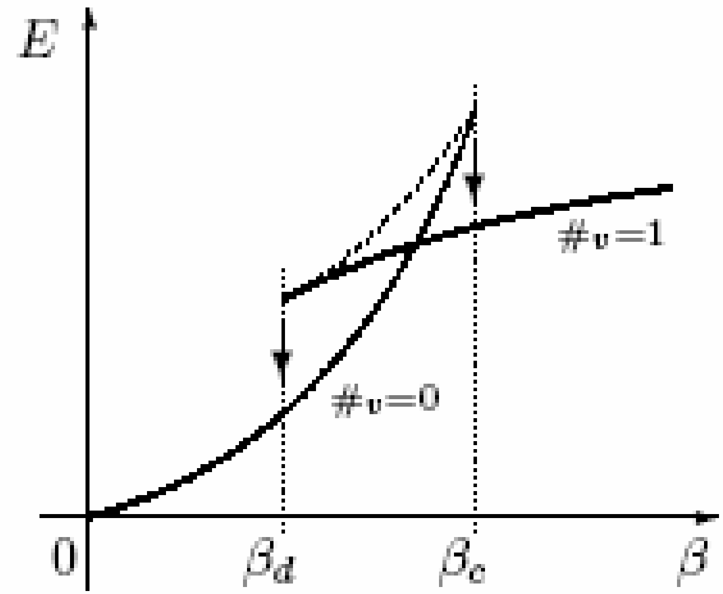
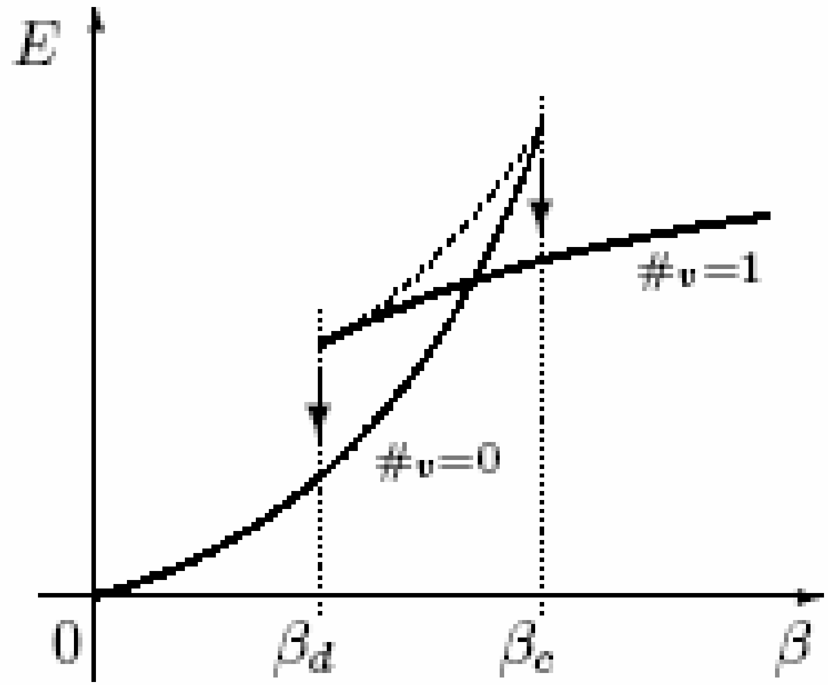


Barenblatt

Advantages of Barenblatt's model:

- predicts fracture onset,
- shows the size effect.



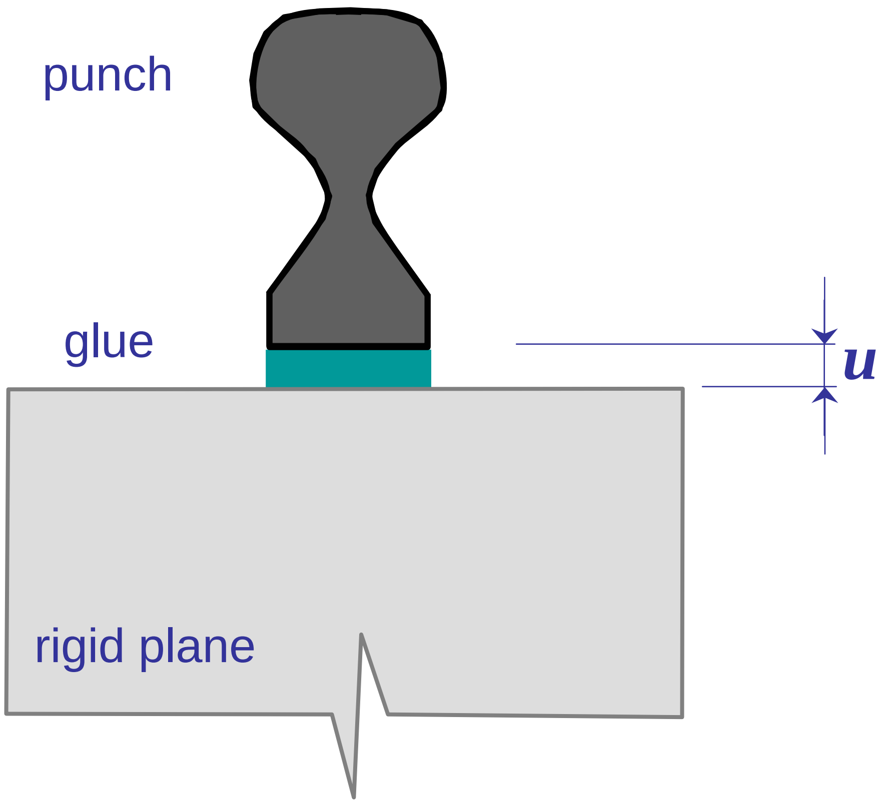
Inconvenience of Barenblatt’s model:

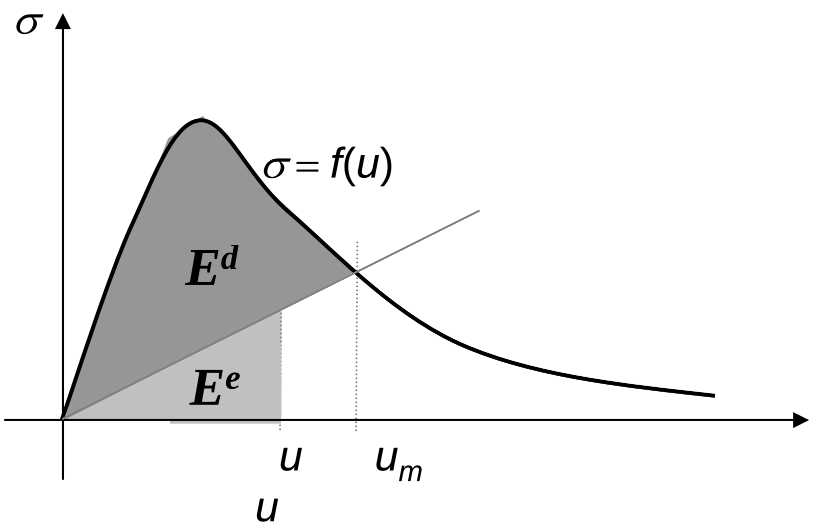
- at unloading, it predicts a jump from $\#_v=1$ to $\#_v=0$, in contradiction with the irreversibility of fracture.

This drawback of the model originates from the fact that in the applications of Barenblatt's theory the energy of a jump is usually considered as a part of the free energy. A more correct response is obtained by assuming that it is instead a dissipated energy

rather than from specific constitutive assumptions, elastic unloading can be obtained from incremental minimization

An adhesive contact problem with elastic unloading





u layer thickness
σ adhesion force
u_m state variable

$$u_m(t) = \max \{ u(s) , s \leq t \}$$

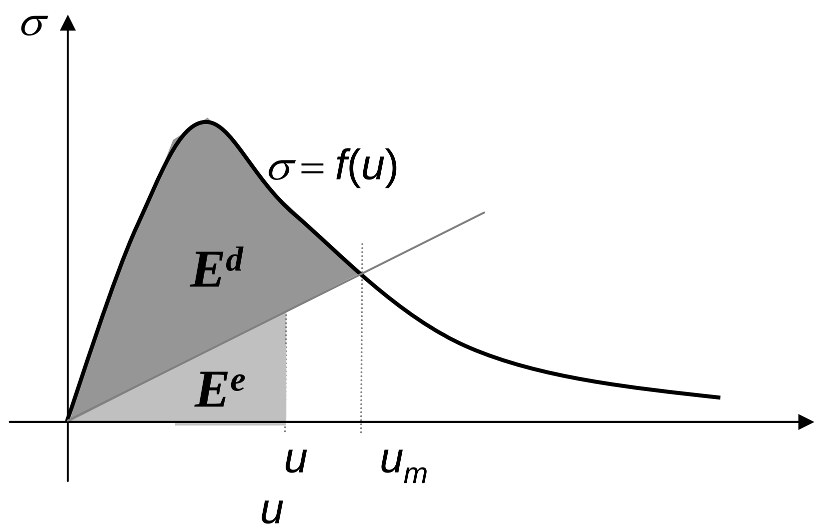
$\sigma = f(u)$ equation of loading curve
$g(u) = f(u) / u$ slope of unloading line

$$g'(u) < 0$$

$$\boldsymbol{E = E^e + E^d}$$

the elastic and the dissipative parts of the energy

$$E^e(u, u_m) = \frac{1}{2}\, g(u_m)\, u^2\,, \qquad E^d(u_m) = \int_0^{u_m} f(v)\, dv - \frac{1}{2}\, g(u_m)\, u_m^2\,.$$



Incremental energy minimization

Given: $t \mapsto u_t\,, \quad u_m(t)\,,$

Minimize

$$E_{t+\varepsilon} - E_t = \varepsilon\,\delta E_t + \frac{1}{2}\,\varepsilon^2\delta^2 E_t + o(\varepsilon^2)$$

under the condition:

$$\dot{E}^d = -\frac{1}{2}\,g'(u_m)\,u_m^2\,\dot{u}_m \geq 0$$

which implies

$$\dot{u}_m \geq 0$$

first-order minimization

$$\delta E_t = \frac{1}{2}\, g'(u_m)\, u^2 \dot{u}_m + g(u_m)\, u\dot{u} - \frac{1}{2}\, g'(u_m)\, u_m^2 \dot{u}_m$$

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$$\boxed{g'(u_m)(u^2 - u_m^2)\, \dot{u}_m = \textbf{min} \,, \qquad \dot{u}_m \geq 0}$$

first-order minimization

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$$\boxed{g'(u_m)(u^2 - u_m^2)\, \dot{u}_m = \mathbf{min} \;, \qquad \dot{u}_m \geq 0}$$

$$u < u_m \;\Rightarrow\; \dot{u}_m = 0$$

$$u = u_m \;\Rightarrow\; \delta E_t = 0$$ **nothing to minimize**

second-order minimization

$$\delta^2 E_t \Big|_{u=u_m} = -g'(u_m)\, u_m(\dot{u}_m^2 - 2\dot{u}_m \dot{u})$$

$$\dot{u}_m(\dot{u}_m - 2\dot{u}) = \min \qquad \dot{u}_m \geq 0 \qquad \boxed{\dot{u}_m \geq \dot{u}}$$

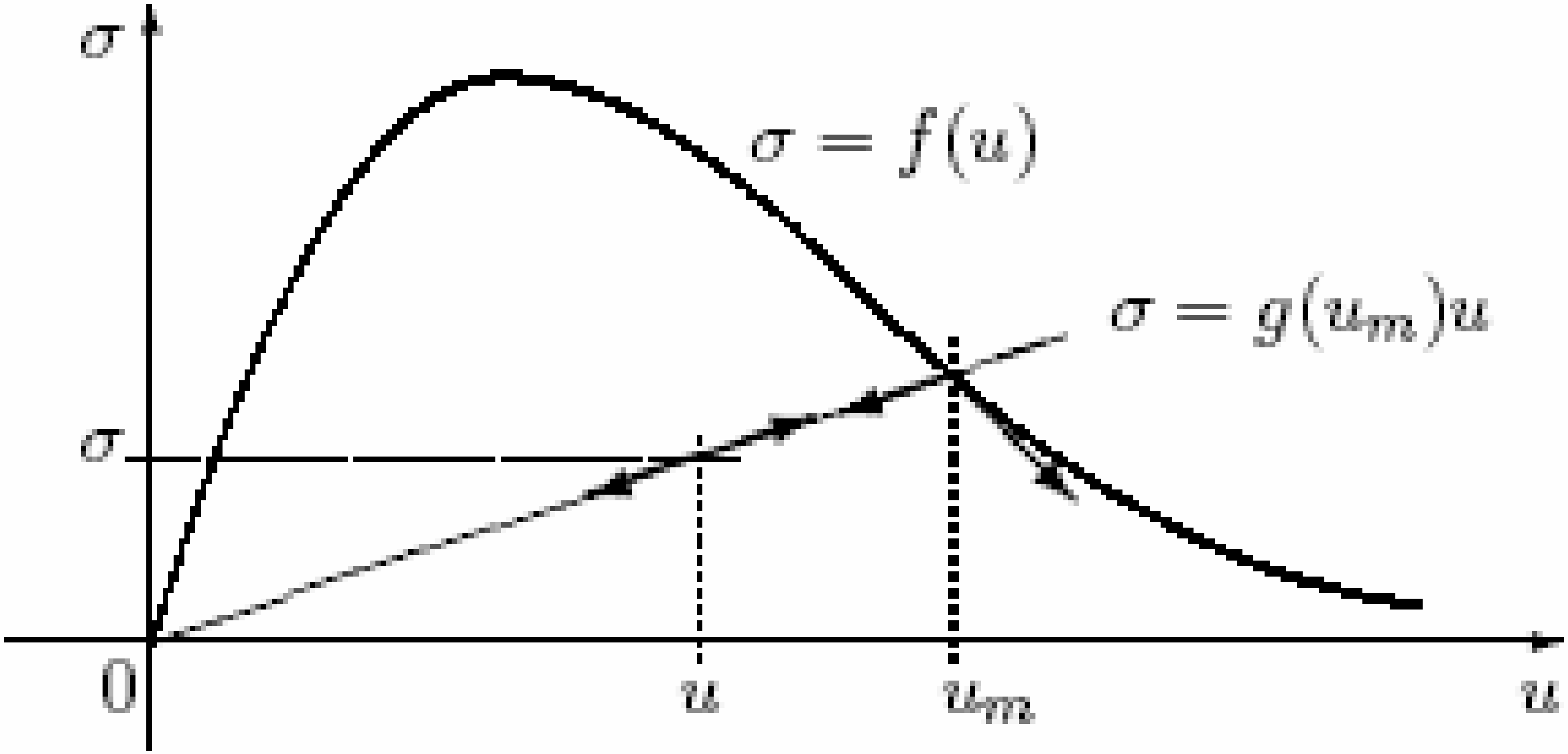
second-order minimization

$$\delta^2 E_t \big|_{u=u_m} = -g'(u_m)\, u_m (\dot{u}_m^2 - 2\dot{u}_m \dot{u})$$

$\dot{u}_m(\dot{u}_m - 2\dot{u}) = \min$ $\qquad \dot{u}_m \geq 0 \qquad \boxed{\dot{u}_m \geq \dot{u}}$

solution:

$$\boxed{\begin{aligned} \dot{u} > 0 \quad &\Longrightarrow \quad \dot{u}_m = \dot{u}\,, \\ \dot{u} < 0 \quad &\Longrightarrow \quad \dot{u}_m = 0\,. \end{aligned}}$$



directions of loading-unloading, as given by incremental energy minimization

Conclusions

- A number of inelastic, path-dependent problems can be solved by incremental energy minimization.

- In general, the domain of the (incremental) energy functional is not a linear space. The associated field equations are piecewise linear. This determines the different responses at loading and at unloading

- Sometimes the first-order incremental problem is trivial, and the solution is provided by the second-order problem.