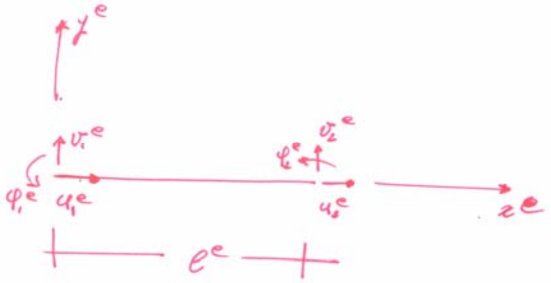
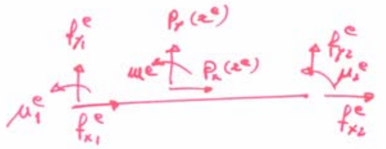
Metodo (algebrico) degli spostamenti per la soluzione delle travature.

Ipotesi:

Tarvatura paiana con forze concentrate nei nodi e forze distribuite lungo gli elementi.
Modello di trave di Eulero-Bernoulli.

I





$$\underbrace{\begin{bmatrix} -EA\dfrac{d^2}{dx^2} & 0 \\ 0 & EJ\dfrac{d^4}{dx^4} \end{bmatrix}}_{K} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} p_x \\ p_y - m' \end{Bmatrix}$$

$$u(0) = u_1 \qquad u(\ell) = u_2$$
$$v(0) = v_1 \qquad v(\ell) = v_2$$
$$v'(0) = \varphi_1 \qquad v'(\ell) = \varphi_2$$

$$K u = b$$

$$u = u_0 + u_p$$

a) Soluzione particolare

$$K u_p = b \qquad \begin{cases} u_p(0) = 0 & u_p(\ell) = 0 \\ v_p(0) = 0 & v_p(\ell) = 0 \\ v_p'(0) = 0 & v_p'(\ell) = 0 \end{cases} \Longrightarrow u_p(x)$$

(dipende da b)

b) Integrale generale

$$K u_0 = 0 \qquad \begin{cases} u_0(0) = u_1 & u_0(\ell) = u_2 \\ v_0(0) = v_1 & v_0(\ell) = v_2 \\ v_0'(0) = \varphi_1 & v_0'(\ell) = \varphi_2 \end{cases}$$

$$\begin{cases} u_0(x) = a_1 + a_2 x \\ v_0(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 \end{cases}$$

Costanti per problema assiale

II

$$\begin{cases} a_1 = u_1 \\ a_1 + a_2 \ell = u_2 \end{cases} \qquad a_2 = \frac{u_2 - u_1}{\ell}$$

$$u_0(x) = u_1 + \frac{u_2 - u_1}{\ell} x$$

$$= \left(1 - \frac{x}{\ell}\right) u_1 + \frac{x}{\ell} u_2$$

$$\boxed{u_0(x) = \Psi_1(x)\, u_1 + \Psi_2(x)\, u_2}$$

Funzioni di interpolazione

$$\Psi_1(x) = 1 - \frac{x}{\ell}$$

$$\Psi_2(x) = \frac{x}{\ell}$$

Costanti per problema flessionale

$$\begin{cases} v(0) = c_1 = v_1 \\ \varphi(0) = c_2 = \varphi_1 \\ v(\ell) = c_1 + c_2 \ell + c_3 \ell^2 + c_4 \ell^3 = v_2 \\ \varphi(\ell) = c_2 + 2c_3 \ell + 3 c_4 \ell^3 = \varphi_2 \end{cases}$$

$$\mathbb{A} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix} \qquad \mathbb{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathbb{A}^{-1} \begin{Bmatrix} v_1 \\ \varphi_1 \\ v_2 \\ \varphi_2 \end{Bmatrix} \qquad \mathbb{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{\ell^2} & -\frac{2}{\ell} & \frac{3}{\ell^2} & -\frac{1}{\ell} \\ \frac{2}{\ell^3} & \frac{1}{\ell^2} & -\frac{2}{\ell^3} & \frac{1}{\ell^2} \end{bmatrix}$$

$$v_0(x) = \psi_3(x)\, v_1 + \psi_4(x)\, \varphi_1 + \psi_5(x)\, v_2 + \psi_6(x)\, \varphi_2$$

$$\psi_3(x) = 1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}$$

$$\psi_4(x) = x - \frac{2x^2}{\ell} + \frac{x^3}{\ell^2}$$

$$\psi_5(x) = \frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3}$$

$$\psi_6(x) = -\frac{x^2}{\ell} + \frac{x^3}{\ell^2}$$

$$u_0^e(x) = \Psi^e(x)\, q^e$$

$$q^e = \left[u_1^e\ v_1^e\ \varphi_1^e\ \ u_2^e\ v_2^e\ \varphi_2^e \right]^T$$

$$\Psi^e(x) = \begin{bmatrix} \psi_1^e(x) & 0 & 0 & \psi_2^e(x) & 0 & 0 \\ 0 & \psi_3^e(x) & \psi_4^e(x) & 0 & \psi_5^e(x) & \psi_6^e(x) \end{bmatrix}$$

$$u^e(x) = \Psi^e(x)\, q^e + u_p^e$$

$$\sigma^e(x) = C^e D\, \Psi^e(x)\, q^e + \underbrace{C D u_p}_{\sigma_p^e}$$

$$\begin{Bmatrix} N^e \\ M^e \end{Bmatrix} = \begin{bmatrix} EA & \\ & EJ \end{bmatrix} \begin{bmatrix} \frac{d}{dx} & 0 \\ 0 & \frac{d^2}{dx^2} \end{bmatrix} \Psi(x)\, q^e + \sigma_p^e(x)$$

Recupero il legame dalle equazioni di equilibrio.

$$-\frac{dM}{dx} - T = m$$

Problema omogeneo $(m = 0)$

$$T_o = -\frac{dM_o}{dx}$$

$$\begin{Bmatrix} N^e(x) \\ T(x) \\ M(x) \end{Bmatrix} = \underbrace{\begin{bmatrix} EA\frac{d}{dx} & 0 \\ 0 & -EJ\frac{d^3}{dx^3} \\ 0 & EJ\frac{d^2}{dx^2} \end{bmatrix} \Psi(x)\, q}_{[N_o(x)\ T_o(x)\ M_o(x)]^T} + \begin{bmatrix} N_p(x) \\ T_p(x) \\ M_p(x) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} EA\,\psi_1' & 0 & 0 & EA\,\psi_2' & 0 & 0 \\ 0 & -EJ\,\psi_3''' & -EJ\,\psi_4''' & 0 & -EJ\,\psi_5''' & -EJ\,\psi_6''' \\ 0 & EJ\,\psi_3'' & EJ\,\psi_4'' & 0 & EJ\,\psi_5'' & EJ\,\psi_6 \end{bmatrix}}_{H(x)} q + \begin{bmatrix} N_p(x) \\ T_p(x) \\ M_p(x) \end{bmatrix}$$

$\psi_1'(x) = -\frac{1}{e}$

$\psi_2'(x) = \frac{1}{e}$

$\psi_3'(x) = -6\frac{x}{e^2} + 6\frac{x^2}{e^3}$ $\qquad$ $\psi_3''(x) = -\frac{6}{e^2} + \frac{12x}{e^3}$ $\qquad$ $\psi_3'''(x) = \frac{12}{e^3}$

$\psi_4'(x) = 1 - \frac{4x}{e} + \frac{3x^2}{e^2}$ $\qquad$ $\psi_4''(x) = -\frac{4}{e} + \frac{6x}{e^2}$ $\qquad$ $\psi_4'''(x) = \frac{6}{e^2}$

$\psi_5'(x) = \frac{6x}{e^2} - \frac{6x^2}{e^3}$ $\qquad$ $\psi_5''(x) = \frac{6}{e^2} - \frac{12x}{e^3}$ $\qquad$ $\psi_5'''(x) = -\frac{12}{e^3}$

$\psi_6'(x) = -\frac{2x}{e} + \frac{3x^2}{e^2}$ $\qquad$ $\psi_6''(x) = -\frac{2}{e} + \frac{6x}{e^2}$ $\qquad$ $\psi_6'''(x) = \frac{6}{e^2}$

Forze nodali

$$
\begin{bmatrix} f_{1x} \\ f_{1y} \\ \mu_1 \\ f_{2x} \\ f_{2y} \\ \mu_2 \end{bmatrix}
=
\begin{bmatrix} -N(0) \\ -T(0) \\ -M(0) \\ N(\ell) \\ T(\ell) \\ M(\ell) \end{bmatrix}
=
\begin{bmatrix} -N_0(0) \\ -T_0(0) \\ -M_0(0) \\ N_0(\ell) \\ T_0(\ell) \\ M_0(\ell) \end{bmatrix}
+
\begin{bmatrix} f_{1xp} \\ f_{1yp} \\ \mu_{1p} \\ f_{2xp} \\ f_{2yp} \\ \mu_{2p} \end{bmatrix}
$$

$\mathbf{f}$ $\qquad$ $\mathbf{f}_0$ $\qquad$ $\mathbf{f}_p = -\mathbf{p}$

reazioni vincolari
trave a nodi fissi
con carico $\mathbf{b}$

$$
\begin{bmatrix} -N_0(0) \\ -T_0(0) \\ -M_0(0) \\ N_0(\ell) \\ T_0(\ell) \\ M_0(\ell) \end{bmatrix}
=
\left[\begin{array}{ccc|ccc}
\frac{EA}{\ell} & 0 & 0 & -\frac{EA}{\ell} & 0 & 0 \\
0 & \frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} & 0 & -\frac{12EJ}{\ell^3} & \frac{6EJ}{\ell^2} \\
0 & \frac{6EJ}{\ell^2} & \frac{4EJ}{\ell} & 0 & -\frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} \\
\hline
-\frac{EA}{\ell} & 0 & 0 & \frac{EA}{\ell} & 0 & 0 \\
0 & -\frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} & 0 & +\frac{12EJ}{\ell^3} & -\frac{6EJ}{\ell^2} \\
0 & \frac{6EJ}{\ell^2} & \frac{2EJ}{\ell} & 0 & -\frac{6EJ}{\ell^2} & \frac{4EJ}{\ell}
\end{array}\right] \mathbf{q}
$$

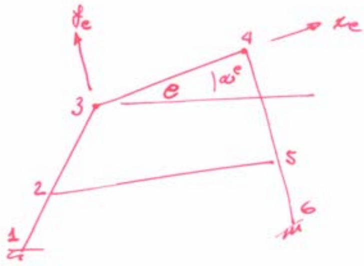
$$
\mathbf{K} = \begin{bmatrix} -\mathbb{H}(0) \\ \mathbb{H}(\ell) \end{bmatrix}
$$

$\mathbf{f}^e = \mathbb{K}^e \mathbf{q}^e - \mathbf{p}^e$ $\qquad$ $\mathbf{p}^e$: forze esterne $\mathbf{b}^e(x)$ ridotte ai nodi

Assemblaggio

$\underset{(6\times 1)}{q^e}$ spostamenti nodali elem. e

$\underset{(3N\times 1)}{q}$ vettore spos. nodali struttura



$$q = \left[u_1 \; v_1 \; \varphi_1 \;\; u_2 \; v_2 \; \varphi_2 \; \ldots \; u_N \; v_N \; \varphi_N \right]^T$$

$$q^e = \left[u_1^e \, v_1^e \, \varphi_1^e \;\; u_2^e \, v_2^e \, \varphi_2^e \right]^T$$

$$\underset{6\times 1}{q^e} = \underset{6\times 6}{T^e}\,\underset{6\times 3N}{\Omega^e}\,\underset{3N\times 1}{q}$$

$$\Omega^e = \left[\begin{array}{ccccc} 0 & \cdots & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} & \cdots \\ 0 & & & & \end{array} \right]$$

posizione nodo 1 in travatura — posiz. nodo 2 in trav.

$$T^e = \left[\begin{array}{c|c} \bar{T}^e & 0 \\ \hline 0 & \bar{T}^e \end{array} \right] \qquad \bar{T}^e = \begin{bmatrix} \cos\alpha^e & \sin\alpha^e & 0 \\ -\sin\alpha^e & \cos\alpha^e & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equilibrio nodi

$$\bar{p} - \sum_{e=1}^{m} \Omega^{e^T} T^{e^T} f^e = 0$$

$\mathbb{T}^{e^T} f^e$ = reaz. vincolari trave e dovute a spostamenti nodali e carichi $b^e(x)$ in rif. globale

$-\Omega^T \mathbb{T}^{e^T} f^e$ = forze scaricate sul nodo dalla trave e (rif globale, num. globale).

$T^{e^T} f^e$ forze interne proiettate nel riferimento globale

$\bar{p}$ forze esterne applicate ai nodi

$$f^e = K^e q^e - p^e$$

$$\underbrace{\bar{p} + \sum_{e=1}^{m} \Omega^{e^T} T^{e^T} p^e}_{\underset{3N\times 1}{\mathbb{P}}} = \underbrace{\sum_{e=1}^{m} \Omega^{e^T} \mathbb{T}^{e^T} K^e T^e \Omega^e}_{\underset{3N\times 3N}{\mathbb{K}}} \, \mathbb{q}$$

$$\mathbb{K} \mathbb{q} = \mathbb{P}$$

Condizioni di vincolo

$$\mathbb{q} = \begin{bmatrix} q_L \\ q_V \end{bmatrix} \qquad \mathbb{P} = \begin{bmatrix} \hat{p} \\ r \end{bmatrix}$$

$\hat{p}$ forze nodali attive

r reazioni vincolari

$$\begin{bmatrix} K_{LL} & K_{LV} \\ K_{VL} & K_{VV} \end{bmatrix} \begin{bmatrix} q_L \\ q_V \end{bmatrix} = \begin{bmatrix} \hat{p} \\ r \end{bmatrix}$$

cedimenti vincolari assegnati

$$K_{LL} q_L + K_{LV} q_V = \hat{p}$$

$$\begin{cases} q_L = K_{LL}^{-1} (\hat{p} - K_{LV} q_V) \\ r = K_{VL} K_{LL}^{-1} (\hat{p} - K_{LV} q_V) + K_{VV} q_V \end{cases}$$

Sollecitazioni negli elementi

$$u^e(x) = \varphi^e(x)\, T^e \Omega^e q + u_p^e(x)$$

$$\sigma^e(x) = C^e D\, \Psi^e(x)\, T^e \Omega^e q + \sigma_p^e(x)$$

Riduzione ai nodi delle forze distribuite

$\mathbb{p}^e$ forze nodali equivalenti alle forze distribuite $\mathbb{b}^e(x^e)$

Si scrive l'identità dei lavori virtuali considerando il campo statico (forze, tensioni) della soluzione particolare e il campo cinematico (deformazioni - spostamenti) della soluzione omogenea.

$$\int_0^e \sigma_p^T \delta\varepsilon_0 \, dx = \int_0^e \mathbb{b}^T \delta u_0 \, dx - p^e \delta q^e \qquad (*)$$

σ_p, $\mathbb{b}^T$ e $f_p^e = -p^e$ sono in equilibrio se l'ILV vale per ogni $\delta\varepsilon_0$, δu_0 e δq^e congruenti.

$$\delta u_0(x) = \Psi(x)\delta q^e$$

Il primo termine della $(*)$ è nullo. Ciò può essere dimostrato applicando il teorema di reciprocità

$$\int_0^e \sigma_p^T \delta\varepsilon_0 \, dx = \int_0^e \delta\sigma_0^T \varepsilon_p \, dx = \int \cancel{\delta \mathbb{b}_0^T} u_p \, dx + \delta f_p^T \cancel{u_p}\Big|_0^e = 0$$

Per l'arbitrarietà di δq^e, la $(*)$ risulta:

$$\boxed{p^e = -f_p^e = \int_0^e \Psi^T(x)\, \mathbb{b}(x)\, dx}$$