Microhydrodynamics

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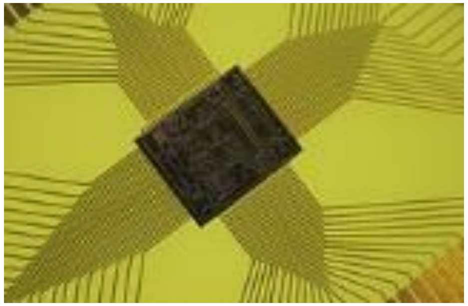
L'Aquila, July 2013

Microhydrodynamics

Flows in micro-devices are characterized by:

- small volumes (μl, nl …) and sizes;
- low energy consumption;
- effects of the microdomain.

Micro-fluidics encompasses many technologies ranging from physics, to chemistry and biotechnology



Microhydrodynamics

Recent applications of flows in micro-devices:

- **Cells-on-chip**

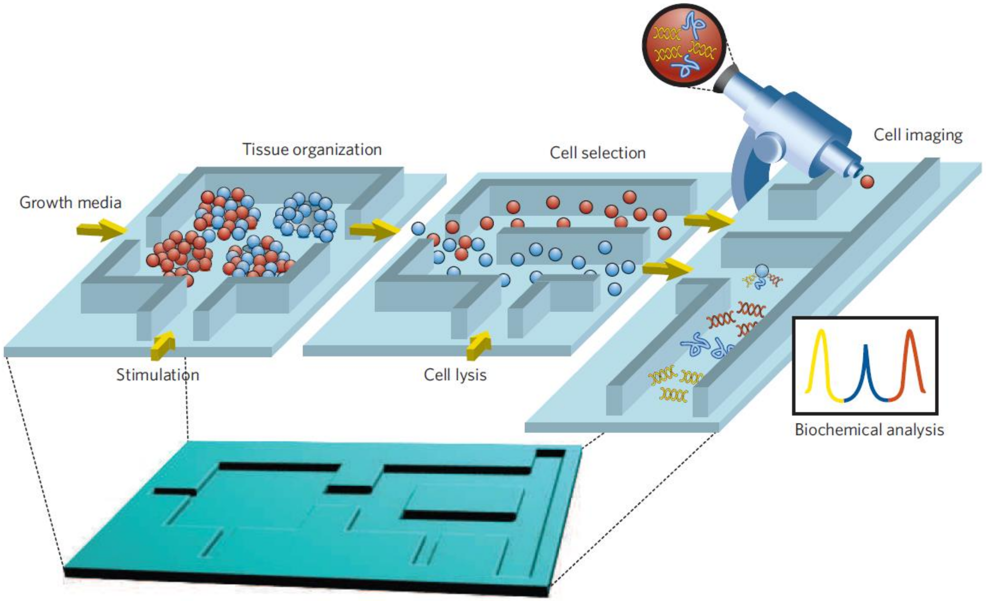


Figure 1 | Tissue organization, culture and analysis in microsystems. Advanced tissue organization and culture can be performed in microsystems by integrating homogeneous and heterogeneous cell ensembles, 3D scaffolds to guide cell growth, and microfluidic systems for transport of nutrients and other soluble factors. Soluble factors — for example, cytokines for cell stimulation — can be presented to the cells in precisely defined spatial and temporal patterns using integrated microfluidic systems. Microsystems technology can also fractionate heterogeneous cell populations into homogeneous populations, including single-cell selection, so different cell types can be analysed separately. Microsystems can incorporate numerous techniques for the analysis of the biochemical reactions in cells, including image-based analysis and techniques for gene and protein analysis of cell lysates. This makes microtechnology an excellent tool in cell-based applications and in the fundamental study of cell biology. As indicated by the yellow arrows, the different microfluidic components can be connected with each other to form an integrated system, realizing multiple functionalities on a single chip. However, this integration is challenging with respect to fluidic and sample matching between the different components, not least because of the difficulty in simultaneously packaging fluidic, optical, electronic and biological components into a single system.

Microhydrodynamics

Recent applications of flows in micro-devices:

- **Selection of CTC**

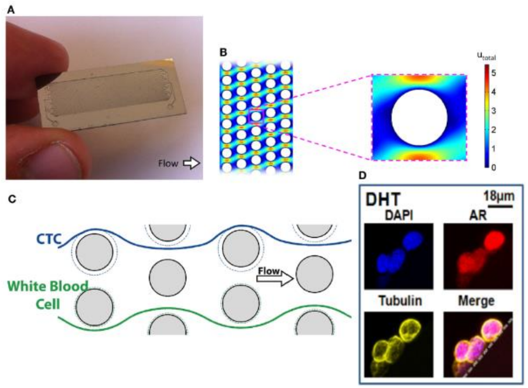


FIGURE 2 | Geometrically-enhanced differential immunocapture (GEDI) microfluidic device. (A) GEDI Chip **(B)** GEDI post-array **(C)** Illustration of laminar flow through GEDI device **(D)** Captured CTCs stained for AR and tubulin.

frontiers in ONCOLOGY

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Isolation and characterization of circulating tumor cells in prostate cancer

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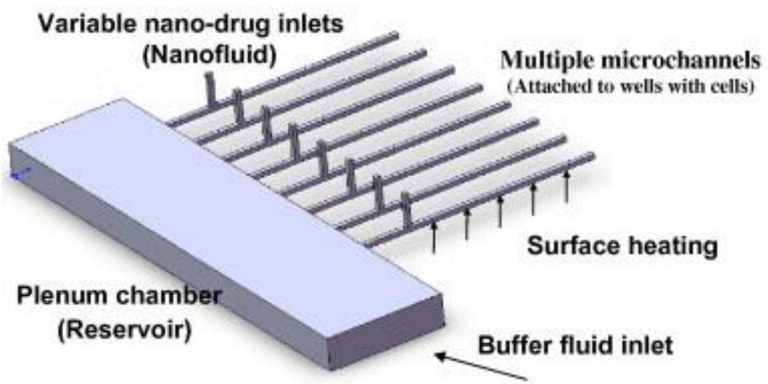
Circulating tumor cells (CTCs) are tumor cells found in the peripheral blood that putatively originate from established sites of malignancy and likely have metastatic potential. Analysis of CTCs has demonstrated promise as a prognostic marker as well as a source of identifying potential targets for novel therapeutics. Isolation and characterization of these cells for study, however, remain challenging owing to their rarity in comparison with other cellular components of the peripheral blood. Several techniques that exploit the unique biochemical properties of CTCs have been developed to facilitate their isolation. Positive selection of CTCs has been achieved using microfluidic surfaces coated with antibodies against epithelial cell markers or tumor-specific antigens such as EpCAM or prostate-specific membrane antigen (PSMA). Following isolation, characterization of CTCs may help guide clinical decision making. For instance, molecular and genetic characterization may shed light on the development of chemotherapy resistance and mechanisms of metastasis without the need for a tissue biopsy. This paper will review novel isolation techniques to capture CTCs from patients with advanced prostate cancer, as well as efforts to characterize the CTCs. We will also review how these analyzes can assist in clinical decision making. **Conclusion:** The study of CTCs provides insight into the molecular biology of tumors of prostate origin that will eventually guide the development of tailored therapeutics. These advances are predicated on high yield and accurate isolation techniques that exploit the unique biochemical features of these cells.

Keywords: prostate cancer, circulating tumor cells (CTCs), prostate-specific membrane antigen (PSMA), microfluidic device, androgen receptor (AR)

Microhydrodynamics

Recent applications of flows in micro-devices:

- **Drug delivery**



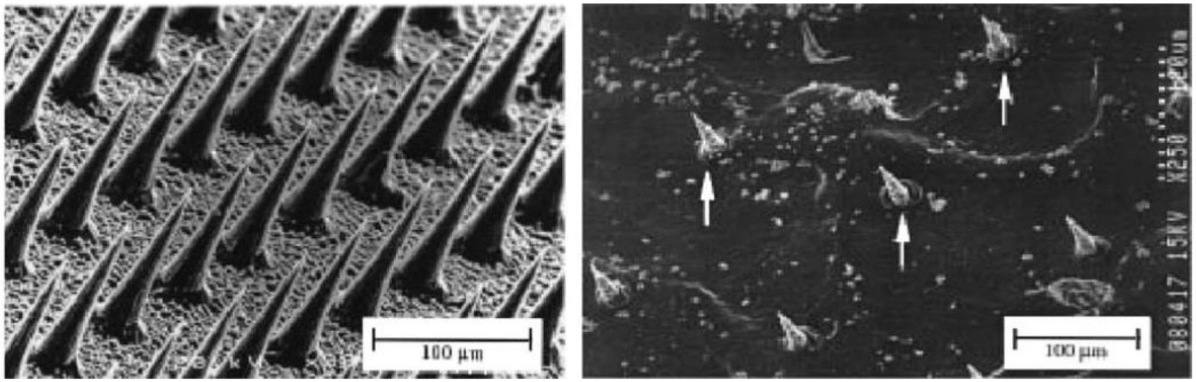


Fig. 4. Microfabricated solid silicon microneedle arrays. Left: A section of a 20 × 20 array of microneedles. Right: Microneedle tips inserted across epidermis (© 1998 IEEE) [28].

Microhydrodynamics

Recent applications of flows in micro-devices:

- **DNA analysis**

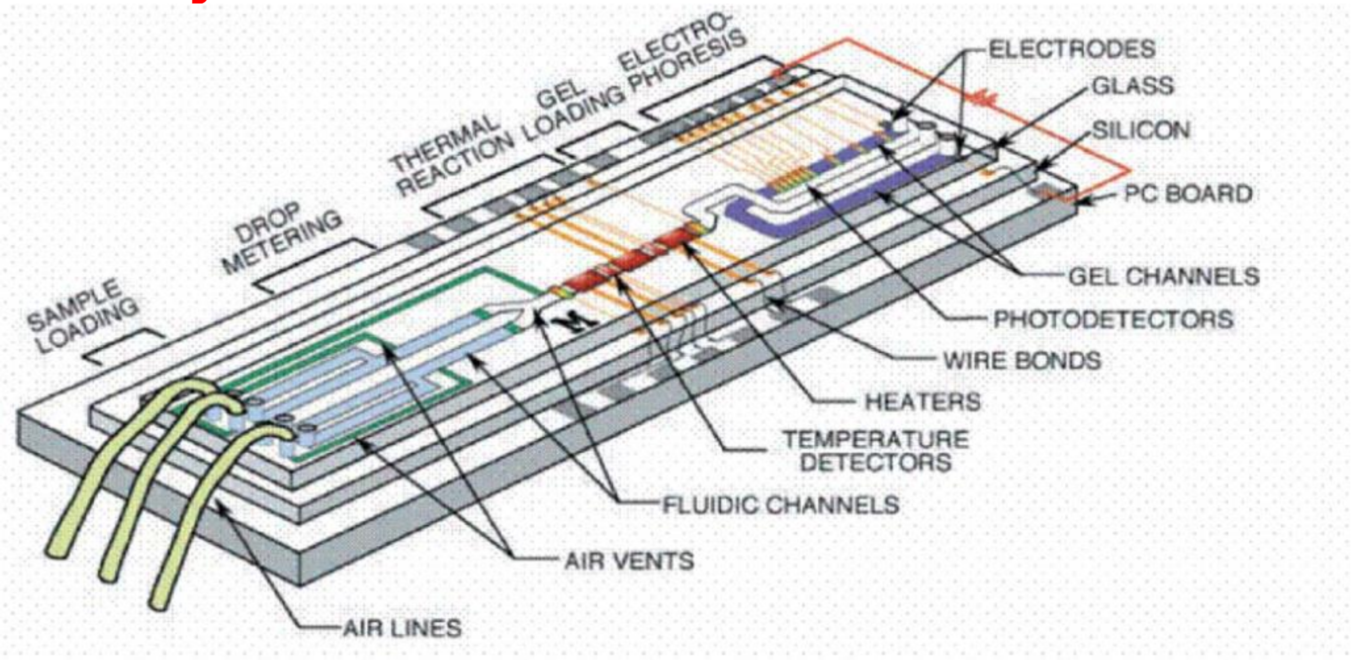
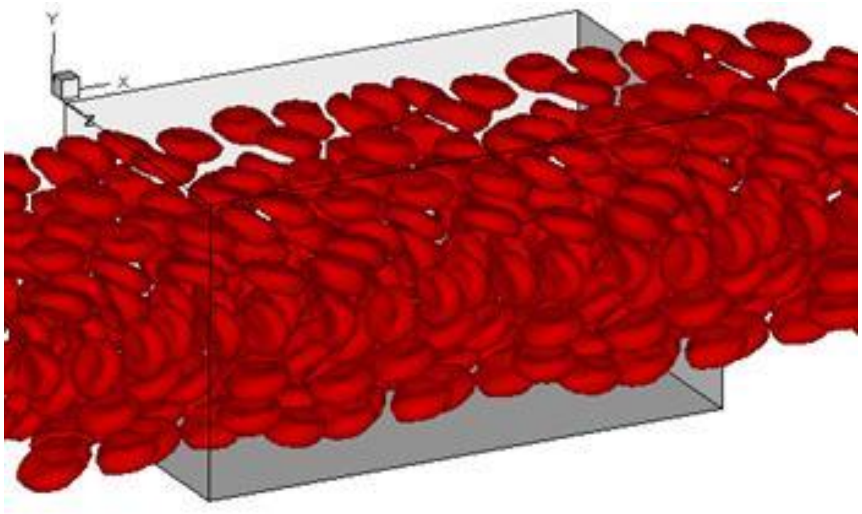


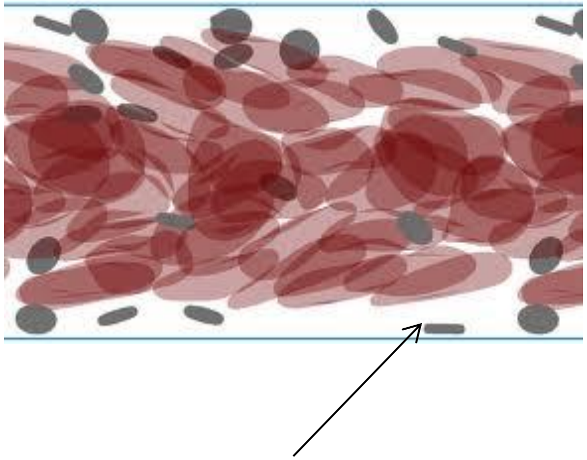
Fig. 8. A schematic of an integrated microfluidic device for DNA analysis

Microhydrodynamics

Other biological applications:

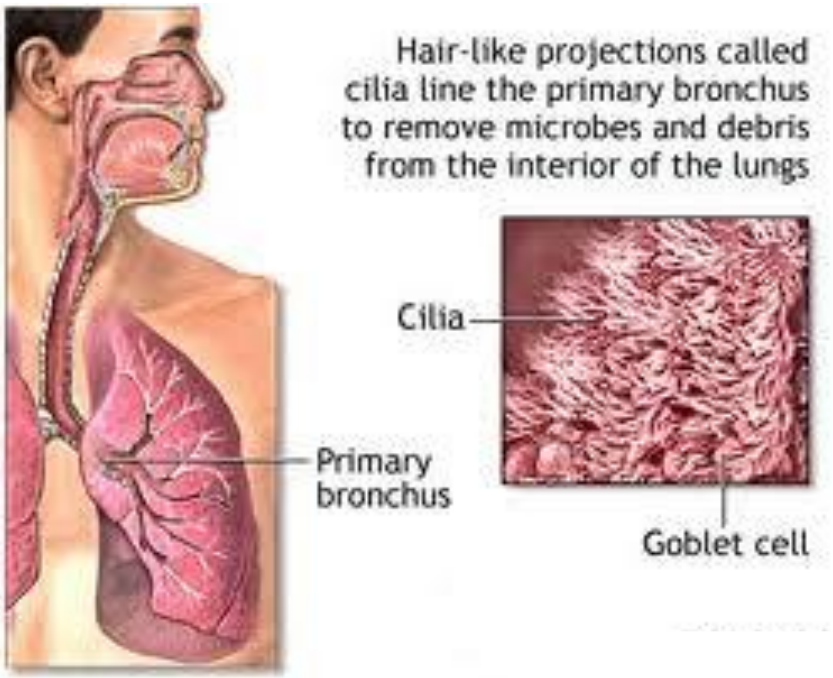
- **Red blood cells, vesicles, capsules …**





Other biological applications:

- **Cilia and flagella**

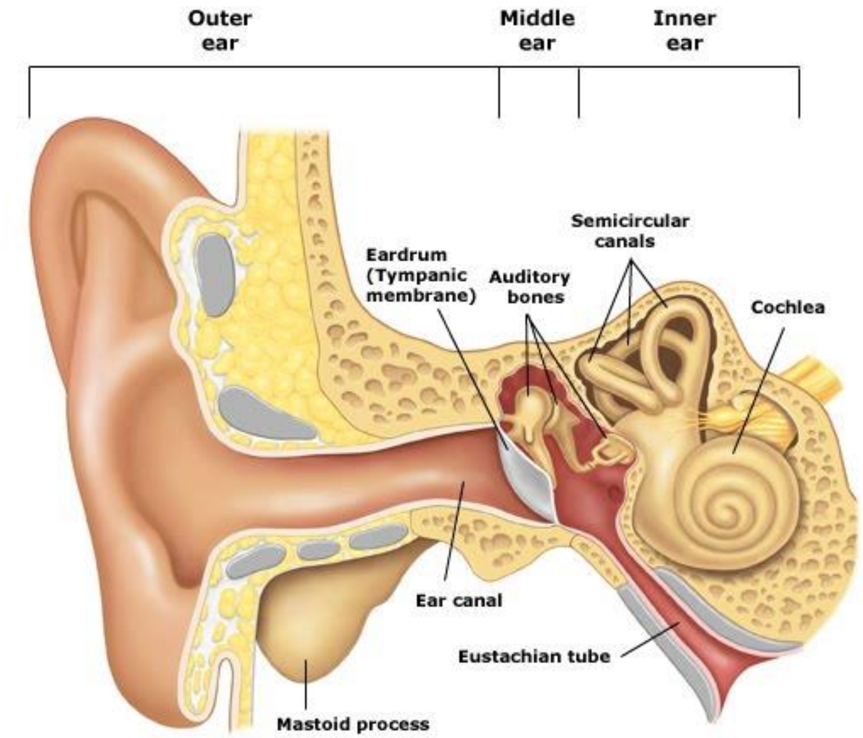


Microhydrodynamics

Other biological applications:

- **Cilia and flagella**

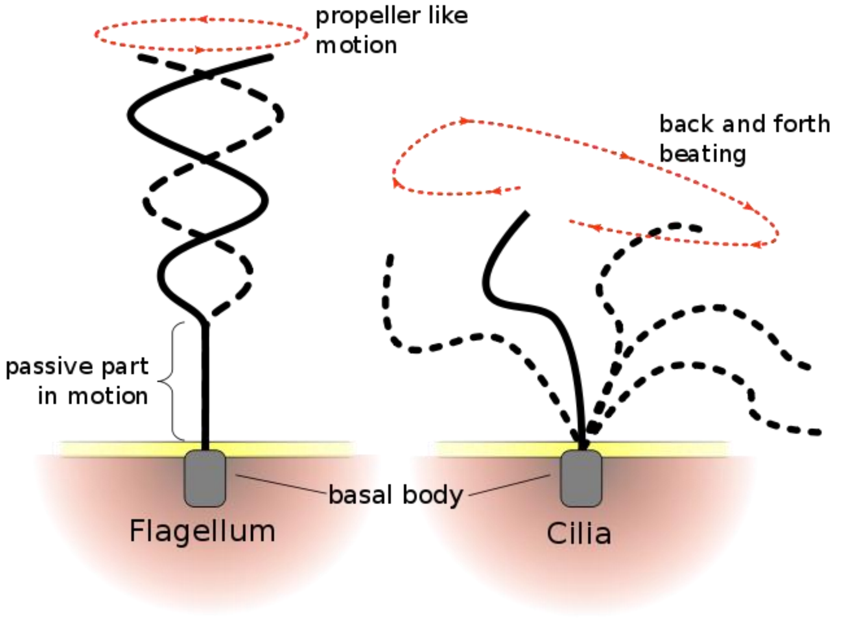
Stereocilia within the cochlea in the inner ear sense vibrations (sound waves) and trigger the generation of nerve signals that are sent to the brain.

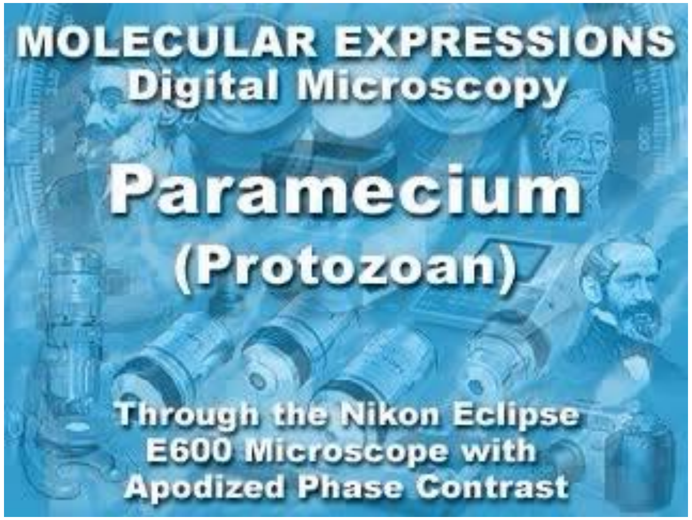


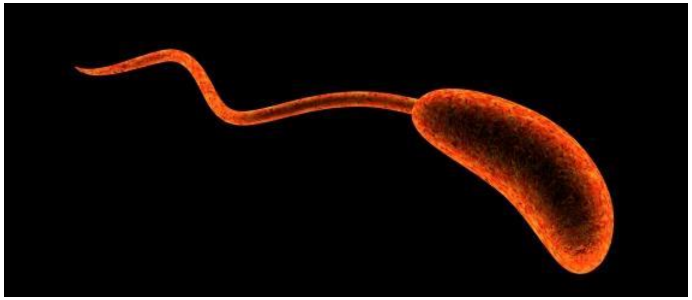
Microhydrodynamics

Other biological applications:

- **Cilia and flagella**

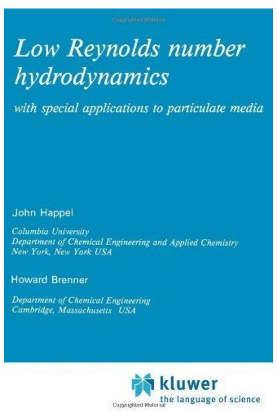


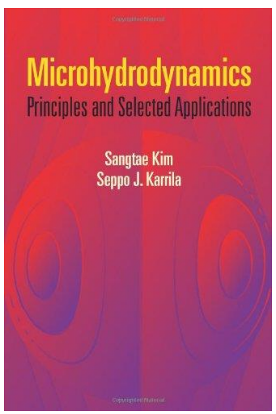


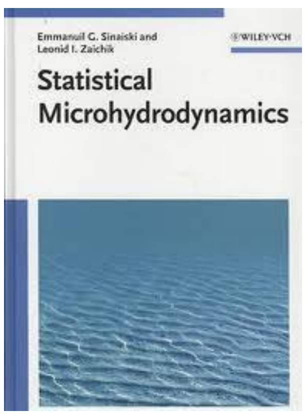


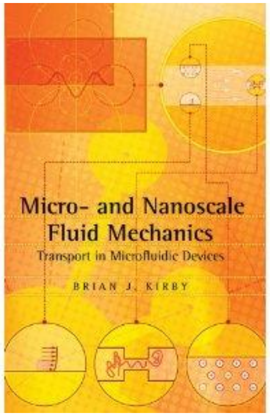
Microhydrodynamics

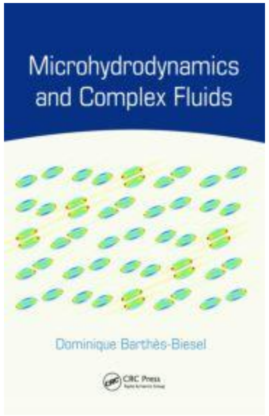
For all these applications (and for many others) it is important to develop an understanding of low Re flows

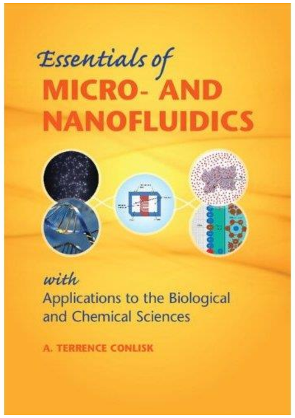












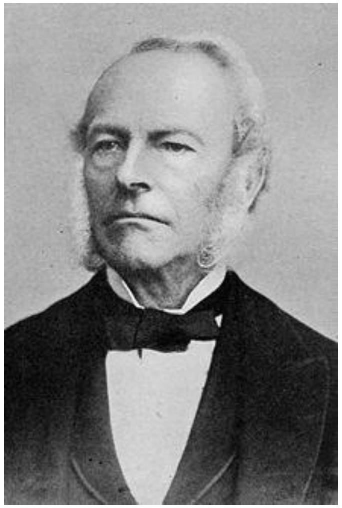
1965 **1991** **2008** **2009** **2010** **2013**

Microhydrodynamics → Creeping Flows

Major learning objectives:

1. Feeling for viscous (and inviscid) flows
2. General solution and theorems for Stokes' flow
3. Derive the complete solution for creeping flow around a sphere (water drop in air, etc.)
4. Flow past a cylinder: Stokes paradox and the Oseen approximation
5. Elementary solutions (Stokeslet, stresslet …)

George Gabriel Stokes (1819-1903)



- Stokes' law, in fluid dynamics
- Stokes radius in biochemistry
- Stokes' theorem, in differential geometry
- Lucasian Professor of Mathematics at Cambridge University at age 30
- Stokes line, in Raman scattering
- Stokes relations, relating the phase of light reflected from a non-absorbing boundary
- Stokes shift, in fluorescence
- Navier–Stokes equations, in fluid dynamics
- Stokes drift, in fluid dynamics
- Stokes stream function, in fluid dynamics
- Stokes wave in fluid dynamics
- Stokes boundary layer, in fluid dynamics
- Stokes phenomenon in asymptotic analysis
- Stokes (unit), a unit of viscosity
- Stokes parameters and Stokes vector, used to quantify the polarisation of electromagnetic waves
- Campbell–Stokes recorder, an instrument for recording sunshine that was improved by Stokes, and still widely used today
- Stokes (lunar crater)
- Stokes (Martian crater)

Creeping *vs* Inviscid Flows

Creeping Flows

Viscosity goes to ∞ (very low Reynolds number)

Left hand side of the momentum equation is not important (can be taken to vanish).

Friction is more important than inertia.

Inviscid Flows

Viscosity goes to zero (very large Reynolds number)

Left hand side of the momentum equation is important. Right hand side of the momentum equation includes pressure only.

Inertia is more important than friction.

Creeping *vs* Inviscid Flows

Creeping Flows

Inviscid Flows

Scaling of pressure

$\mu V / L$

ρV^2

Creeping *vs* Inviscid Flows

Creeping Flow Solutions

Use the partial differential equations. Apply transform, similarity, or separation of variables solution.

Use no-slip condition.

Use stream functions for conservation of mass.

Inviscid Flow Solutions

Use flow potential, complex numbers.

Use "no normal velocity."

Use velocity potential for conservation of mass.

In both cases, we will assume incompressible flow, $\nabla \cdot \mathbf{v} = 0$

Incompressible Navier-Stokes equations

$$\nabla \cdot \vec{V} = 0$$

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	$\{L\}$
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

Scaling

- We define ***nondimensional variables*** using the scaling parameters from the table in the previous slide

$$t^\star = ft \qquad \vec{x}^\star = \frac{\vec{x}}{L} \qquad \vec{V}^\star = \frac{\vec{V}}{V}$$

$$P^\star = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^\star = \frac{\vec{g}}{g} \qquad \nabla^\star = L\nabla$$

- To plug the nondimensional variables into the NSE, we need to first rearrange the equations in terms of the dimensional variables

$$t = \frac{1}{f}t^\star \qquad \vec{x} = L\vec{x}^\star \qquad \vec{V} = V\vec{V}^\star \qquad \nabla = \frac{1}{L}\nabla^\star$$

$$P = P_\infty + (P_0 - P_\infty)P^\star \qquad \vec{g} = g\vec{g}^\star$$

Scaling

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$$P = P_\infty + (P_0 - P_\infty)P^\star \qquad \vec{g} = g\vec{g}^\star$$

Scaling

- Now we substitute into the NSE to obtain

$$\rho V f \frac{\partial \vec{V}^\star}{\partial t^\star} + \frac{\rho V^2}{L}\left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\frac{P_0 - P_\infty}{L}\nabla^\star P^\star + \rho g \vec{g}^\star + \frac{\mu V}{L^2}\nabla^{\star 2}\vec{V}^\star$$

- Every additive term has primary dimensions $\{m^1L^{-2}t^{-2}\}$. To nondimensionalize, we multiply every term by $L/(\rho V^2)$, which has primary dimensions $\{m^{-1}L^2t^2\}$, so that the dimensions cancel. After rearrangement,

$$\left[\frac{fL}{V}\right]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\left[\frac{P_0 - P_\infty}{\rho V^2}\right]\nabla^\star P^\star + \left[\frac{gL}{V^2}\right]\vec{g}^\star + \left[\frac{\mu}{\rho V L}\right]\nabla^{\star 2}\vec{V}^\star$$

Dimensionless numbers

Terms in [ ] are nondimensional parameters

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\nabla^\star P^\star + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^\star + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{\star 2}\vec{V}^\star$$

Strouhal number

Euler number

Inverse of Froude number squared

Inverse of Reynolds number

$$\boxed{[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star + \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star}$$

Navier-Stokes equation in nondimensional form

Nondimensionalization vs. Normalization

- NSE are now *nondimensional*, but not necessarily *normalized.* What is the difference?
- ***Nondimensionalization*** concerns only the *dimensions* of the equation - we can use *any* value of scaling parameters L, V, etc.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters *L,V,* etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables are of order of magnitude unity,*** i.e., their minimum and maximum values are close to 1.0.

$$t^\star \sim 1 \qquad \vec{x}^\star \sim 1 \qquad \vec{V}^\star \sim 1 \qquad P^\star \sim 1 \qquad \vec{g}^\star \sim 1 \qquad \nabla^\star \sim 1$$

If we have properly normalized the NSE, we can compare the relative importance of the terms in the equation by comparing the relative magnitudes of the nondimensional parameters St, Eu, Fr, and Re.

The Reynolds number

$$\boxed{Re = \frac{\rho VL}{\mu}}$$

In water ($\rho \approx 10^3$ kg m^{-3}, $\mu \approx 10^{-3}$ Pa s), a swimming bacterium such as *E. coli* with $V \approx 10\ \mu\text{m s}^{-1}$ and $L \approx 1$–$10\ \mu$m has a Reynolds number $Re \approx 10^{-5}$–10^{-4}.

A human spermatozoon with $V \approx 200\ \mu\text{m s}^{-1}$ and $L \approx 50\ \mu$m moves with $Re \approx 10^{-2}$.

The larger ciliates, such as *Paramecium*, have $V \approx 1$ mm s^{-1} and $L \approx 100\ \mu$m, and therefore $Re \approx 0.1$.

The Reynolds number

$$\boxed{Re = \frac{\rho VL}{\mu}}$$

Low Re:

1. $Re = \frac{f_{inertial}}{f_{viscous}}$ $\longrightarrow$ viscous forces dominate in the fluid

2. $Re = \frac{\tau_{diff}}{\tau_{adv}}$ $\longrightarrow$ fluid transport dominated by viscous diffusion

3. For a given fluid $\mathcal{F} = \mu^2/\rho$ is a force, and any body acted upon by a force $\mathcal{F}$ will experience a Reynolds number of order 1 (whatever its size). Easy to see that $Re = f_{\text{viscous}} / \mathcal{F}$ and $Re = (f_{\text{inertial}} / \mathcal{F})^{1/2}$, so that if $Re = 1 \longrightarrow f_{\text{viscous}} = f_{\text{inertial}} = \mathcal{F}$.

   A body moving at low Re therefore experiences forces smaller than $\mathcal{F}$, where $\mathcal{F} = 1$ nN for water.

Leading order terms

- To simplify NSE, assume St ~ 1, Fr ~ 1

$$\boxed{[Eu]\,\nabla^{\star} P^{\star} = \left[\frac{1}{Re}\right] \nabla^{\star 2} \vec{V}^{\star}}$$

Pressure forces

Viscous forces

- Since $P^{\star} \sim 1, \quad \nabla^{\star} \sim 1$

$$Eu = \frac{P_0 - P_{\infty}}{\rho V^2} \sim \frac{1}{Re} = \frac{\mu}{\rho V L} \Rightarrow \boxed{P_0 - P_{\infty} \sim \frac{\mu V}{L}}$$

General equations for creeping flows

- This is important $\boxed{P_0 - P_\infty \sim \frac{\mu V}{L}}$ (cf. slide 15)
  - Very different from inertia dominated flows where
    $$P_0 - P_\infty \sim \rho V^2$$
  - Density has completely dropped out of NSE. To demonstrate this, convert back to dimensional form.

$$\boxed{\nabla P = \mu \nabla^2 \vec{V}}$$

  - This is now a **LINEAR EQUATION** which can be solved for simple geometries.

General equations for creeping flows

$$\nabla P = \mu \nabla^2 \mathbf{v}$$

Taking the curl of the equation above we have:

$$\boxed{\nabla^2 \boldsymbol{\zeta} = \mathbf{0}}$$

or

$$\nabla \times \nabla^2 \mathbf{v} = -\nabla \times \nabla \times \nabla \times \mathbf{v} = -\nabla \times \nabla \times \boldsymbol{\zeta} = \mathbf{0}$$

given the vector identity: $\nabla \times \nabla \times \mathbf{v} = -\nabla^2 \mathbf{v} + \nabla(\nabla \cdot \mathbf{v})$

⇒ the vorticity field $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$ is harmonic

General equations for creeping flows

$$\nabla P = \mu \nabla^2 \mathbf{v}$$

Taking the divergence of the equation above we have:

$$\boxed{\nabla^2 P = \mathbf{0}}$$

since

$$\nabla \cdot \nabla^2 \mathbf{v} = \nabla^2 (\nabla \cdot \mathbf{v}) = \mathbf{0}$$

on account of the solenoidal velocity field $\mathbf{v}$

$\Rightarrow$ the pressure field P is harmonic

(and the velocity field satisfies the biharmonic equation …)

General equations for creeping flows

Laplacians appear everywhere:

$$\left\{ \begin{aligned} &\nabla^2 \mathbf{v} = \frac{1}{\mu} \nabla P \\ &\nabla^2 \boldsymbol{\zeta} = \mathbf{0} \\ &\nabla^2 P = 0 \end{aligned} \right\}$$

Boundary conditions act as localized sources, and ∇P acts as a distributed source.

and this points to the **non-locality** of Stokes flows: the solution at any point is determined by conditions over the entire boundary. Dependence on remote boundary points can be quantified by Green's functions.

Properties of Stokes' flows

- The solutions of Stokes' equation are unique
- They can be added because of linearity
- The solutions represent states of minimal dissipation
- The solutions are reversible (*scallop* theorem)

Linearity and reversibility

Let $\{\mathbf{v}_1(\mathbf{r}); P_1(\mathbf{r})\}$ and $\{\mathbf{v}_2(\mathbf{r}); P_2(\mathbf{r})\}$ be two solutions of Stokes equations which satisfy respectively the conditions $\mathbf{v}_1 = \mathbf{U}_1(\mathbf{r})$ and $\mathbf{v}_2 = \mathbf{U}_2(\mathbf{r})$ on the boundary S of the domain.

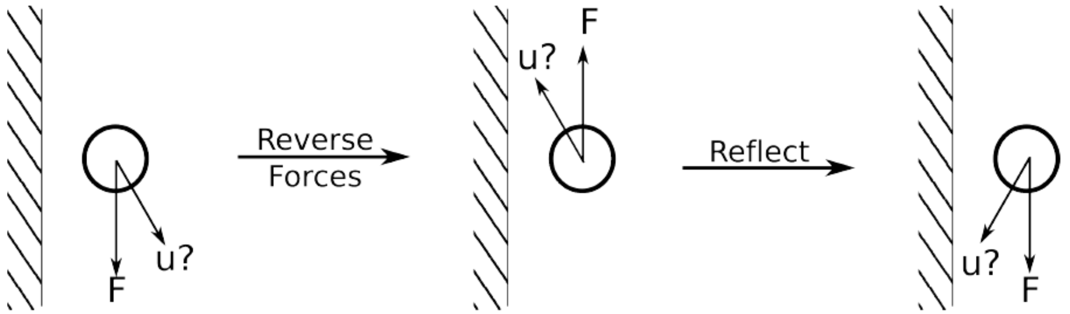
Then $\left\{\alpha\, \mathbf{v}_1(\mathbf{r}) + \beta\, \mathbf{v}_2(\mathbf{r}); \alpha\, P_1(\mathbf{r}) + \beta\, P_2(\mathbf{r})\right\}$ is also a solution with boundary condition $\mathbf{u} = \alpha\, \mathbf{U}_1(\mathbf{r}) + \beta\, \mathbf{U}_2(\mathbf{r})$ over S.

In particular $\alpha = -1$, $\beta = 0$ is a solution (we inverse the motion of a boundary ($\mathbf{U}$ becomes $-\mathbf{U}$) and the fluid velocity is also inverted ($\mathbf{v}(\mathbf{r})$ becomes $-\mathbf{v}(\mathbf{r})$) $\longrightarrow$ **reversibility of Stokes eq.**

Linearity and reversibility

$$\nabla^2 \mathbf{v} = \frac{1}{\mu} \nabla P + \boldsymbol{F}$$

If the sign of all forces $\boldsymbol{F}$ changes so does the sign of the velocity field $\mathbf{v}$. This can be used together with symmetry arguments to rule out something:



Unique solution

This can be demonstrated assuming two different solutions for same boundary conditions and analysing their difference …

(see, *e.g*. D. Barthès-Biesel, *Microhydrodynamics and Complex Fluids*, CRC Press, 2012)

Minimal dissipation

$\frac{dK}{dt} = P^{ext} - \Phi,$ K the total kinetic energy of the flow,

and Φ the rate of energy dissipation.

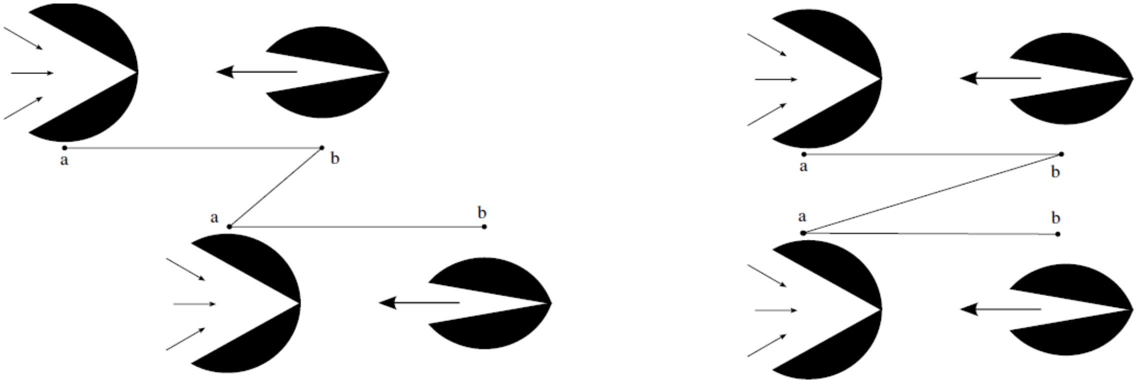
In Stokes flow it is $P^{ext} = \Phi$, and it is easy to

show that $\Phi = \int_V \sigma_{ij}\, \epsilon_{ij}\, dV$, with $\sigma_{ij} = -p\, \delta_{ij} + 2\mu\, \epsilon_{ij}$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ rate of strain

is **minimal** (when compared to another solenoidal flow with same boundary conditions, *cf.* Barthès-Biesel, 2010)

Scallop theorem



Re > 1

Re << 1

Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)

American Journal of Physics, Vol. 45, No. 1, January 1977

Scallop theorem

Theorem

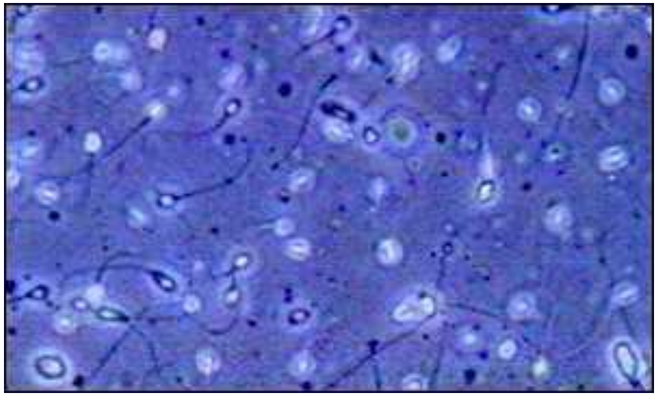
Suppose that a small swimming body in an infinite expanse of Newtonian fluid is observed to execute a periodic cycle of configurations, relative to a coordinate system moving with constant velocity $\boldsymbol{U}$ relative to the fluid at infinity. Suppose that the fluid dynamics is that of Stokes flow. If the sequence of configurations is indistinguishable from the time reversed sequence, then $\boldsymbol{U} = 0$ and the body does not locomote.

Other formulation:

To achieve propulsion at low Reynolds number in Newtonian fluids a swimmer must deform in a way that is not invariant under time-reversal.

Scallop theorem

Re << 1, micro-organisms use *non-reciprocal waves* to move (no inertia → symmetry under time reversal); 1 DOF in the kinematics is not enough!



General solution for Stokes' flows

$$\begin{aligned} \nabla p(\mathbf{r},t) - \mu \nabla^2 \mathbf{u}(\mathbf{r},t) &= \underline{\mathbf{f}_{ext}(\mathbf{r})} \\ \nabla \cdot \mathbf{u}(\mathbf{r},t) &= 0 \end{aligned}$$

Assume the external force acts on a single point $\mathbf{r}'$ in the fluid:

$$\mathbf{f}_{ext}(\mathbf{r}) = \mathbf{F}_0 \delta(\mathbf{r} - \mathbf{r}')$$

because of linearity of Stokes flow, the answer must be linear in $\mathbf{F}_0$:

$$\boxed{\begin{aligned} \mathbf{u}(\mathbf{r}) &= \mathbf{T}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{F}_0 \\ p(\mathbf{r}) &= \mathbf{g}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{F}_0 \end{aligned}}$$

$\mathbf{T}$ is the Oseen tensor

General solution for Stokes' flows

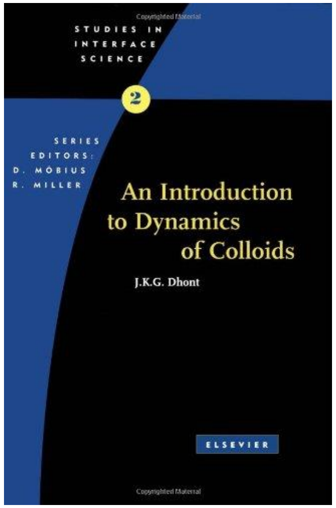
Now, assume a <u>continuously distributed force density</u> in the fluid; again because of linearity/superposition:

$$\mathbf{u}(\mathbf{r}) = \int \mathbf{T}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{f}(\mathbf{r}')\, \mathrm{d}^3 r'$$

$$p(\mathbf{r}) = \int \mathbf{g}(\mathbf{r}-\mathbf{r}') \cdot \mathbf{f}(\mathbf{r}')\, \mathrm{d}^3 r'$$

The Green's function can be evaluated formally by Fourier transform (*e.g*. J.K.G. Dhont, *An Introduction to Dynamics of Colloids*, Elsevier, Amsterdam 1996) and the result is:

$$\mathbf{T}(\mathbf{r}) = \frac{\mathbf{G}(\mathbf{r})}{8\pi\mu} \quad \text{with} \quad \mathbf{G}(\mathbf{r}) = \frac{1}{r}\hat{\mathbf{I}} + \frac{\mathbf{r}\mathbf{r}}{r^3}$$



General solution for Stokes' flows

The Green's function $\mathbf{G}(\mathbf{r})$ of a point disturbance in a fluid is known as *Stokeslet* (or *Stokes propagator* since it describes how the flow field is propagated throughout the medium by a single localized point force acting on the fluid in $\mathbf{r}'$ as a singularity); it is a **tensor**.

Also the pressure Green's function (a **vector**) can be found analytically:

$$\mathbf{g}(\mathbf{r}) = \frac{1}{4\pi}\frac{\mathbf{r}}{r^3}$$

The velocity field decays in space as r^{-1} and the pressure goes like r^{-2}.

General solution for Stokes' flows

From the Stokeslet many other solutions can be obtained the complete set of singularities for viscous flow can be obtained by differentiation (A.T. Chwang & T.Y.T. Wu, Hydromechanics of low-Reynolds-number flow: II. Singularity method for Stokes flows *J. Fluid Mech.* **67** (1975) 787–815). One derivative leads to force dipoles, with flow fields decaying as r^{-2}. Two derivatives lead to source dipoles and force quadrupoles, with velocity decaying in space as r^{-3}. Higher-order singularities are easily obtained by further differentiation.

A well-chosen distribution of such singularities can then be used to solve exactly Stokes' equation in a variety of geometries. For example, the Stokes flow past a sphere is a combination of a **Stokeslet** and an irrotational (!) **point source dipole** at the center of the sphere.

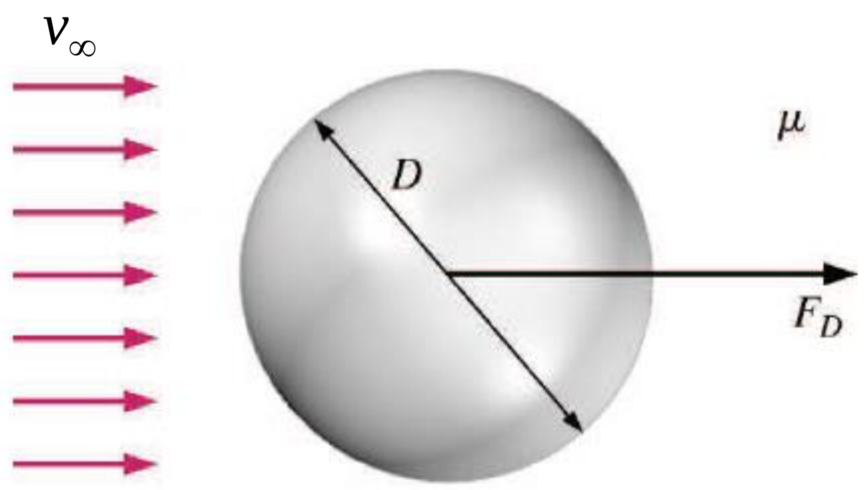
The boundary integral method

A linear superposition of singularities is also at the basis of the *boundary integral method* to computationally solve for Stokes flows using solely velocity and stress information at the boundary (*e.g*. C. Pozrikidis, *Boundary Integral and Singularity Methods for Linearized Viscous Flow,* Cambridge University Press, 1992)

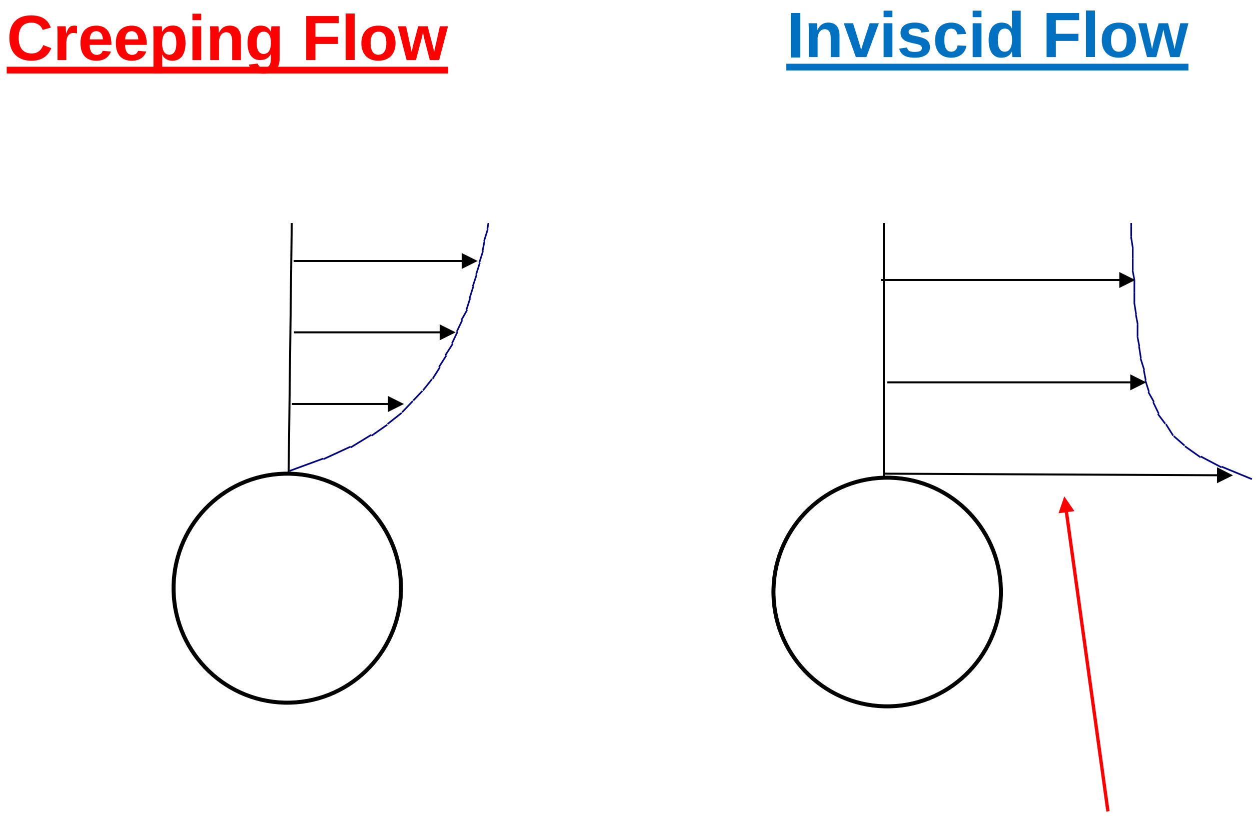
Stokes’ flow

Let’s focus on a special case which admits

a well-known analytical solution



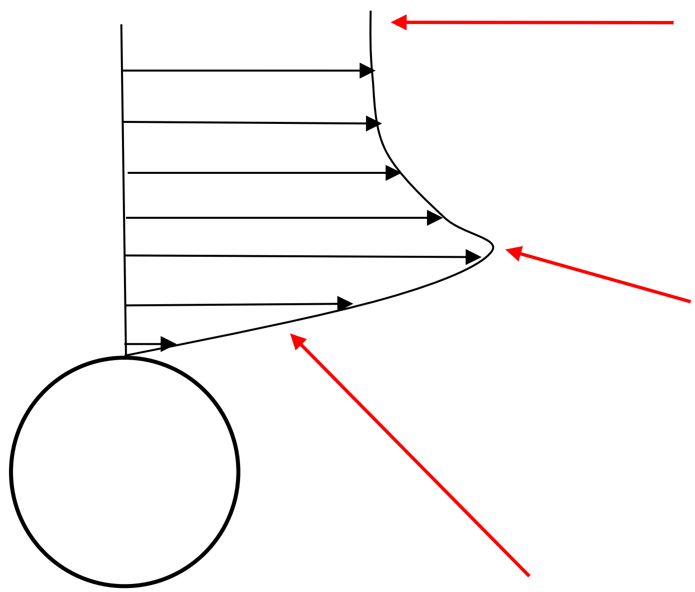
A special case: ***FLOW AROUND A SPHERE***



Larger velocity near the sphere is an inertial effect.

Flow around a Sphere

General case:



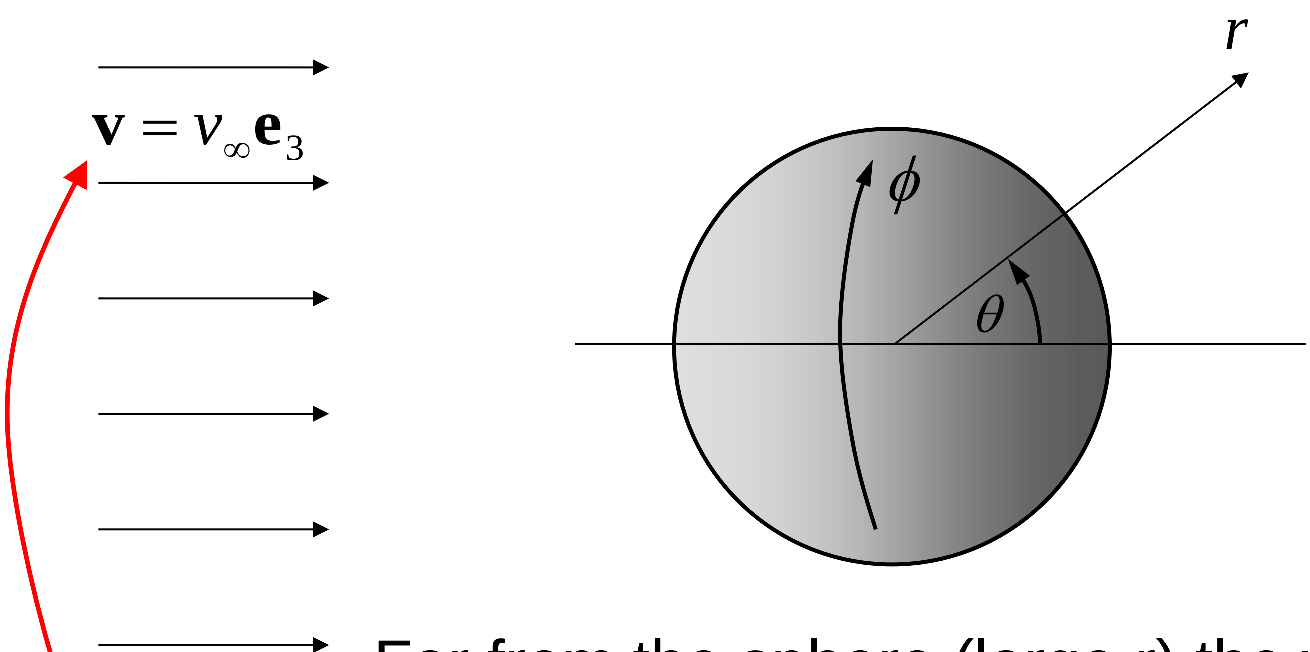
Incident velocity is approached far from the sphere.

Increased velocity as a result of inertia terms.

Shear region near the sphere caused by viscosity and no-slip.

Stokes Flow: The Geometry

Use Standard Spherical Coordinates: r, θ, and ϕ



θ = polar angle

$0 \leq \theta \leq \pi$

ϕ = azimuthal angle

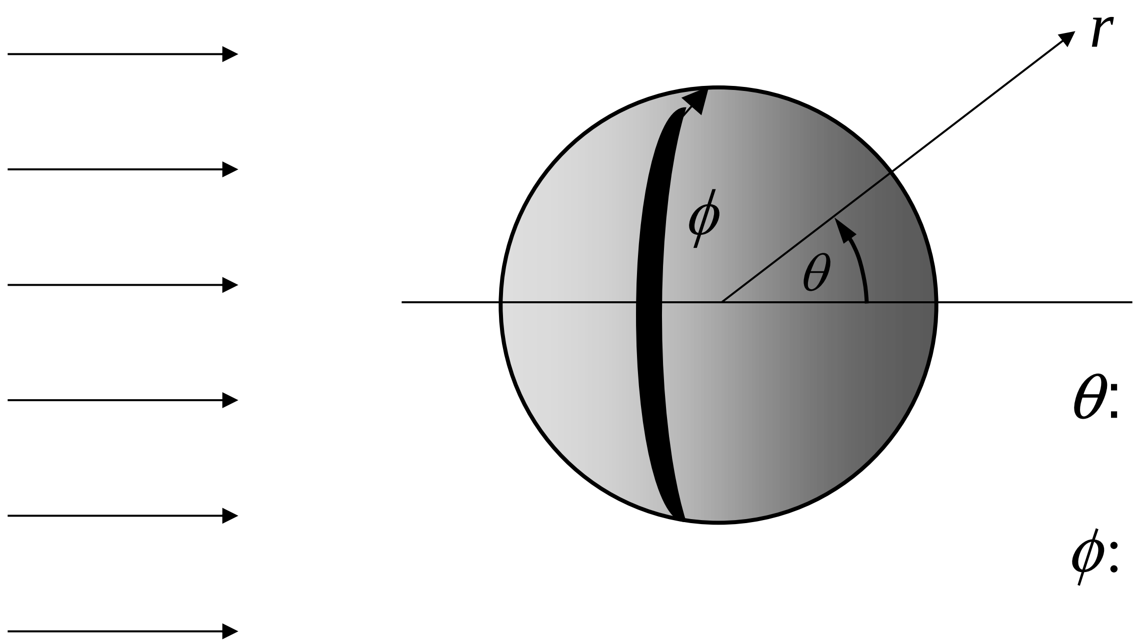
$0 \leq \phi \leq 2\pi$

Far from the sphere (large r) the velocity is uniform in the rightward direction; $\mathbf{e}_3$ is the Cartesian (rectangular) unit vector. It does not correspond to the spherical unit vectors.

Flow past a sphere: objectives

1. Obtain the velocity field around the sphere
2. Use this velocity field to determine pressure and shear stress at the sphere surface
3. From the **pressure** and the **shear stress**, determine the **drag** force on the sphere as a function of the sphere's velocity
4. Analyze similar flow cases …

Symmetry of the Geometry



θ: inclination or polar

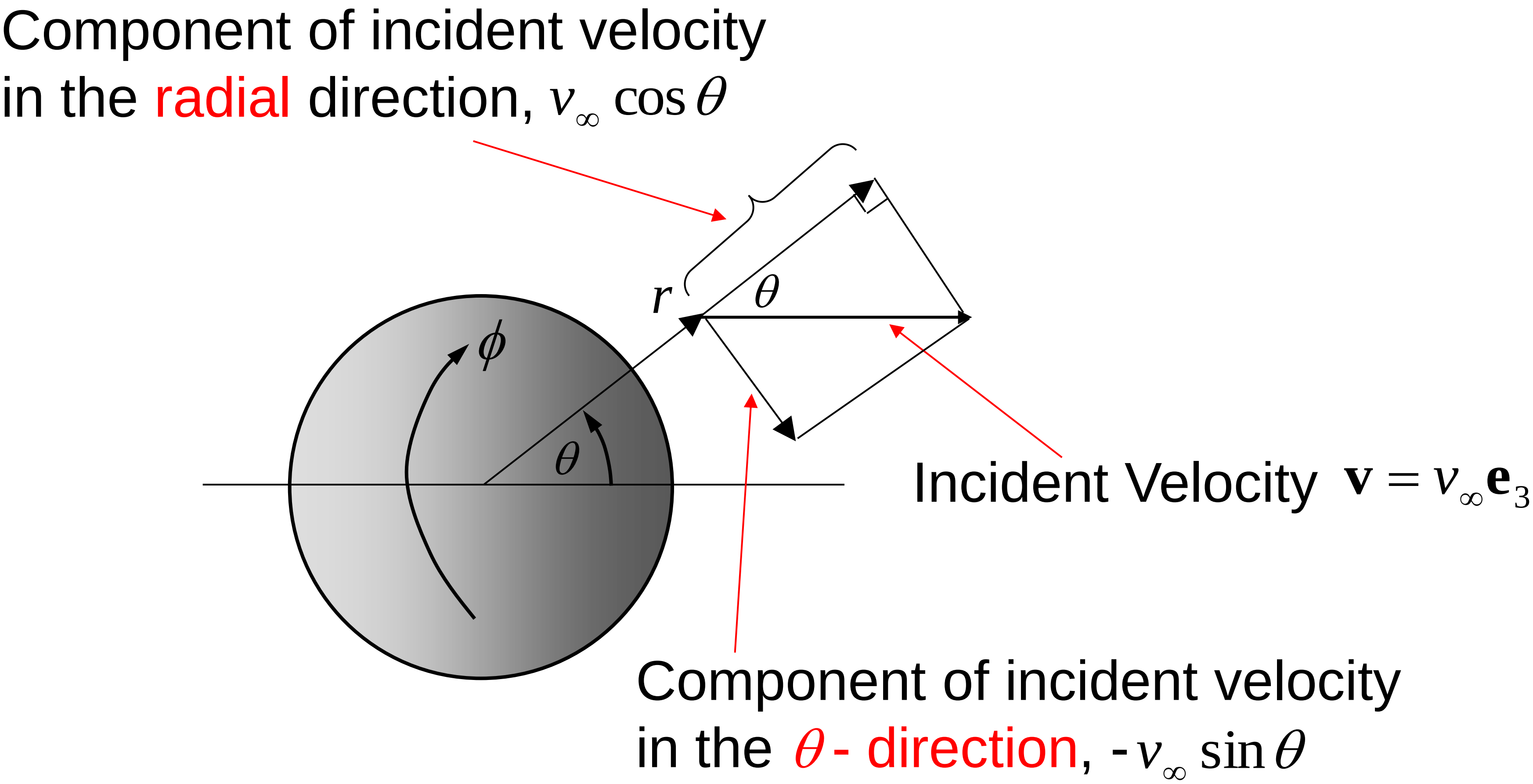
$0 \leq \theta \leq \pi$

ϕ: azimut

$0 \leq \phi \leq 2\pi$

The flow will be symmetric with respect to ϕ

Components of the Incident Flow



Reynolds Number

One can use the kinematic (ν) or the dynamic (μ) viscosity, so that the Reynolds number may be

$$\mathrm{Re} = \frac{VL}{\nu} \quad \text{or} \quad \mathrm{Re} = \frac{\rho VL}{\mu}$$

In the case of creeping flow around a sphere, we use v_∞ for the characteristic velocity, and we use the sphere diameter as the characteristic length scale. Thus,

$$\boxed{\mathrm{Re} = \frac{\rho v_\infty D}{\mu}}$$

Summary of Equations to be Solved

Conservation of mass

$$\nabla \cdot \mathrm{v} = 0$$

takes the following form in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\rho v_\theta \sin\theta\right) + \boxed{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\rho v_\phi\right) = 0}$$

or $\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right) + \frac{1}{r\,\mathbf{sin}\theta}\frac{\partial}{\partial\theta}\left(\rho v_\theta\, \mathbf{sin}\theta\right) = 0}$ when $v_\phi = 0$ and $\frac{\partial}{\partial\phi} = 0$

Summary of Equations (Momentum)

Because there is symmetry in ϕ, we only worry about the radial and circumferential components of momentum.

$$-\nabla P + \nabla \cdot \tau = 0$$ (incompressible, Newtonian Fluid)

In spherical coordinates:

Radial

$$-\frac{\partial p}{\partial r} + \mu\left(\mathcal{H}v_r - \frac{2}{r^2}v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2}{r^2}v_\theta \cot\theta\right) = 0$$

Azimuthal

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\mathcal{H}v_\theta + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2 \sin^2\theta}\right) = 0$$

$$\text{with} \quad \mathcal{H} \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right)$$

Comments

Three equations, one first order, two second order.

Three unknowns (v_r, v_θ and P).

Two independent variables (r and θ).

Equations are linear (there is a solution).

Stream Function Approach

We will use a stream function approach to solve these equations.

The stream function is a differential form that automatically solves the conservation of mass equation and reduces the problem from one with 3 variables to one with two variables.

Stream Function (Cartesian)

Cartesian coordinates, the two-dimensional continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

If we define a stream function, ψ, such that:

$$u = \frac{\partial \psi(x, y)}{\partial y}, \quad v = -\frac{\partial \psi(x, y)}{\partial x} = 0$$

Then the two-dimensional continuity equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Summary of the Procedure

1. Use a stream function to satisfy conservation of mass.
   a. Form of ψ is known for spherical coordinates.
   b. Gives 2 equations (r and θ momentum) and 2 unknowns (ψ and pressure).
   c. Need to write B.C.s in terms of the stream function.
2. Obtain the momentum equation in terms of velocity.
3. Rewrite the momentum equation in terms of ψ.
4. Eliminate pressure from the two equations (gives 1 equation (momentum) and 1 unknown, ψ).
5. Use B.C.s to deduce a form for ψ (equivalently, assume a separable solution).

Procedure (Continued)

6. Substitute the assumed form for ψ back into the momentum equation to obtain ordinary differential equations, whose solutions yield ψ.
7. Use the definition of the stream function to obtain the radial and tangential velocity components from ψ.
8. Use the radial and tangential velocity components in the momentum equation to obtain pressure.
9. Integrate the $\mathbf{e}_3$ component of both types of forces (pressure and viscous stresses) over the surface of the sphere to obtain the drag force on the sphere.

Streamfunction

Recall the following form for conservation of mass:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta \sin\theta\right)=0$$

Slide 52

If we define a function $\psi(r,\theta)$ as:

$$\boxed{v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}}$$

then the equation of continuity is automatically satisfied. We have combined 2 unknowns into 1 and eliminated 1 equation.

Note that other forms work for rectangular and cylindrical coordinates.

Momentum Eq. in Terms of ψ

Use $v_r = \dfrac{1}{r^2 \sin\theta} \dfrac{\partial \psi}{\partial \theta}, \;\; v_\theta = \dfrac{-1}{r \sin\theta} \dfrac{\partial \psi}{\partial r}$

and conservation of mass is satisfied (procedure step 1).

Substitute these expressions into the steady flow momentum equation (slide 53) to obtain a partial differential equation for ψ from the momentum equation (procedure step 3):

$$\boxed{\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)\right]^2 \psi = 0}$$

Elimination of Pressure

The final equation on the last slide requires several steps. The first is the elimination of pressure in the momentum equations. The second was substitution of the form for the stream function into the result.

How do we eliminate pressure from the momentum equation? We have:

$$-\nabla P+\mu\nabla\cdot\left(\nabla\mathbf{v}\right)=\mathbf{0}$$

We take the curl of this equation to obtain:

$$\nabla\times\nabla\cdot\left(\nabla\mathbf{v}\right)=\nabla\times\nabla^{2}\mathbf{v}=\mathbf{0}$$

Exercise: Elimination of Pressure

Furthermore:

$$\nabla \times \nabla^2 \mathbf{v} = -\nabla \times \nabla \times \nabla \times \mathbf{v} = -\nabla \times \nabla \times \boldsymbol{\zeta} = \mathbf{0}$$

given the vector identity: $\nabla \times \nabla \times \mathbf{v} = -\nabla^2 \mathbf{v} + \nabla(\cancel{\nabla \cdot \mathbf{v}})$

It can be shown (straightforward, see Appendix A…) that:

$$\boxed{\mathbf{v} = \nabla \times \left(\frac{\psi\, \mathbf{e}_\phi}{r \sin\vartheta} \right)}$$

Exercise: Elimination of Pressure

Furthermore:

$$\nabla \times \nabla^2 \mathbf{v} = -\nabla \times \nabla \times \nabla \times \mathbf{v} = -\nabla \times \nabla \times \boldsymbol{\zeta} = \mathbf{0}$$

given the vector identity: $\nabla \text{ x } \nabla \text{ x } \mathbf{v} = -\nabla^2 \mathbf{v} + \nabla(\cancel{\nabla \cdot \mathbf{v}})$

It can be shown (straightforward, see Appendix A…) that:

$$\boxed{\mathbf{v} = \nabla \text{ x } \left(\frac{\psi\, \mathbf{e}_\phi}{r \sin\vartheta} \right)}$$

Momentum in Terms of ψ

Given that:

$$\boldsymbol{\zeta} = \nabla\times\nabla\times\left(\frac{\psi\ \ \mathbf{e}_\phi}{r\ \sin\vartheta}\right) = -\frac{\mathbf{e}_\phi}{r\ \sin\vartheta}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\vartheta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\vartheta}\frac{\partial\psi}{\partial\vartheta}\right)\right)$$

(see Appendix B)

from $\nabla\times\nabla\times\boldsymbol{\zeta} = \mathbf{0}$ it follows:

$$\nabla\times\nabla\times\nabla\times\nabla\times\left(\frac{\psi\ \ \mathbf{e}_\phi}{r\ \sin\vartheta}\right) = 0 \quad \rightarrow \quad \left(E^2\right)^2\psi = 0$$

Momentum in Terms of ψ

$$E^4\psi = 0, \text{ where } E^2 \equiv \left\{ \frac{\partial}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right\}.$$

$$\text{Thus } \left\{ \frac{\partial}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right\}^2 \psi = 0.$$

This equation was given on slide 60.

N.B. The operator E^2 is NOT the laplacian …

Boundary Conditions in Terms of ψ

From

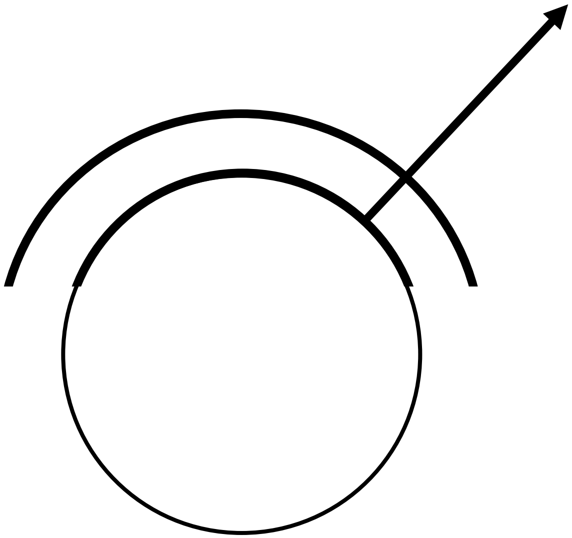
$$v_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = 0 \text{ at } r = R, \quad v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = 0 \text{ at } r = R$$

$\frac{\partial\psi}{\partial r}$ and $\frac{\partial\psi}{\partial\theta}$ must be zero for all θ at $r = R$. Thus, ψ must be constant along the curve $r = R$. But since the constant of integration is arbitrary, we can take it to be zero at that boundary, i.e.

$$\psi = 0 \text{ at } r = R$$

Question

Consider the following curves. Along which of these curves must velocity change with position?

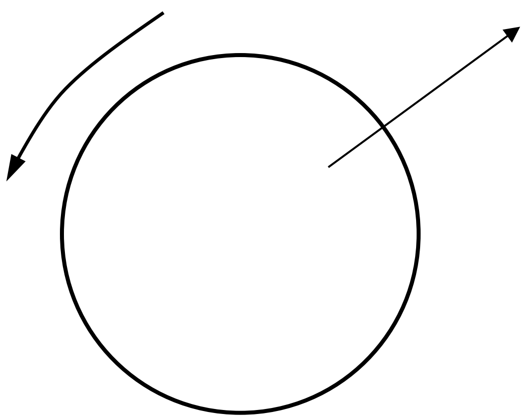


Comment

A key to understanding the previous result is that we are talking about the surface of the sphere, where r is fixed.

Because $v_r = 0, \; \dfrac{1}{r^2 \sin\theta}\dfrac{\partial \psi}{\partial \theta} = 0.$ And so because $\dfrac{\partial \psi}{\partial \theta} = 0$ $\underline{\textit{for all } \theta}$, ψ must be constant along that curve.

ψ does not change as θ changes.



As r changes, however, we move off of the curve $r = R$, so ψ *can* change.

Boundary Conditions in Terms of ψ

From $v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}, \quad \frac{\partial \psi}{\partial \theta} = v_r r^2 \sin\theta$

At $r \to \infty, \; v_r \to v_\infty cos\theta$ (see slide 49)

Thus, as $r \to \infty, \frac{\partial \psi}{\partial \theta} \to (v_\infty cos\theta) r^2 sin\theta = v_\infty r^2 cos\theta \; sin\theta$

In contrast to the surface of the sphere, ψ will change with θ far from the sphere.

Boundary Conditions in Terms of ψ

From $v_r = \dfrac{1}{r^2 \sin\theta}\dfrac{\partial\psi}{\partial\theta}, \quad \dfrac{\partial\psi}{\partial\theta} = v_r r^2 \sin\theta$

$$\psi\Big|_{r\to\infty} = \int_0^\theta \frac{\partial\psi}{\partial\theta}\,d\theta = \int_0^\theta v_r r^2 sin\theta\,d\theta = r^2\int_0^\theta \left(v_\infty cos\theta\right) sin\theta\,d\theta$$

$$= \frac{1}{2} r^2 v_\infty sin^2\theta$$

which suggests the θ-dependence of the solution.

$$\boxed{\psi = f(r)\; v_\infty sin^2\theta}$$

Comment on Separability

For a separable solution we assume that the function ψ is the product of one function that depends only on r and another one that depends only on θ, i.e.

$$\psi(r,\theta) = \mathcal{R}(r)\Theta(\theta)$$

Whenever the boundary conditions can be written in this form, it is advisable to search for a solution written in this form. Since the equations are linear, the solution will be unique.

Comment on Separability

In our case, the boundary condition at $r=R$ is:

$$\psi(R,\theta) = \mathcal{R}(R)\Theta(\theta) = 0$$

and the boundary condition at $r \to \infty$ is:

$$\psi(\infty,\theta) = \frac{1}{2} v_{\infty} r^2 \sin^2 \theta$$

Both of these forms can be written as a function of r times a function of θ. (For $r=R$ we take $\mathcal{R}(R)$=0). The conclusion that the θ dependence like sin$^2\theta$ is reached because <u>these two boundary conditions must hold for all θ</u>. A similar statement about the r-dependence cannot be reached.

Momentum Equation

The momentum equation:

$$\nabla P = \mu \nabla^2 \mathrm{v}$$

is 2 equations with 3 unknowns (P, v_r and v_θ). We have used the stream function (i.e. the fact that $\mathbf{v}$ is solenoidal) to get 2 equations and 2 unknowns (P and ψ). We then used these two equations to eliminate P (step 4 on slide 57).

Substitute back into momentum eq.

With $\psi = f(r)\, v_\infty sin^2\theta$

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0$$

(slide 65) becomes:

$$sin^2\theta\left[\frac{d^4 f}{dr^4} - \frac{4}{r^2}\frac{d^2 f}{dr^2} + \frac{8}{r^3}\frac{df}{dr} - \frac{8f}{r^4}\right] = 0$$

(cf. calculations in Appendix C)

Substitute back into momentum eq.

The resulting ODE is an *equidimensional equation* for which:

$$r^4 \frac{d^4 f}{dr^4} - 4r^2 \frac{d^2 f}{dr^2} + 8r \frac{df}{dr} - 8f = 0$$

$$\boxed{f(r) = ar^n}$$

Substitution of this form back into the equation yields:

$$f(r) = \frac{1}{4\,r}\left[\,2\,r^3 - 3\,R\,r^2 + R^3\right]$$

(details in Appendix D)

Solution for Velocity Components and Vorticity

From the definition of streamfunction and vorticity we have:

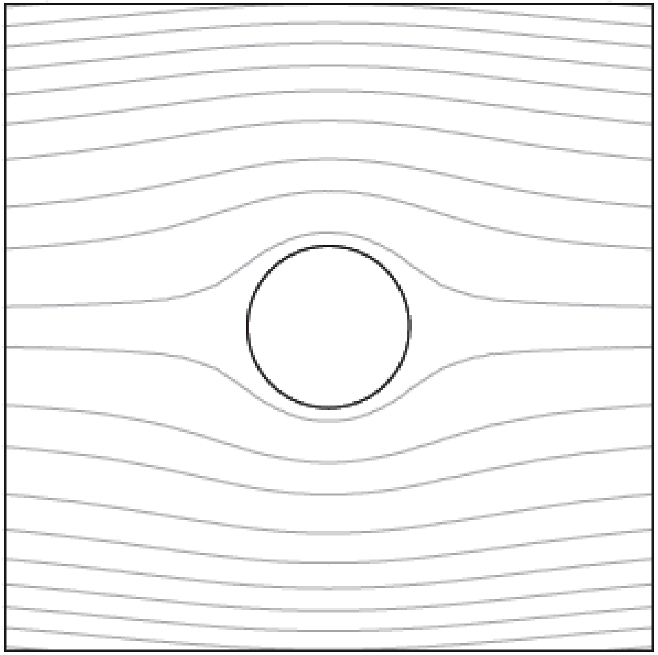
$$\frac{v_r}{v_\infty} = \left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3 \right] cos\,\theta$$

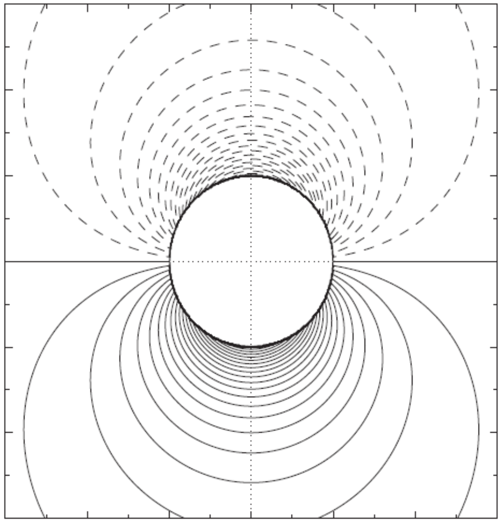
$$\frac{v_\theta}{v_\infty} = \left[-1 + \frac{3}{4}\frac{R}{r} + \frac{1}{4}\left(\frac{R}{r}\right)^3 \right] sin\,\theta$$

$$\zeta = -\frac{3\ v_\infty\ R\ sin\,\theta}{2\ r^2}$$

(cf. slides 64, 66)

Solution for Streamfunction and Vorticity





Streamlines and contour lines of the vorticity

(dashed/solid lines indicate opposite signs of ζ).

Notice the symmetry!

Dragging slowly the sphere from right to left …

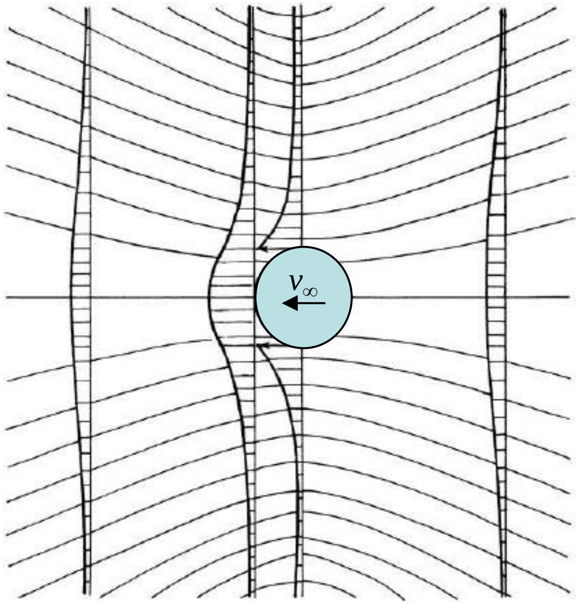
i.e. adding a uniform velocity v_∞

the streamfunction becomes:

$$\psi = f(r)\, v_\infty\, sin^2\vartheta$$

with

$$f(r) = \frac{1}{4\ r}\left[\,2\, r^3 - 3\ R\ r^2 + R^3\right]$$



upstream-downstream symmetry is the result of the neglect of non-linearities

Multipolar solutions

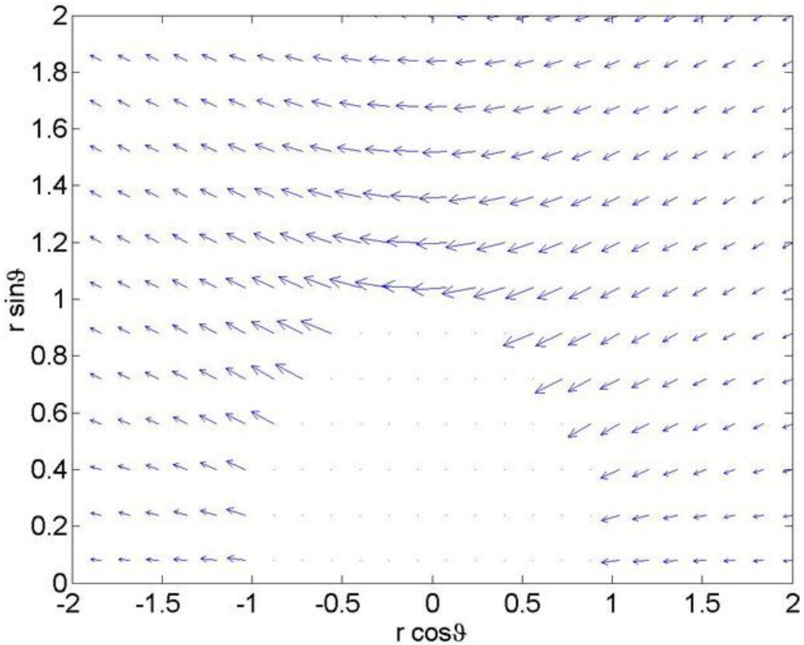
Stokes flow past a sphere is comprised by three terms:

$$\frac{v_r}{v_\infty} = \left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right] \cos\theta$$

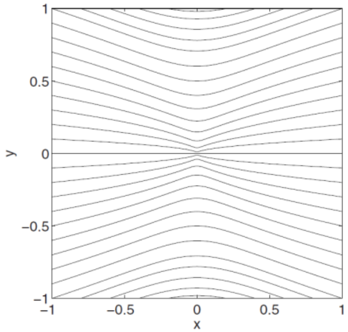
$$\frac{v_\theta}{v_\infty} = \left[-1 + \frac{3}{4}\frac{R}{r} + \frac{1}{4}\left(\frac{R}{r}\right)^3\right] \sin\theta$$

The terms relate to the multipolar solutions arising from the solution of Laplace equation in spherical coordinates. The constant term refers to the uniform free-stream velocity v_∞; this is the flow that would be observed if the sphere were absent. The term proportional to R/r is the Stokeslet term; it corresponds to the response of the flow caused by a point force of $F_{Stokes} = 6\,\pi\,R\,\mu\,v_\infty$ applied to the fluid at the center of the sphere. The term proportional to R^3/r^3 is a stresslet.

Stokeslet



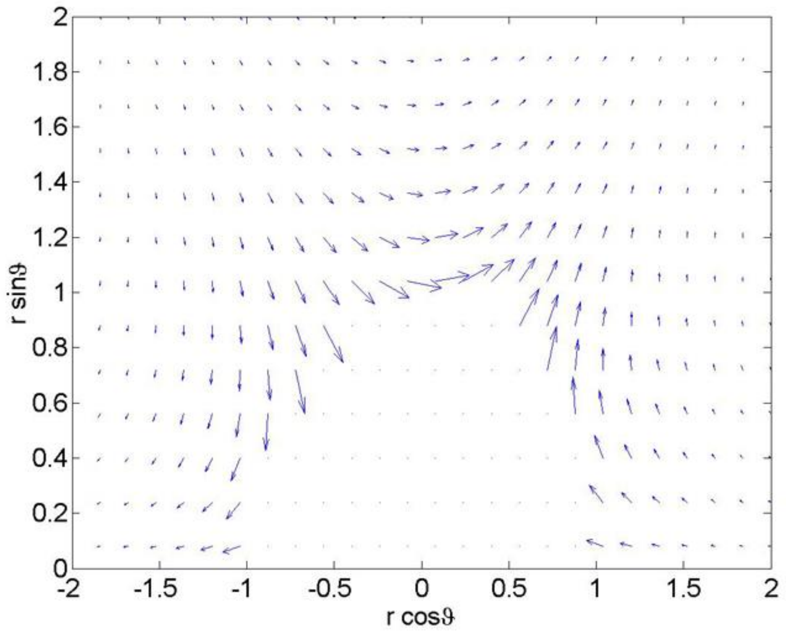
The Stokeslet term describes the viscous response of the fluid to the no-slip condition at the particle surface, and this term contains all of the vorticity caused by the viscous action of the particle.



The Stokeslet component of Stokes flow around a sphere moving from right to left along the x-axis. The velocity magnitude along the horizontal axis (right to left) is:

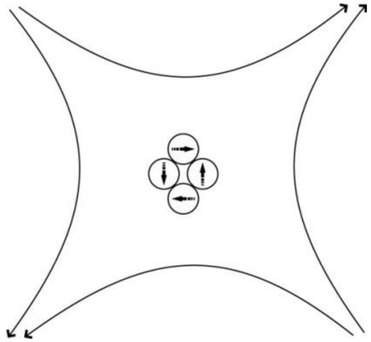
$$u_h = -\; v_r \cos\theta + v_\theta \sin\theta = \tfrac{3}{4}\, v_\infty\, R/r\,(1 + \cos^2\theta) = \frac{F_{Stokes}}{8\,\pi\mu r}\,(1 + \cos^2\theta)$$

Stresslet



The stresslet term is not related to the viscous force of the sphere (it is an **irrotational** term), and is caused by the finite size of the sphere. It satisfies Stokes equation with P constant.

The stresslet (or point source dipole) component of Stokes flow around a sphere moving from right to left along the x-axis. Note, in comparison to the previous figure, how quickly the velocities decay as the distance from the surface increases.



Multipolar solutions

Since the stresslet term decays proportional to r^{-3} while the Stokeslet term decays proportional to r^{-1}, the primary long-range effect of the particle is induced by the Stokeslet. Thus, the net force on the fluid induced by the sphere is required to prescribe the flow far from a sphere, rather than the particle size or velocity alone.

Far from a sphere moving in a Stokes flow, the flow does not distinguish between the effects of one particle that has velocity v_∞ and radius 2 R and another that has velocity 2 v_∞ and radius R, since these two spheres have the same drag force. Close to these spheres, of course, the two flows are different, as distinguished by the different stresslet terms.

Effective Pressure

To obtain the *effective* pressure, we go back to the momentum equation:

$$\nabla P = \mu \nabla^2 \mathbf{v}$$

Once v_r and v_θ are known, they are replaced into the equation above to yield:

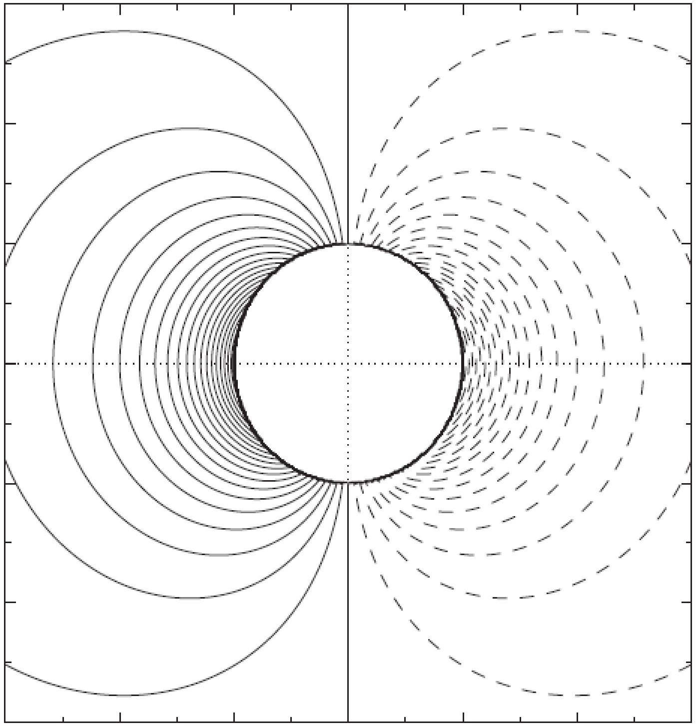
$$\frac{\partial P}{\partial r} = 3\mu v_\infty R \frac{\cos\theta}{r^3} \qquad \frac{\partial P}{\partial \theta} = \frac{3}{2}\mu v_\infty R \frac{\sin\theta}{r^2}$$

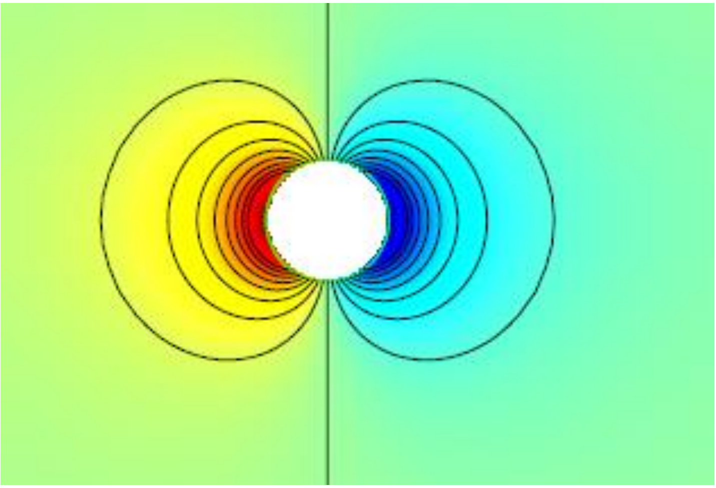
Integration yields:

$$P = p_0 - \frac{3}{2}\mu v_\infty R \frac{\cos\theta}{r^2}$$

with p_0 constant of integration. Decay as r^{-2} related to Stokeslet.

Effective Pressure





CFD
solution
for Re = 0.1

Contours of the *effective* pressure $P - p_0$

(solid-dashed lines correspond to opposite signs of $P - p_0$)

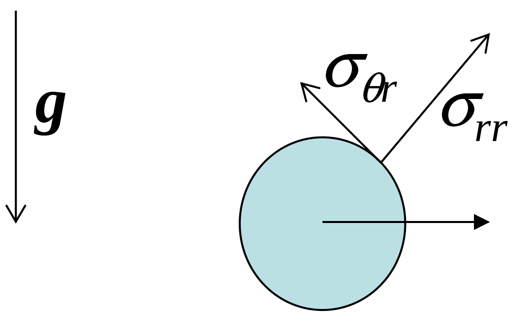
Drag Force

To obtain the drag force on the sphere ($r = R$), we must remember that it is caused by both the pressure and the viscous stress:

$$\mathbf{F}\cdot\mathbf{e}_3 = R^2\int_0^{2\pi} d\phi \int_0^{\pi}\left[\sigma_{rr}(R,\theta)\cos\theta + \sigma_{\theta r}(R,\theta)\left(-\sin\theta\right)\right]\sin\theta\, d\theta =$$

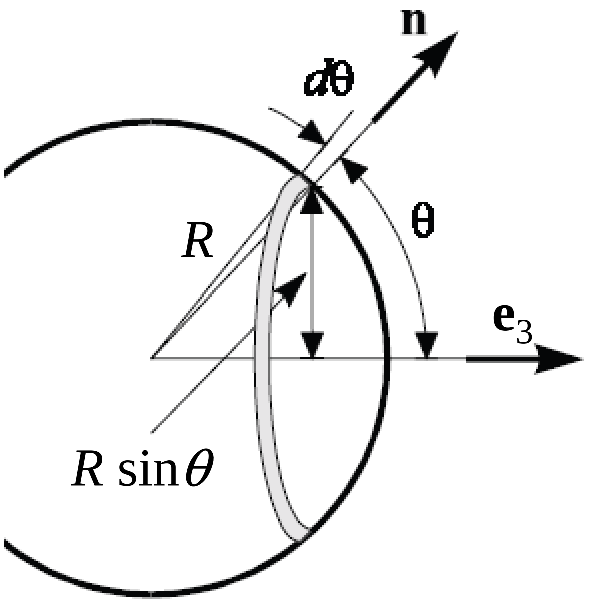
$$= 2\,\pi\,R^2\int_0^{\pi}\left[\sigma_{rr}(R,\theta)\,\cos\theta\,\sin\theta - \sigma_{\theta r}(R,\theta)\,\sin^2\theta\right]d\theta$$

used to get the $\mathbf{e}_3$ component



$\mathbf{e}_3$ is the direction the sphere is moving relative to the fluid.

Drag Force



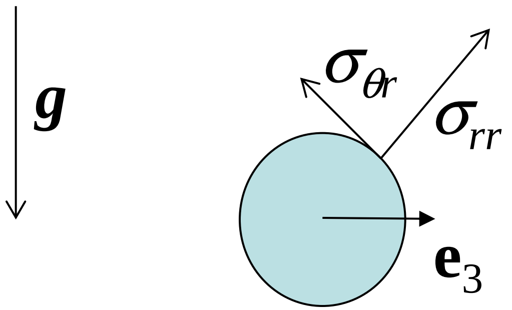
$$dS = 2\pi\,(R\,\sin\theta)\,(R\,d\theta)$$

Drag Force

The radial and tangential components of the force per unit area exerted on the sphere by the fluid are:

$$\sigma_{rr}(R,\theta) = \left(-p + \cancel{2\mu \frac{\partial v_r}{\partial r}} \right)_{r=R} = -p(R,\theta) = -p_0 + \frac{3\ \mu\ v_\infty\ cos\ \theta}{2\ R}$$

$$\sigma_{\theta r}(R,\theta) = \mu \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)_{r=R} = \mu\ \zeta = -\frac{3\ \mu\ v_\infty\ sin\ \theta}{2\ R}$$



Integration gives the fluid force on the sphere along $\mathbf{e}_3$ to be equal to $\boxed{6\ \pi R\ \mu\ v_\infty}$ which is the celebrated Stokes formula.

Drag Force

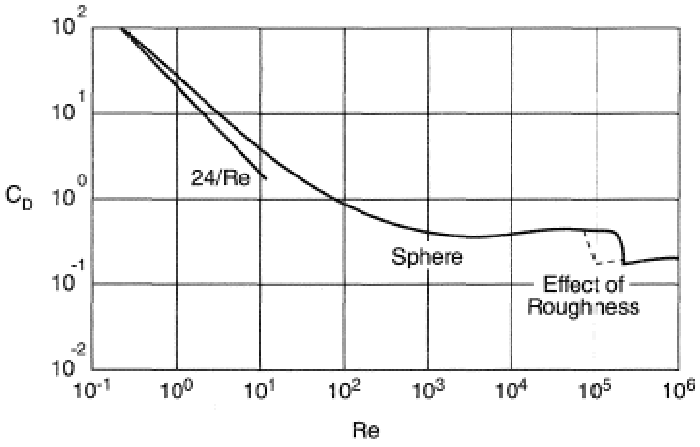
It can be easily found that the drag force can be split as

$2\,\pi R\,\mu\,v_\infty$ contribution due to pressure forces

$4\,\pi R\,\mu\,v_\infty$ contribution due to viscous forces (skin friction drag)

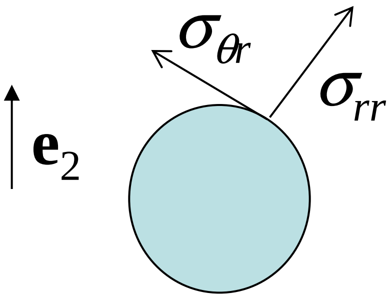
The drag coefficient is $C_D = 24/Re$

($Re = 2\,v_\infty R\,/\,\nu$)



Vertical force

In the vertical balance equation one should also account for the buoyancy force due to the weight of the fluid displaced by the sphere!



$$\mathbf{F} \cdot \mathbf{e}_2 = \frac{4}{3} \pi \; \rho \; g \; R^3$$

$$\boldsymbol{g} = -g\,\mathbf{e}_2$$

Potential flow (for comparison)

No vorticity → a velocity potential ϕ can be defined

The continuity equation:

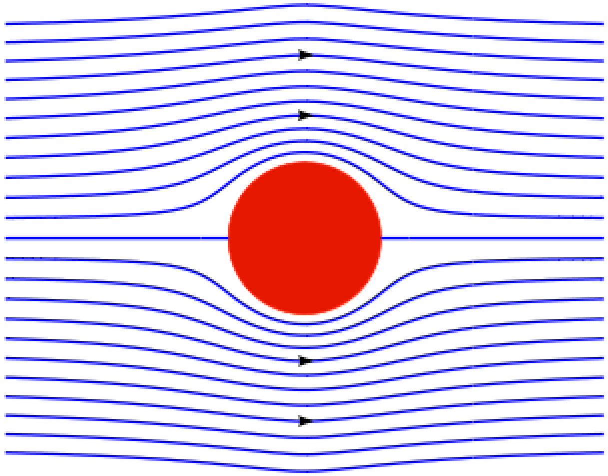
$$\nabla \cdot \mathbf{v} = 0$$

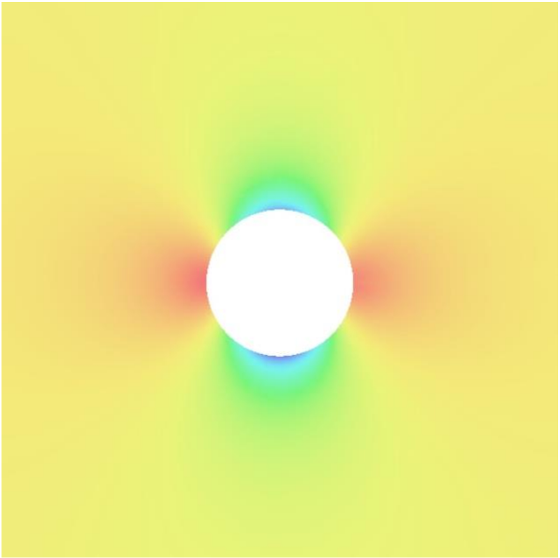
becomes:

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

Therefore potential flow reduces to finding solutions to Laplace's equation.

Potential flow (for comparison)

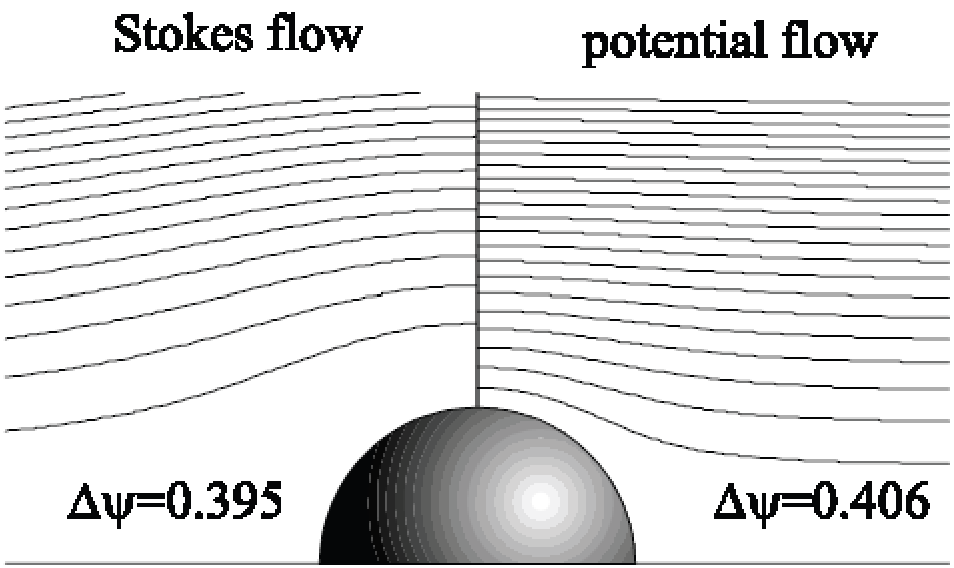




Streamlines are similar, **isobars are not**.

$C_D = 0$ → D'Alembert paradox

Stokes vs potential flow



In fact, streamlines are not so similar! In creeping flows even distant streamlines are significantly displaced, an effect of the *non-locality* of Stokes equation.

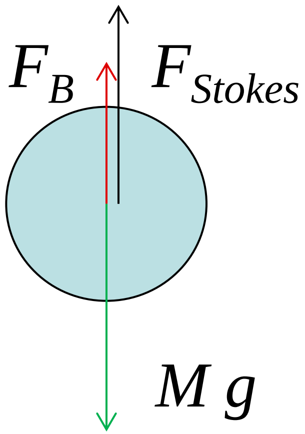
Back to Stokes flow past a sphere

Vertical forze balance over a vertically *falling sphere* with Re << 1 yields:

$F_3 = M\,g$ (with M the mass of the sphere of density $\bar{\rho}$)

so that the terminal velocity of the sphere is:

$$\boxed{v_\infty = \frac{2}{9}\frac{\rho g R^2}{\mu}\left(1 - \frac{\bar{\rho}}{\rho}\right)}$$



and the sphere move downwards when $\bar{\rho} > \rho$ (i.e., when it is denser than the fluid)

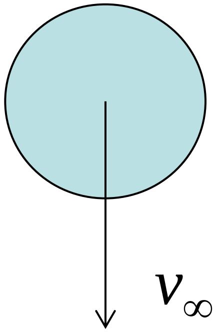
Stokes flow past a sphere

The Reynolds number of the fluid in the vicinity of the sphere is:

$$Re = \frac{2\rho\, v_\infty R}{\mu} = \frac{4}{9}\,\frac{gR^3}{\nu^2}\left|1 - \frac{\bar{\rho}}{\rho}\right|$$

with $\nu = \mu / \rho$ the fluid's kinematic viscosity.

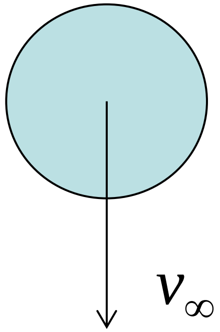
Example: grain of sand falling through water at 20°C, we have $\bar{\rho}/\rho \simeq 2$ and $\nu = 1.0 \times 10^{-6}$ m²/s. Hence, Re = $(R/6 \times 10^{-5})^3$, where R is measured in meters. For Re $\cong 1$ this yields a radius R of the sand grain of about 60 μm and the corresponding velocity is $v_\infty = 8 \times 10^{-3}$ m/s.



Stokes flow past a sphere

Another example: droplet of water falling through air at 20°C and atmospheric pressure, we have $\bar{\rho}/\rho \simeq 780$ and $\nu = 1.5 \times 10^{-5}$ m²/s. Hence, $\mathrm{Re} = (R/4 \times 10^{-5})^3$, where R is measured in meters. For $\mathrm{Re} \cong 1$ this yields a radius R of the water droplet of about 40 μm and the corresponding velocity is $v_\infty = 0.2$ m/s.

Microparticles achieve *equilibrium* quickly.



Stokes flow past a sphere

Given a system with small but finite Reynolds number, the *instantaneity* of the particle response can be quantified by calculating the Stokes number Sk, which is the ratio of the particle lag time to the characteristic time over which the flow changes. The characteristic flow time can come from the characteristic time of an unsteady boundary condition, or from the ratio v_∞/l of the characteristic velocity and length scale from a steady boundary condition in a nonuniform flow. Choosing the latter, we have

$$\text{Sk} = \frac{\tau_p}{l/v_\infty} = \frac{2\,R^2\rho_p\,v_\infty}{9\mu l}.$$

Particles with Stokes number Sk ≪ 1 can be assumed to be always in steady-state with a local velocity field given by the idealized solution derived earlier.

Exercise: estimate τ_p (Appendix E)

Stokes flow past a sphere

But what happens when we are far from the body, i.e. $r / R \rightarrow \infty$?

Stokes flow for a moving sphere in quiescent fluid

Sufficiently far from the sphere the Stokeslet dominate:

$$\frac{v_r}{v_\infty} = \left[-\frac{3}{2}\frac{R}{r} + \cancel{\frac{1}{2}\left(\frac{R}{r}\right)^3} \right] \cos\theta$$

$$\frac{v_\theta}{v_\infty} = \left[\frac{3}{4}\frac{R}{r} + \cancel{\frac{1}{4}\left(\frac{R}{r}\right)^3} \right] \sin\theta$$

Hence:

$$[\rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v}]_r = \rho\left[v_r\,\frac{\partial\, v_r}{\partial r} + \frac{v_\theta}{r}\,\frac{\partial v_r}{\partial\theta} - \frac{v_\theta^2}{r} \right] \sim \quad \frac{\rho\, {v_\infty}^2 R}{r^2}$$

Stokes flow for a moving sphere in quiescent fluid

The viscous term scales like:

$$(\mu \nabla^2 \mathbf{v})_r \sim \mu \frac{\partial^2 v_r}{\partial r^2} \sim \frac{\mu\, v_\infty R}{r^3}$$

Hence:

$$\frac{[\rho\,(\mathbf{v}\cdot\nabla)\,\mathbf{v}]_r}{(\mu \nabla^2 \mathbf{v})_r} \sim \frac{\rho\, v_\infty r}{\mu} \sim \text{Re}\,\frac{r}{R} \quad \text{as} \;\; r \to \infty$$

and even if Re << 1, inertia inevitably dominates viscosity at sufficiently large $\frac{r}{R}$, and the Stokes approx breaks down.

Oseen approximation

The remark above has first been made by Carl Wilhelm Oseen; he suggested to look at the flow as a uniform component plus a small disturbance: $\boxed{\boldsymbol{u} = v_\infty \boldsymbol{i} + \boldsymbol{u}'}$ (valid <u>far</u> from body …)

The *linearized* momentum equation becomes:

$$\underline{\rho\, v_\infty \frac{\partial \boldsymbol{u}'}{\partial x}} = -\nabla p + \mu \nabla^2 \boldsymbol{u}'$$ Oseen's eq.

For a moving sphere the boundary conditions are

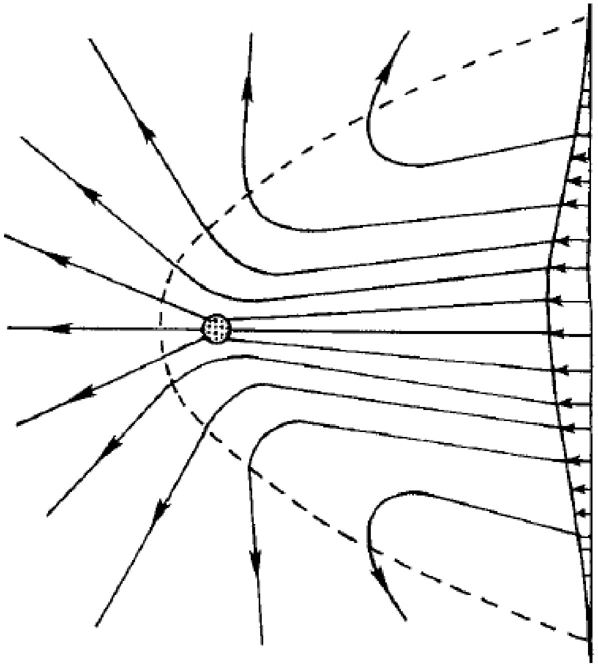
$u' = v' = w' = 0$ at infinity,

$u' = -v_\infty, \quad v' = w' = 0$ at the surface,

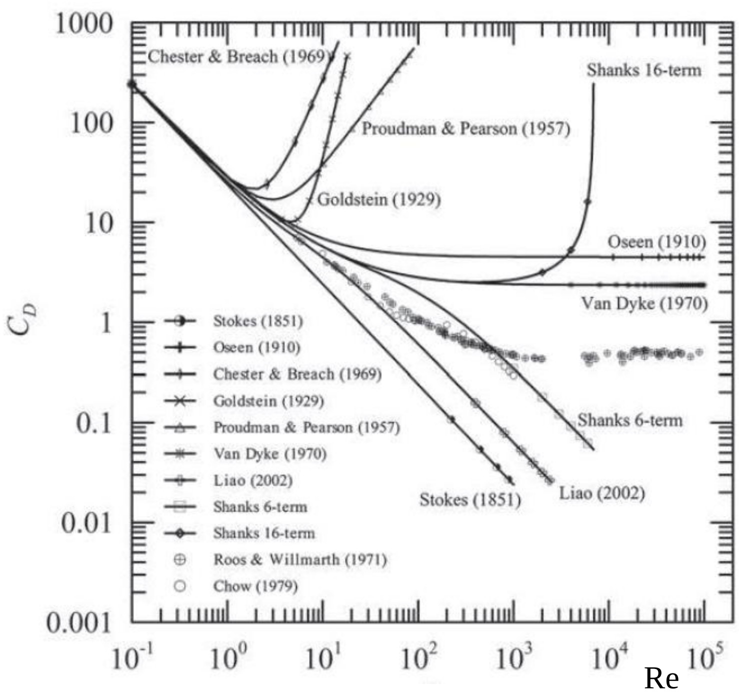
and Oseen was able to find an analytical solution (1910).

Oseen approximation

$$C_D = \frac{24}{\mathrm{Re}}\left(1 + \frac{3}{16}\mathrm{Re}\right)$$



Oseen asymmetric solution for a moving sphere



Axisymmetric Stokes flow in and around a fluid sphere (i.e. a drop)

Suppose a drop moves at constant speed V in a surrounding fluid, and suppose the two fluids are immiscible. Transform to a frame of reference in which the drop is stationary and centred at the origin; further, assume that Re both immediately outside and inside the drop are much less than unity (→ Stokes flow). Same analysis as before yields:

$$\psi(r,\theta) = \sin^2\theta\left(\frac{A}{r} + B\,r + C\,r^2 + D\,r^4\right)$$ outside the drop

$$\psi(r,\theta) = \sin^2\theta\left(\frac{\overline{A}}{r} + \overline{B}\,r + \overline{C}\,r^2 + \overline{D}\,r^4\right)$$ inside the drop

Axisymmetric Stokes flow in and around a fluid sphere (i.e. a drop)

The velocity components (both inside and outside the drop)

Have the same form, i.e.

$$v_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta}\left(\frac{\psi}{r}\right) = 2 \cos\theta \; (A/r^3 + B/r + C + Dr^2)$$

$$v_\theta = -\frac{1}{r} \frac{\partial}{\partial\theta}\left(\frac{\psi}{\sin\theta}\right) = -\sin\theta \; (-A/r^3 + B/r + 2\,C + 4\,Dr^2)$$

Axisymmetric Stokes flow in and around a fluid sphere (i.e. a drop)

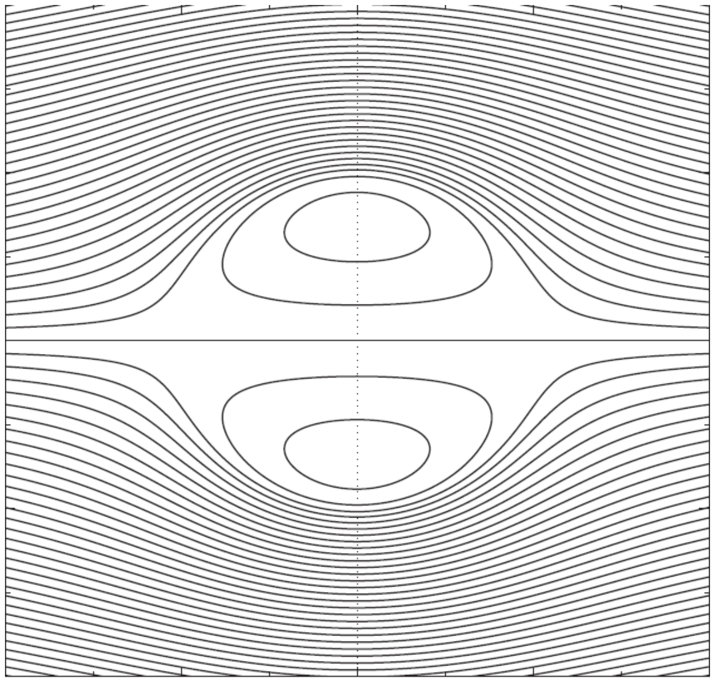
Boundary conditions (Appendix F) yield:

$$\psi(r,\theta) = \frac{1}{4} V a^2 \sin^2\theta \left[\left(\frac{\overline{\mu}}{\mu + \overline{\mu}} \right) \frac{a}{r} - \left(\frac{2\mu + 4\overline{\mu}}{\mu + \overline{\mu}} \right) \frac{r}{a} + 2 \left(\frac{r}{a} \right)^2 \right]$$

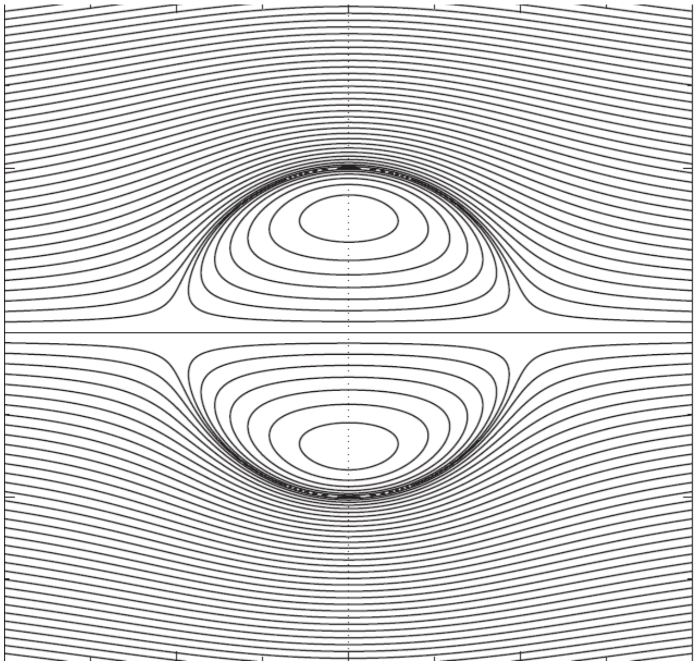
$$\psi(r,\theta) = \frac{1}{4} V a^2 \sin^2\theta \left(\frac{\mu}{\mu + \overline{\mu}} \right) \left(\frac{r}{a} \right)^2 \left[1 - \left(\frac{r}{a} \right)^2 \right]$$

(the drop radius is a)

Axisymmetric Stokes flow in and around a fluid sphere (i.e. a drop)



$\overline{\mu}/\mu = 10$



$\overline{\mu}/\mu = 1/10$

Axisymmetric Stokes flow in and around a fluid sphere

If the drop is falling under the effect of gravity the discontinuity in radial stress across the drop boundary is:

$$\sigma_{rr}(a_+,\theta) - \sigma_{rr}(a_-,\theta) = \overline{p}_0 - p_0 + \underline{(\rho - \overline{\rho})\, g\, a\, \cos\theta} - 3\,\mu\,\frac{V}{a}\left[\frac{\mu + (3/2)\,\overline{\mu}}{\mu + \overline{\mu}}\right]\cos\theta$$

i.e.

$$\overline{p}_0 - p_0 = \frac{2\,\gamma}{a}$$

γ: surface tension

$$V = \frac{a^2\, g}{3\,\nu}\left(1 - \frac{\overline{\rho}}{\rho}\right)\left[\frac{\mu + \overline{\mu}}{\mu + (3/2)\,\overline{\mu}}\right]$$

Axisymmetric Stokes flow in and around a fluid sphere

$$\boxed{V = \frac{a^2 g}{3\,\nu}\left(1 - \frac{\overline{\rho}}{\rho}\right)\left[\frac{\mu + \overline{\mu}}{\mu + (3/2)\,\overline{\mu}}\right]}$$

Limiting cases:

$\boxed{\overline{\mu} \gg \mu}$ $$V = \frac{2}{9}\frac{a^2 g}{\nu}\left(1 - \frac{\overline{\rho}}{\rho}\right)$$ i.e. the drop acts like a solid sphere

$\boxed{\overline{\mu} \ll \mu \quad \text{and} \quad \overline{\rho} \ll \rho}$ $$V = \frac{a^2 g}{3\,\nu}$$ i.e. the drop behaves like an air bubble rising through a liquid

Axisymmetric Stokes flow in and around a fluid sphere

Computation of the drag force gives:

$$\boxed{F_D = 2\,\pi\,\mu\,a\,V\left(\frac{2\,\mu + 3\,\overline{\mu}}{\mu + \overline{\mu}}\right)}$$

Limiting cases:

$\boxed{\overline{\mu} \gg \mu}$ $\quad F_D = 6\,\pi\,\mu\,a\,V$ i.e. the drop acts like a solid sphere

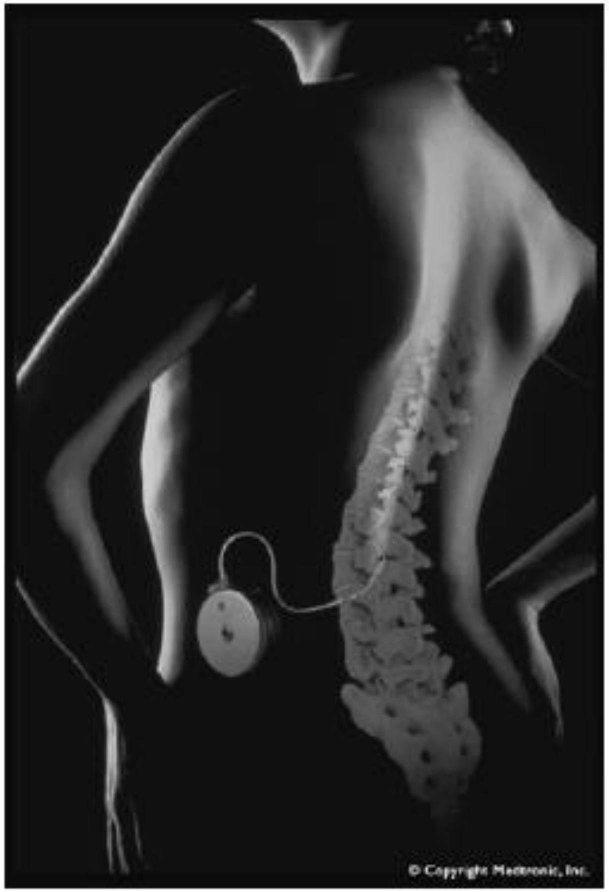
$\boxed{\overline{\mu} \ll \mu}$ $\quad F_D = 4\,\pi\,\mu\,a\,V$ i.e. the drop behaves like an air bubble rising through a liquid

Stokes flow past a cylinder: the Stokes' paradox

In theory, low Re flow around a circular cylinder can be approached in the same way as for a sphere:

So far we have just started to scratch the surface …

Figure 8 Patients that have spinal cord lesions can now be healed effectively thanks to the injection of a product into the cereberospinal fluid. The efficacy of this mode of injection is far greater than by oral means. The company Medtronic has commercialized these injection pumps, which are generally implanted below the abdomen and connected to the zone to be treated using a 500 µm diameter catheter, which the neurosurgeon must manipulate with great dexterity. There are also implanted pumps for the injection of insulin into the liver for the treatment of diabetes.



Appendix A (cf. slide 63)

In spherical coordinates :

$$\nabla \times \mathbf{q} = \mathbf{e}_r \frac{1}{r\ \sin\theta}\left[\frac{\partial}{\partial\vartheta}\left(q_\phi\ \sin\theta\right) - \frac{\partial q_\theta}{\partial\phi}\right] +$$

$$\mathbf{e}_\vartheta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial q_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r\ q_\phi\right)\right] +$$

$$\mathbf{e}_\phi \frac{1}{r}\left[\frac{\partial\left(r\ q_\vartheta\right)}{\partial r} - \frac{\partial q_r}{\partial\theta}\right]$$

from which: $\mathbf{v} = \nabla \times \left(\frac{\psi\, \mathbf{e}_\phi}{r\, \sin\vartheta}\right) = \mathbf{e}_r \left(\underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}}_{v_r}\right) + \mathbf{e}_\theta \left(\underbrace{\frac{-1}{r\ \sin\theta}\frac{\partial\psi}{\partial r}}_{v_\theta}\right)$

Appendix B (cf. slide 64)

$$\nabla \times \mathbf{v} = \nabla \times \nabla \times \left(\frac{\psi\, \mathbf{e}_\phi}{r \sin\vartheta} \right) = \nabla \times \left[\mathbf{e}_r \left(\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \right) - \mathbf{e}_\theta \left(\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \right) \right] =$$

$$\mathbf{e}_\phi \; \frac{-1}{r \sin\theta} \left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \vartheta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial r} \right) \right]$$

only component of the vorticity vector: ζ

Appendix C (cf. slide 49)

The operator is applied a first time …

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left(f(r)v_\infty\sin^2\theta\right)$$

$$=v_\infty\left[\sin^2\theta\frac{\partial^2 f(r)}{\partial r^2}+f(r)\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\sin^2\theta}{\partial\theta}\right)\right]$$

$$=v_\infty\left[\sin^2\theta\frac{\partial^2 f(r)}{\partial r^2}+f(r)\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{2\sin\theta\cos\theta}{\sin\theta}\right]$$

$$=v_\infty\left[\sin^2\theta\frac{\partial^2 f(r)}{\partial r^2}+2f(r)\frac{\sin\theta}{r^2}\frac{\partial\cos\theta}{\partial\theta}\right]$$

$$=v_\infty\left[\sin^2\theta\frac{\partial^2 f(r)}{\partial r^2}-2\frac{\sin^2\theta}{r^2}f(r)\right]$$

Appendix C (cf. slide 74)

… and then a second time …

$$\left[\frac{\partial^2}{\partial r^2}+\ \frac{\sin\theta}{r^2}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial r}\right)\right]\left[v_\infty\left(\sin^2\theta\ f^{''}-2\frac{\sin^2\theta}{r^2}\ f\right)\right]=$$

$$v_\infty\ \sin^2\theta\left[\ f^{''''}-2\frac{\partial}{\partial r}\left(-2\frac{f}{r^3}+\frac{f^{'}}{r^2}\right)\right]+v_\infty\ \ \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left[2\ \cos\theta\ \ f^{''}-4\ \cos\theta\ \ \frac{f}{r^2}\right]=$$

$$v_\infty\ \sin^2\theta\left[\ f^{''''}-4\frac{f^{''}}{r^2}+8\frac{f^{'}}{r^3}-8\frac{f}{r^4}\right]$$

Appendix D (cf. slide 52)

The details of the solution of the *equidimensional equation* are:

$$r^4 \frac{d^4 ar^n}{dr^4} - 4r^2 \frac{d^2 ar^n}{dr^2} + 8r \frac{dar^n}{dr} - 8ar^n = 0$$

$$r^4 (n)(n-1)(n-2)(n-3)ar^{n-4} - 4r^2 (n)(n-1)ar^{n-2} + 8r(n)ar^{n-1} - 8ar^n$$

divide by ar^n

$$(n)(n-1)(n-2)(n-3) - 4(n)(n-1) + 8n - 8 = 0$$

This is a 4^{th} order polynomial which can also be written as :

$$[n\ (n - 1) - 2][(n-2)(n-3) - 2] = 0$$

and there are 4 possible values for n which turn out to be

-1, 1, 2 and 4.

Appendix D (cf. slide 75)

Thus:

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$ which means that

$$\psi(r,\theta) = v_\infty \sin^2\theta \left[\frac{A}{r} + Br + Cr^2 + Dr^4 \right];$$

the boundary conditions state that

$$f(R) = 0, \quad f(\infty) = \frac{1}{2} r^2$$ (cf. slide 72)

from which:

$$A = \frac{1}{4} R^3, \quad B = -\frac{3}{4} R, \; C = \frac{1}{2}, \quad D = 0$$

Appendix E (cf. slide 96)

<u>How much does a spherical particle *coast* after removing thrust</u>?

From Stokes flow solution, the force on the particle $F = 6\,\pi\,\mu\,V\,a$ tends to decelerate it, *i.e.* $M\,\mathrm{d}V/\mathrm{d}t = 6\,\pi\,\mu\,V\,R$, with M and ρ_p the particle's mass and density, yielding:

$$\mathrm{d}V/V = \frac{9}{2}\frac{\mu}{\rho_p R^2}\,\mathrm{d}t.$$

This leads to an exponential solution: $V(t) = V_0 \exp(t\,/\,\tau_p)$ with

$$\boxed{\tau_p = \frac{2}{9}\frac{\rho_p R^2}{\mu}}$$

Appendix E (cf. slide 96)

A micro-organism (length scale $R \approx 1\ \mu m$) moving in water at a characteristic speed of 30 $\mu m/s$ will coast for a time equal to

$$\boxed{\tau_p = 0.2\ \mu s}$$

over a coasting distance $V_0 \tau_p$ equal to

$$\boxed{0.07\ \mathring{A}}$$

Purcell (1977) states that "if you are at very low Reynolds number, what you are doing at the moment is entirely determined by the forces that are exerted on you *at the moment*, and by nothing in the past."

In a footnote he adds that "in that world, Aristotle's mechanics is *correct*!"

Appendix F (cf. slide 104)

Boundary conditions for the "drop" case:

1. Inside the bubble we have

- $\bar{A} = \bar{B} = 0$ so that v_r and v_θ are finite at $r = 0$
- at $r = a$ the radial velocity is zero, so that $\bar{C} + \bar{D}a^2 = 0$

2. Outside the bubble we have:

- $D = 0$, $C = V/2$ (see previous problem)
- at $r = a$ the radial velocity vanishes, so that $A/a^3 + B/a = -V/2$

Appendix F (cf. slide 104)

3. <u>At the interface</u>:

- The circumferential velocities inside and outside the sphere must be the same, $v_{\theta 1} = v_{\theta 2}$ at $r = a$, leading to:

$$2\,\bar{C} = \frac{A}{a^3} - \frac{B}{a} - V$$

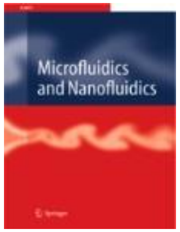
- The circumferential shear stresses at the interface inside and outside must be equal in magnitude and in opposite directions. Since v_r = 0 on the interface, this condition is $\mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)_{outside} = -\,\overline{\mu}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)_{inside}$ leading to

$$6\,\overline{C}\,\overline{\mu} = \mu\,(4\,A/a^3 - 2\,B/a - V)$$

Further readings

Beyond the textbooks shown on slide 11, and the few references given throughout the slides, other material of interest include:

- E. Lauga & T.R. Powers, The hydrodynamics of swimming microorganisms, *Rep. Prog. Phys*. **72** (2009) 096601
- http://www.math.nyu.edu/faculty/childres/chpseven.PDF
- http://www.mit.edu/~zulissi/courses/slow_viscous_flows.pdf
- P. Tabeling, *Introduction to Microfluidics*, Oxford U. Press (2005)
- Dongquing Li, Encyclopedia of Microfluidics and Nanofluidics, Springer (2008)
- Micro and NanoFluidics, Springer
- Lab on a Chip, Royal Soc. of Chemistry



Further documentation

Absolutely “can’t-miss”:

National Committee for Fluid Fluid Mechanics Films

http://web.mit.edu/hml/ncfmf.html