Lower bound limit analysis of masonry bridges including arch–fill interaction

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Abstract

Collapse tests on full and model scale masonry bridges have shown the structural role of fill and spandrels, which has to be taken into account to obtain realistic evaluations of the load carrying capacity of existing bridges. A plane model of multi-span masonry bridge is proposed in which the vault–fill interaction effects are considered, whose lower bounds on collapse load are obtained by a finite element application of the lower bound theorem of limit analysis. Arches and piers are modelled as beams made up of no tension, ductile in compression material and the fill as a cohesive-frictional material with a tension cut-off. The fill domain is discretized by triangular elements connected by interface elements in order to increase the ratio of the unknowns to the conditions of static admissibility; arches and piers are discretized by two-node straight beam elements. By linearisation of the conditions of plastic admissibility, a Linear Programming problem is formulated and lower bounds on the collapse load are obtained. The procedure is successfully applied to two example bridge models, where a comparison with the results obtained from the kinematic approach is made. The first example is a simulation of a collapse test on a single span bridge; the second concerns a multi-span bridge and highlights the capability of the procedure to describe complex interactions between the arch–pier structural system and the fill at collapse.

Keywords: Masonry bridges; Arch–fill interaction; Limit analysis; Lower bound

1. Introduction

Collapse mechanisms of masonry bridges are the result of complex interactions between all their components. In particular, experimental collapse tests on full scale and model scale bridges [1] have shown the relevant increase of collapse load due to the strength of fill and spandrels.

The analysis of the interaction between arch, fill and spandrels needs detailed three-dimensional models [2] with suitable constitutive models able to describe the behaviour of the masonry under triaxial stress states. On the other hand, when the transverse effects can be neglected and the spandrel walls are not considered, the structural system can be assumed as transversally homogeneous and idealised by a plane model to obtain a simpler description of the arch–fill interaction. This makes it possible to assume simplified constitutive models for the masonry, thereby reducing the sources of uncertainties; moreover, the bridge behaviour can be more synthetically described. Two-dimensional models based on the no-tension assumption for the masonry, originally proposed in the pioneering works by Castigliano [3], Kooharian [4] and Heyman [5], have been extended by various authors to take into account the fill (see e.g. [6]): Crisfield [7] and Choo [8] modelled the fill by one-dimensional devices; Crisfield and Packam [9] and Hughes et al. [10] have extended the mechanism method by directly applying horizontal forces to the arch representing the fill containing effect; Bicanic et al. [11] have made use of DDA (Discontinuous Deformation Analysis) modelling in which the fill behaviour is described by means of interacting blocks. Cavicchi [12] and Cavicchi and Gambarotta [13] have proposed a model in which the fill is described as a cohesive-frictional continuum interacting with the arch–pier system represented by no-tension, perfectly plastic beams; upper estimates of the collapse load and the corresponding collapse mechanisms have been obtained by seeking the minimum kinematically admissible load multiplier of a compatible FE discretization of the model. Unfortunately,
the unconservative nature of the results obtained from the kinematic approach may represent a problem for the applications of the results so obtained.

The aim of this paper is to overcome this limitation by applying the lower bound theorem of limit analysis to a statically admissible FE discretization of the mechanical model defined in Cavicchi and Gambarotta [13], so that lower bounds on the collapse load are obtained together with the corresponding stress fields. In this case, the effects of the approximations due to FE discretization do not affect the safety of the result. Moreover, the static approach makes it possible to take into account the effects of a limited transverse constraining action of the spandrel walls on the fill by introducing a condition which limits the transverse compressive stress under an admissible value; this feature provides a simplified means to evaluate the dependence of the collapse load on the transverse effects and release and discuss the plane strain assumption [13].

The fill is assumed in plane state and the statically admissible stress field is described by the three in-plane components and the out-of-plane principal stress. Mohr–Coulomb criterion with cut-off is assumed and the transverse stress component is limited within a compressive admissible range. The limit value assumed for this admissible range can be referred to the conditions which limits the transverse compressive stress under an admissible value; this feature provides a simplified means to evaluate the dependence of the collapse load on the transverse effects and release and discuss the plane strain assumption [13].

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2. Plane equilibrium model of the masonry bridge

A longitudinal cross section of a typical masonry bridge, a three-dimensional structural system composed of arch barrels, piers, fill, haunchings, spandrel walls and abutments, is diagrammatically described in Fig. 1(a). In order to get the most relevant effects of the arch–fill interaction through a simplified two-dimensional approach, the structural model described in Fig. 1(b) is assumed, in which the spandrel walls are not considered. The three-dimensional region occupied by the fill is represented by the two-dimensional domain \( \Omega_f \), representing a longitudinal section of the fill, while arch barrels and piers are described by plane curved beams: the \( i \)-th beam occupies the three-dimensional region \( \Omega_i \), and is represented by its centre line \( Z_i \), as shown in Fig. 1(b). The piers at the base and the arch barrels at springings are considered built in and the connections between the arch springings and the top of the piers are assumed rigid. Finally, the fill is restrained at the opposite ends of the bridge.

2.1. Applied forces

The external forces are assumed parallel to the plane of the longitudinal cross section of the bridge and uniform across the width. The self-weight per unit area of the fill is denoted by vector \( \mathbf{b} = \{b_1 \ b_2\}^T \) and line tractions distributed over the upper boundary of the fill \( \partial \Omega_f \) are denoted by vector \( \mathbf{p} = \{p_1 \ p_2\}^T \); the self-weight per unit length of the arch barrels and the piers is referred to the centre line and denoted by vector \( \mathbf{b}_b = \{b_{b1} \ b_{b2}\}^T \), external couples are ignored. The load vector \( \mathbf{p} \) is decomposed as \( \mathbf{p} = \mathbf{p}_0 + s \mathbf{p} \), where \( \mathbf{p}_0 \) is fixed and \( s \mathbf{p} \) represents the live load, \( s \) being the multiplier of the reference live load \( \mathbf{p} \).

2.2. Stress field assumptions

Among the three-dimensional stress fields in equilibrium with the prescribed loads, only plane fields are considered, obtained by imposing the homogeneity across the width, namely \( \sigma_{i,j,3} = 0 \), and vanishing tangential stresses acting on the plane normal to \( x_3 \) axis, \( \sigma_{31} = \sigma_{32} = 0 \); the stress field is therefore represented by vector \( \mathbf{\sigma}(\mathbf{x}) = \{\sigma_{11} \ \sigma_{11} \ \tau_{12} \ \sigma_3\}^T \), defined on the domain \( \Omega_f \), \( \sigma_3 \) being principal stress.

The out-of-plane stress component \( \sigma_3 \) in the fill is induced by smooth boundaries which represent the effects of either the spandrel walls or the tie rods inserted in the fill and connecting the opposite spandrel walls to strengthen masonry bridges [14]. In the latter case, the interaction between the fill, the spandrels and the tie rods can be assumed to generate self-equilibrated stress states with no effects on the equilibrium of the arch barrel; if the tie rods are not present, the lateral

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Fig. 1. (a) Longitudinal cross section and (b) two-dimensional model of the bridge.
pressure generated on the spandrel walls may be reduced to a resultant force and couple per unit length acting on the lateral boundaries of the arch barrel extrados, thereby making the present analysis approximated as will be inferred by further assumptions. It is worth noting that the equilibrium equations and traction boundary conditions in the $x_1$ and $x_2$ directions must be considered to obtain statically admissible stress fields, while the fill is self-equilibrated in the transverse direction, independently of the value of the $\sigma_3$ component.

According to the previous assumptions, the mutual actions between the fill and the extrados of the arch barrels at the interfaces $\mathcal{J}$ are assumed lying on the $(x_1, x_2)$ plane and uniform across the bridge width, so that they can be represented by the normal $\sigma_n$ and the tangential $\tau$ components of the interface stress resultant vector $\boldsymbol{\sigma}_\mathcal{J} = [\sigma_n \tau]^T$.

Finally, a simple description of the stress field in arch barrels and piers is obtained by ignoring the membrane and shear forces and the bending moment acting on planes parallel to the longitudinal cross section and the twisting moment as well; the generalized stress vector results in $\boldsymbol{\sigma}_\mathcal{S} = \begin{bmatrix} N & V & M \end{bmatrix}^T$, $N$ and $V$ being the axial and the shear forces, respectively, and $M$ the bending moment.

2.3. Constitutive assumptions

Both the fill and the masonry arches and piers are assumed made of no-tension rigid perfectly plastic materials. The fill is assumed as a Mohr–Coulomb cohesive-frictional no-tension material, whose conditions of plastic admissibility read

\[
\max \left[ |\sigma_a - \sigma_\beta| + (\sigma_a + \sigma_\beta) \sin \varphi \right] - 2c \cos \varphi \leq 0, \quad \sigma_a \neq \sigma_\beta, \quad \alpha \neq \beta, \quad \max(\sigma_a) \leq 0, \quad \varphi, c \text{ being the friction angle and the cohesion of the fill, respectively, and } \alpha, \beta = 1, 3 \text{ denoting the principal stress directions. Moreover, a constraint on the transverse stress is assumed}
\]

\[
\sigma_3 \geq -\sigma_c, \quad (3)
\]

where the limiting value $\sigma_c \geq 0$ can represent either the prescribed maximum allowable pressure on the spandrel walls or the maximum allowable tensile strength of the tie rods per unit influence area of the lateral surface of the fill.

Since equilibrium in the transverse direction is implicitly satisfied, the stress component $\sigma_3$ only depends on the inequalities (1)–(3) that provide a range of admissible values of $\sigma_3$, function of $\sigma_1$, $\sigma_2$ and $\sigma_c$. As a consequence, it is unnecessary to maintain $\sigma_3$ in the formulation, but it is sufficient to ensure that its admissible range exists. The resulting conditions of plastic admissibility can therefore be expressed in the form

\[
|\sigma_1 - \sigma_2| + (\sigma_1 + \sigma_2) \sin \varphi - 2c \cos \varphi \leq 0, \quad \sigma_\alpha \leq 0, \quad \alpha = 1, 2, \quad -\sigma_\alpha + \sigma_c^* - r\sigma_c \leq 0, \quad \alpha = 1, 2,
\]

where $\sigma_c^* = -2c \cos \varphi/(1 - \sin \varphi)$ and $\sigma_c^* = 2c \cos \varphi/(1 + \sin \varphi)$ being the uniaxial compressive and tensile strength, respectively, and $r = -\sigma_c^*/\sigma_c^* = (1 + \sin \varphi)/(1 - \sin \varphi)$; the corresponding plastically admissible stress domain is shown in Fig. 2(a). The admissible range of $\sigma_3$ is obtained a posteriori as a function of $\sigma_1$ and $\sigma_2$; the corresponding minimum admissible compressive value is

\[
\begin{align*}
(a) & \quad \text{if } \min(\sigma_\alpha, \alpha = 1, 2) \geq \sigma_c^*, \quad \text{then } \sigma_3 = 0; \\
(b) & \quad \text{else } \sigma_3 = \sigma_c^* + \frac{1}{r} \min(\sigma_\alpha, \alpha = 1, 2).
\end{align*}
\]

Case (a) corresponds to plane stress; as the limit transverse stress $\sigma_c$ tends to infinity, condition (6) turns out to be not active and the no-tension Mohr–Coulomb plane strain criterion is obtained (inequalities (4) and (5)).

In analogy, the no-tension Coulomb yield criterion is assumed at the interface between the fill and the extrados of arches.

The yield function of the cross sections of arch barrels and piers is obtained by assuming vanishing tensile strength across the mortar joints orthogonal to the centre line and rigid–ideal plastic response under compression. The first assumption turns out to be acceptable for both stone arches made of ashlar and brick masonry arches having radial mortar joints continuous from extrados to intrados. In the present analysis, sliding failure is neglected. As a consequence the condition of plastic admissibility referred to the beam cross section is $f_b(\sigma_b) = M + 2N(1 + \bar{N}) \leq 0$, where $\bar{N} = N/N_p$, $M = M/M_p$, $N_p = bh\sigma_c$, $M_p = 1/4bh^2\sigma_c$, $\sigma_c \geq 0$ being the compressive strength of the masonry and $b$ and $h$ the width and the depth of the rectangular cross section (Fig. 2(b)).

3. Finite element lower bound evaluation of the collapse load

Lower bound estimates of collapse load multiplier $s_c$ are obtained by a finite element application of the Lower Bound Theorem of Limit Analysis [15,16]. In Fig. 3(a), the two-dimensional finite element discretization of the model is shown: the fill is approximated by three-node triangular elements and stress discontinuities are allowed at their shared edges [17,18]; arches and piers are approximated by straight beam elements. The presence of the discontinuities allows for enriching the stress field and thus increasing the ratio of the unknowns to the conditions of static admissibility.

A description of the assumed finite elements and the resulting Linear Programming problem providing lower bound estimates of the collapse load multiplier with the corresponding statically admissible stress fields are given in the following sections.
3.1. Fill discretization

The fill domain \( \Omega_f \) is discretized by three-node triangular elements; statically admissible stress discontinuities are allowed at the edges shared by adjacent triangular elements [17]. The stress field in the elements is assumed linearly dependent on the nodal stresses \( \sigma^r = [\sigma^r_{11}, \sigma^r_{22}, \tau^r_{12}]^T, \quad r = h, k, l \), so that the in-plane differential equilibrium equations of the \( t \)-th element are expressed as linear equations of the nodal stresses in the matrix form

\[
C^h_t \sigma^h + C^k_t \sigma^k + C^l_t \sigma^l + c_t = 0,
\]

where the vector \( c_t \) depends on the applied load.

In order to obtain a linear programming formulation preserving the lower bound property of the result, the conditions of plastic admissibility \((4)–(6)\) are piecewise linearized in the space of stress components by an internal polyhedron obtained as intersection of three polytopes with an even number \( p_t \) of faces, defined by inequalities

\[
f_t^\alpha(\sigma) = A^\alpha_t \sigma^h + B^\alpha_t \sigma^k + C^\alpha_t \sigma^l + D^\alpha_t \leq 0,
\]

\[
z = 1 \ldots p_t, \quad \alpha = 1, 2, 3,
\]

where the superscript \( \alpha = 1, 2, 3 \) refers to conditions \((4), (5)\) and \((6)\), respectively, and

\[
A^1_t = \cos \phi \sin \varphi + \cos(2\phi z),
\]

\[
A^2_t = \cos \phi + \cos(2\phi z),
\]

\[
A^3_t = -\cos \phi - \cos(2\phi z),
\]

\[
B^1_t = \cos \phi \sin \varphi - \cos(2\phi z),
\]

\[
B^2_t = \cos \phi - \cos(2\phi z),
\]

\[
B^3_t = -\cos \phi + \cos(2\phi z),
\]

\[
C^1_z = 2 \sin(2\phi z),
\]

\[
C^2_z = 2 \sin(2\phi z),
\]

\[
C^3_z = -2 \sin(2\phi z),
\]

\[
D^1_z = -2c \cos \phi \cos \varphi,
\]

\[
D^2_z = 0,
\]

\[
D^3_z = 2(\sigma^r - r \tilde{\sigma}_z) \cos \phi,
\]

where \( \phi = \pi / p_t \). The resulting polyhedron has \( p_f = 3p_t \) faces and the corresponding \( p_f \) inequalities are collected in the matrix form

\[
f_t(\sigma) = N_t^T \sigma - r_t \leq 0, \quad r = h, k, l.
\]

The linear interpolation of the stress field makes it sufficient to impose the conditions of plastic admissibility at element nodes

\[
f_t(\sigma^r) = N_t^T \sigma^r - r_t \leq 0, \quad r = h, k, l.
\]
resultant nodal stress vectors at the edges. Because of the linear interpolation of the stress field, the unilateral Coulomb conditions $f^{\mu}(\sigma_{\mu}) = \pm t + \sigma_{\nu} \tan \varphi - c \leq 0$ and $f^{\nu}(\sigma_{\nu}) = \sigma_{v} - \sigma_{t} \leq 0$ at the shared edges can be imposed only at nodes, so that the condition of plastic admissibility becomes
\[ f_{b}^r = N_{b}^{T} \sigma_{b}^r - r_{b} \leq 0, \quad r = h, k, \] (12)

involving the nodal stresses of one of two adjacent elements.

3.2. Arch and pier discretization

Arches and piers are discretized by two-node straight beam elements having constant section height and self-weight per unit length $b_{h}$. The nodal force at the $r$-th node is represented by vector $\bar{s}_{b_{r}} = \{F_{b_{r}}^{e}, F_{b_{r}}^{v}, C_{b_{r}}^{T}\}^{T}$, $F_{b_{r}}^{e}$ and $F_{b_{r}}^{v}$ being the components of the nodal force and $C^{T}$ the nodal couple with respect to the local reference $(\xi, \nu)$ (Fig. 6). Each beam element discretizing the arches interacts with the edge of a triangular element of fill (Fig. 3(b)), whose stress field resolved at the edge corresponds to the action on the fill of the exadmos of the beam; the generalized forces acting on the beam axis are then obtained by simply taking into account the effect of the beam height on the bending moment. The linearity of the actions coming from the fill allows for writing the overall equilibrium equations of the beam in the matrix form
\[ C_{b}^{l} s_{b_{l}}^{l} + C_{b}^{h} s_{b_{h}}^{h} + C_{b}^{k} s_{b_{k}}^{k} + c_{b} = 0, \] (13)
where $\sigma^{h}$ and $\sigma^{k}$ are the nodal stresses of the edge of the triangular element and vector $c_{b}$ depends on the applied body force $b_{b}$ (see Fig. 3(b)).

Equilibrium allows for obtaining the stress vector at section $\xi$ as
\[ \sigma_{b}(\xi) = D_{b}^{h} s_{b_{h}}^{h} + D_{b}^{k}(\xi) \sigma^{h} + D_{b}^{k}(\xi) \sigma^{k} + d_{b}(\xi), \] (14)
where matrices $D_{b}^{h}$ ($r = i, h, k$) depend on the position $\xi$ and $d_{b}$ collects the contributions of the dead load applied to the element.

To obtain a linear formulation, the plastic limit envelope in the $(N, M)$ plane is piecewise linearised by an internal polygon with an even number $p_{b}$ of sides, as shown in Fig. 4(b), which is defined by the inequalities
\[ f_{b}^{\pm}(N, M) = \pm \left( \frac{N_{z}^{z+1} - N_{z}^{z}}{N_{p}M_{p}} \right) N + \left( \frac{N_{z}^{z} - N_{z}^{z+1}}{N_{p}M_{p}} \right) M \]

and
\[ \pm \left( \frac{N_{z}^{z+1}M_{z} - N_{z}^{z}M_{z+1}}{N_{p}M_{p}} \right) \leq 0, \quad z = 1 \ldots p_{b}/2, \] (15)

where $N_{z} = -2\frac{z-1}{p_{b}}N_{p}$, $M_{z} = -\frac{2M_{p}}{N_{p}}\left( \frac{N_{z}}{N_{p}} + 1 \right)$. The $p_{b}$ inequalities (15) are collected in the matrix form $f_{b}(\bar{\xi}) = N_{b}^{T} \sigma_{b}(\bar{\xi}) - r_{b} \leq 0$, representing the condition of plastic admissibility, which is imposed at a discrete number of sections $\xi_{t} = \frac{t-1}{p_{b}-1}l$, $t = 1 \ldots p_{b}$, $l$ being the beam element length, obtaining a linearised condition. As known, this approximation does not guarantee the lower bound property of the load multiplier provided by the solution of the resulting linear programming problem. On the other hand, the mesh refinement necessary for a good description of fill imposes a discretization of the arches that makes negligible the effects of $p_{b}$ on the result (see the examples described in Section 4). According to this approximation, the condition of plastic admissibility of the beam element reads $f_{b}^{0} = N_{b}^{T} \sigma_{b}(\bar{\xi}) - r_{b} \leq 0, \quad t = 1 \ldots p_{s}$, that, by substituting equilibrium Eqs. (14), becomes
\[ f_{b}^{i} = \left( N_{b}^{T} D_{b}^{h} \right) s_{b}^{h} + \left( N_{b}^{T} D_{b}^{k}(\xi_{t}) \right) \sigma^{h} + \left( N_{b}^{T} D_{b}^{k}(\xi_{t}) \right) \sigma^{k} + \left( N_{b}^{T} d_{b}(\xi_{t}) \right) - r_{b} \leq 0, \quad t = 1 \ldots p_{s}, \] (16)
involving the nodal force vector $s_{b}$ and the triangle nodal stress vectors $\sigma^{h}, \sigma^{k}$.

3.3. Linear Programming problem

The generalized nodal stresses in the triangular and beam elements, collected in vector $\sigma$, and the load multiplier $s$ are the discrete variables which define the finite element equilibrium model. The equilibrium equations are collected in the linear matrix equation $C_{\sigma} = c$, in which the equilibrium equations of nodes where beams converge have been included. The boundary conditions on the nodes where active and reactive forces are applied are expressed in the linear form $Q_{\sigma} - s_{q} = q_{b}$, where $q_{b}$ and $s_{q}$ are the nodal stress vectors depending on the applied dead and live loads, respectively. Moreover, the conditions of plastic admissibility (11), (12) and (16) are collected in the matrix inequality $f = N_{p}^{T} \sigma - r \leq 0$, so that the largest lower bound $s_{b}$ on the collapse load multiplier is
obtained as the solution of the Linear Programming problem

\[
\begin{align*}
\mathbf{s}_b &= \max(s), \\
\mathbf{Ca} &= \mathbf{c}, \\
\mathbf{Qa} + s\mathbf{q} &= \mathbf{q}_0, \\
\mathbf{N}^T \mathbf{a} + \mathbf{a} &= \mathbf{r}, \\
a &\geq 0.
\end{align*}
\]

The total number \(n_u\) of unknowns is \(n_u = 9n_t(1 + p_t) + 9b(6 + p_b p_b) + 6n_s + 1\), \(n_t\) being the number of triangular elements and \(n_b\) the number of beam elements. The total number \(n_{eq}\) of equations is approximately \(n_{eq} \approx n_t(2 + 9p_t) + n_b(6 + p_b p_b) + 10n_s, n_s\) being the number of edges of adjacent triangular elements. Observing that \(n_s \approx 3/2, n_t\) [19], the approximations \(n_{eq} \approx n_t(17 + 9p_t) + n_b(6 + p_b p_b)\) and \(n_u \approx 9n_t(2 + p_t) + n_b(6 + p_b p_b) + 17\) hold. The number \(n_{in}\) of conditions of non-negativity is \(n_{in} = 9p_t n_t + 6n_s + p_b p_b n_b \approx 9n_t(p_t + 1) + n_b p_b p_b\); the dual of (17) contains \(n_{eq} = 9n_t + 6n_b + 1\) equations, \(n_{in} = 9p_t n_t + 6n_s + p_b p_b p_b n_b \approx 9n_t(p_t + 1) + n_b p_b p_b\) conditions of non-negativity and \(n_u \approx n_t(9p_t + 17) + n_b(p_b p_b + 6)\) unknowns; the reduced number of equations makes the dual problem (18) more convenient to solve than the primal (17). The solution of the problem (18) is obtained from the large scale algorithm provided by MATLAB, based on the Mehrotra algorithm [20].

In order to check the reliability of the lower bound numerical procedure here proposed, a preliminary test simulating the Prandtl’s problem was successfully carried out by means of the domain discretizations proposed in [17]. While the solution of this problem exhibits mesh sensitivity and requires proper mesh geometries [17] to obtain good evaluations, this dependence was observed to be limited in the problems considered in this paper, where the interest is focused on global limit stress states that involve both fill and arch.

### 4. Numerical examples

**4.1. Example 1: Prestwood bridge**

The first example refers to Prestwood bridge, a single span bridge tested up to collapse [21] within the experimental research on masonry bridges supported by the Transport Research Laboratory (TRL). The geometry of the bridge and the position of the live load are described in Fig. 5; more details about the loading system used during the test and an analysis of their possible influence on the results of the test will be given at the end of the example. The experimental arch collapse mechanism exhibits four hinges; the mechanism takes place with negligible material crushing and the experimental collapse load is \(P_{\text{exp}} = 228\) kN.

**Fig. 6** shows the numerical results obtained by assuming plane strain fill and the mechanical parameters given in Table 1; the values of the angle of internal friction \(\varphi\) and the compressive masonry strength \(\sigma_c\) come from experimental evaluations [21], while data about cohesion \(\gamma\) furnished in [21] are not sufficient; in order to avoid arbitrary evaluations, the cohesion is chosen in order to obtain the experimental collapse load \(P_{\text{exp}} = 228\) kN as approximately the average between the lower bound \(P_{\text{lb}} = 209\) kN and the upper bound \(P_{\text{ub}} = 254\) kN and an analysis of the sensitivity of the results on the mechanical parameters has been carried out and will be discussed shortly. The load is assumed applied through a punch to better simulate the experimental test; the objective function is therefore the resultant of the vertical pressure under the load line. The finite element model used is defined by the following parameters, resulting from a convergence analysis: \(n_t = 1182, p_t = 18, n_b = 80, p_b = 2, p_b = 48\). Some data about the influence of the admissible domain discretizations on the collapse load estimations and CPU time are shown in Table 2.

**Table 1**

<table>
<thead>
<tr>
<th>Material property</th>
<th>Prestwood bridge: constitutive parameters; experimental and numerical collapse loads</th>
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<tbody>
<tr>
<td></td>
<td>Masonry density (\gamma_m)</td>
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<tr>
<td>-------------------------</td>
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<td></td>
<td>20 kN/m(^3)</td>
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**Table 2**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
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<tbody>
<tr>
<td>(P_{\text{lb}})</td>
<td>Plane strain lower bound</td>
</tr>
<tr>
<td>(P_{\text{ub}})</td>
<td>Plane strain upper bound</td>
</tr>
<tr>
<td>(P_{\text{exp}})</td>
<td>Experimental collapse load</td>
</tr>
</tbody>
</table>

**Fig. 6(a)** shows the collapse mechanism obtained by assuming heavy but non-resistant fill and distributing the live load on the arch by assuming \(\eta = 30\); the circles in the arch are the generalised hinges. The corresponding collapse load is \(P_{\text{exp}} = 46\) kN and agrees with the result obtained by Crisfield [9]. The limit stress state and the collapse mechanism [13] obtained by assuming plane strain fill are shown in **Fig. 6(b)** (a) and (c), respectively (the finite element model is shown in **Fig. 6(c)**). In **Fig. 6(b)** the thick line in the depth of the arch represents the thrust line, while the straight lines in the fill represent the principal directions of the stress field, whose values are shown in **Fig. 7**; the contour plot in the fill in **Fig. 6(c)** show the maximum shear strain rate field due to the Mohr–Coulomb criterion. A good qualitatively agreement between the results from the equilibrium and
compatible models is obtained. In particular, the maximum eccentricities of the line of thrust are attained in correspondence with the positions of the generalized hinges. The limit state described by the numerical results appears to well reproduce the experimental behaviour; in particular, the location of the hinges from the analysis are a good approximation of the experimental positions.

The contour plot of the minimum compressive stress field $\sigma_3$ is shown in Fig. 8. The minimum values are reached in small regions located below the live load and at springings; within
Fig. 9. Influence of the allowable transverse stress $\tilde{\sigma}_c$ on the lower estimates of collapse load (bold line); estimates obtained by imposing plane strain in the fill under the punch (thin line); estimates obtained by assuming heavy but non-resistant fill (dashed line).

most of the left side of the fill and at the bottom of the right side $\sigma_3 \geq -10$ kPa holds, while the in-plane stress field in the regions in white is compatible with plane stress fill.

The effects of the maximum allowable transverse stress $\tilde{\sigma}_c$ on the lower estimates of the collapse load are shown in the graph of Fig. 9 (bold line). For $\tilde{\sigma}_c \geq 63$ kPa the plane strain limit stress state is obtained, while for $\tilde{\sigma}_c < 63$ kPa the limit condition (6) affects the result by reducing the lower estimates of the collapse load until the value $P_{lb}^{\text{stress}} = 57$ kN obtained by assuming $\tilde{\sigma}_c = 0$ kPa (plane stress fill).

The results obtained by assuming $\tilde{\sigma}_c = 20$ kPa are shown in Fig. 10. The corresponding lower bound is $P_{lb}^{u} = 149$ kN and the stress field is similar to the case of plane strain fill, with lower values of $\sigma_3$. The limit state due to the limitation of the transverse stress $\sigma_3$ is first reached by the elements just under the live load, so that the load is equilibrated by a uniform distribution of the maximum compressive stress $\sigma_c^* = r \tilde{\sigma}_c$ under the live load position. The decreased estimate of the collapse load is therefore due to a local effect; this aspect is well shown by the results obtained by keeping the cone under the live load in-plane strain and varying $\tilde{\sigma}_c$ through the rest of the fill. In this case the lower bound on the collapse load (Fig. 9, thin line) decreases only for very low values of $\tilde{\sigma}_c$, and the value corresponding to plane stress $P_{lb}^{\text{stress}} = 189.5$ kN, 91% of $P_{lb}^{\text{strain}} = 208.8$ kN corresponding to plane strain. This result shows that the effect of $\sigma_3$ is relevant only in the region just under the live load, while the diffused reduction of fill strength has a slight effect on the estimates of the collapse load.

The effects of the cohesion $c$ and the angle of friction $\varphi$ of the fill on the lower and upper bound estimates of collapse load by assuming plane strain are shown in the graphs of Fig. 11(a) and (b), respectively; the effect of the masonry compressive strength $\sigma_c$ is shown in Fig. 12. The contribution of the fill resistance to the collapse load estimates is important also for low values of cohesion and friction. In particular, the effects of a decrease in the cohesion are limited, and the lower estimate of the collapse load is still higher than 150 kN by assuming $c = 5$ kPa. Finally, the effects of the masonry compressive strength $\sigma_c$ are negligible only for $\sigma_c \geq 6$ MPa.

The particular procedure followed to apply the load during the experimental test [21] makes it necessary to evaluate its possible effects on the results. In particular, dead load to provide reaction for the jacks was provided by concrete blocks supported above the bridge by three beams resting on concrete block plinths at each abutment (the positions of plinths [21] are shown in Fig. 13). The vertical forces $V_l$ and $V_r$ (Fig. 13) applied on the fill depend on the weight of the concrete blocks (380 kN) and the supporting structure and on the force exerted by the jacks. The results obtained by introducing these forces

Fig. 10. Results obtained by assuming $\tilde{\sigma}_c = 20$ kPa ($P_{lb}^{u} = 149$ kN): (a) in-plane minimum principal stress directions; (b) contour plot of the minimum principal stress field; (c) contour plot of the minimum compressive stress $\sigma_3$. 
in the numerical simulation have shown that their effect is negligible when the fill parameters specified in Table 1 are used, with a difference in the collapse load evaluation less than 1%. Moreover, the results have shown that only large increments of the weight of the loading system can significantly affect the results.

Finally, an analysis of the effect of the fill constrains at the abutments has been carried out by considering a reduced domain of the fill, corresponding to the shaded region in Fig. 13. Also in this case, the differences in the upper and lower bounds on the collapse load provided by the model in plane strain condition with respect to the results obtained by the reference one are less than 1%. This results shows that, for this particular load condition, the introduction of the piles present at the abutments during the experimental test (see Fig. 13 for their position) in the simulation would not have affected the results.

4.2. Example 2: Multi-span bridge

A second example concerning a multi-span bridge [13] is considered in order to evaluate the capability of the proposed procedure to describe more complex fill–arch–pier interactions. The geometry of the model and the live load are described in Fig. 14.

The collapse mechanism obtained by assuming heavy but non-resistant fill is shown in Fig. 15, while the results obtained from the equilibrium and the compatible models, respectively, by assuming plane strain fill and the parameters of materials given in Table 3 are shown in Fig. 16(a) and (b). The finite element model is defined by the following parameters, resulting from a convergence analysis: \( n_t = 1207, p_t = 18, n_b = 77, p_s = 2, p_b = 48 \). The influence of the admissible domain discretizations on the collapse load estimations and CPU time are shown in Table 4.

With reference to the discretization shown in Fig. 16(b), the lower bound on the collapse load is \( P_{lb}^{\text{strain}} = 1495 \, \text{kN} \) and the upper bound \( P_{ub}^{\text{strain}} = 1809 \, \text{kN} \). Also in this example,
limit stress state is obtained, while for \( \bar{\sigma}_c < 212 \) kPa the limit condition (6) affects the result by reducing the lower estimates of the collapse load. The minimum principal stress directions obtained by assuming \( \bar{\sigma}_c = 100 \) kPa are represented in the diagram of Fig. 20, while their values are shown in the contour plot of Fig. 21; the contour plot of Fig. 22 shows the \( \sigma_3 \) distribution. The corresponding value of the collapse load estimate, \( P^{lb}_{\text{strain}} = 1060 \) kN, is a good lower bound approximation of the resultant force \( P = 1108 \) kN of a uniform distribution of the maximum compressive stress \( \sigma^*_c - r\bar{\sigma}_c \) under the live load position; also in this example, in fact, the decrease in the collapse load estimate is due to a local limit state under the line of application of the live load. The collapse load evaluations obtained by keeping the plane strain in the region below the live load and varying \( \bar{\sigma}_c \) in the rest of the fill are shown in the graph of Fig. 19 (thin line); in this case the effect of \( \bar{\sigma}_c \) is very limited.

5. Conclusions

A two-dimensional finite element model and a numerical procedure have been presented to analyse the collapse behavior of masonry bridges taking into account the interaction between the arch–pier structural system and the fill. The method is based
Fig. 16. (a) Limit stress state and line of thrust (plane strain fill, $P_{\text{strain}} = 1495$ kN); (b) collapse mechanism from compatible model (plane strain fill, $P_{\text{strain}} = 1809$ kN).

Fig. 17. Contour plot of the minimum in-plane principal stress.

Fig. 18. Contour plot of the minimum compressive stress $\sigma_3$. 
The capabilities of the procedure have been evaluated through two examples representing different realistic bridge configurations. In the first example the simulation of an experimental collapse test [21] has allowed for a comparison between the numerical and experimental results; the multi-span bridge model of the second example has shown the capability of the procedure to represent the complex interaction between piers, arches and fill at collapse. The results obtained by varying the admissible transverse stress have highlighted the important influence of this parameter on the collapse load and the validity limits of the plane strain assumption. Finally, the availability of the compatible model [13] has made comparison possible; the collapse states predicted by the two approaches are very similar. Lower and upper bounds on the collapse load have been obtained by varying the parameters of the materials.

The model presents some limitations, most of them due to the two-dimensional idealisation of the bridge, which involves a reduced and simplified description of the transverse effects due to the spandrel walls. Moreover, the effects of the assumptions of ductility and associated flow rule, on which limit analysis theorems are based, should be investigated.

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Fig. 22. Contour plot of the minimum compressive stress $\sigma_3$ for $\bar{\sigma}_c = 100$ kPa ($P_u = 1060$ kN).

References