A crash course in fluid mechanics

La Spezia, 27th February, 2015

University of Genoa, DICCA

Dipartimento di **I**ngegneria **C**ivile, **C**himica e **A**mbientale

wolf dynamics

Spin off
Università degli Studi di Genova

Your Lecturer

Alessandro Bottaro

http://www.dicca.unige.it/bottaro

alessandro.bottaro@unige.it

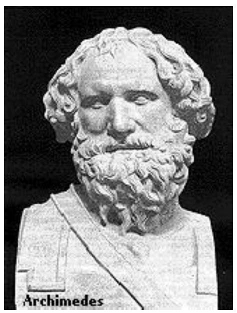
UNIVERSITÀ DEGLI STUDI
DI GENOVA

bottaro@wolfdynamics.com

wolf dynamics

Introduction
Fluid Mechanics

Faces of Fluid Mechanics : some of the greatest minds of history have tried to solve the mysteries of fluid mechanics

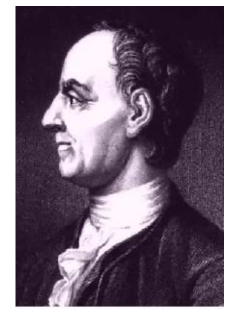


Archimedes


Da Vinci


Newton

Leibniz

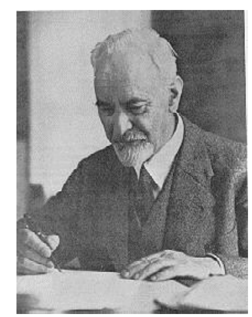

Euler


Bernoulli


Navier


Stokes


Reynolds


Prandtl

Introduction
Fluid Mechanics

- From mid-1800’s to 1960’s, research in fluid mechanics focused upon
  - Analytical methods
    - Exact solution to Navier-Stokes equations (~ 80 known for simple problems, e.g., laminar pipe flow)
    - Approximate methods, e.g., Ideal flow, Boundary layer theory
  - Experimental methods
    - Scale models: wind tunnels, water tunnels, towing-tanks, flumes,...
    - Measurement techniques: pitot probes; hot-wire probes; anemometers; laser-doppler velocimetry; particle-image velocimetry
    - Most man-made systems (e.g., airplane) engineered using build-and-test iteration.
- 1950’s – present : rise of computational fluid dynamics (CFD)

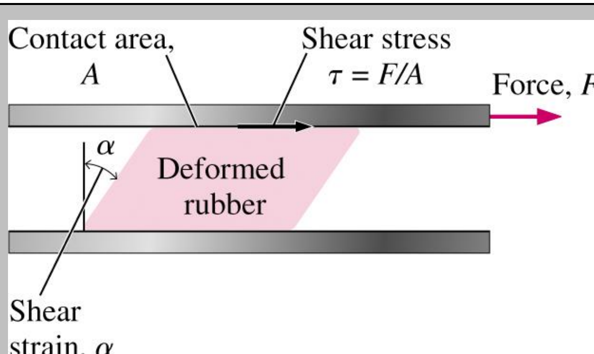
Basic concepts

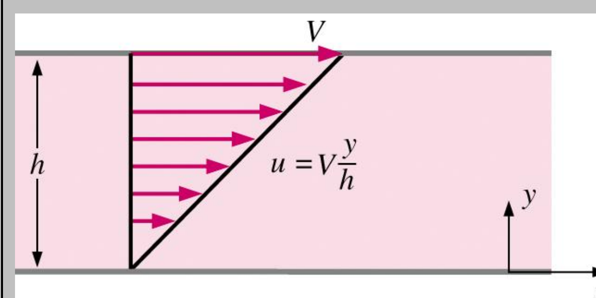
What is a fluid?

- A fluid is a substance in the gaseous or liquid form
- Distinction between solid and fluid?
  - Solid: can resist an applied shear by deforming. Stress is proportional to strain
  - Fluid: deforms continuously under applied shear. Stress is proportional to strain rate

Solid

$$\tau = \frac{F}{A} \propto \alpha$$

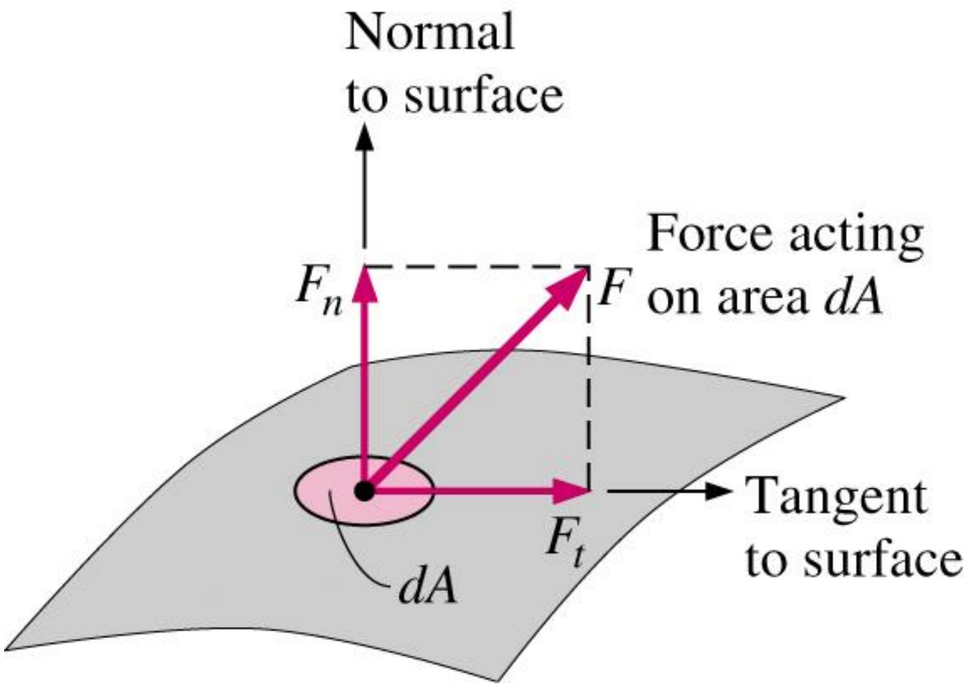




Fluid

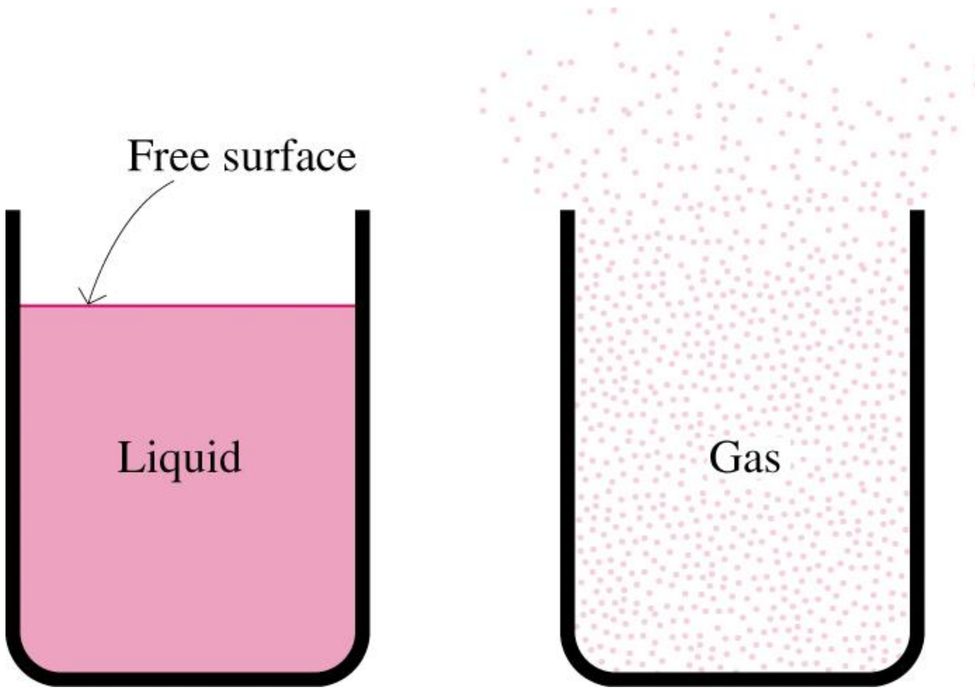
$$\tau = \frac{F}{A} \propto \mu \frac{V}{h}$$

What is a fluid?



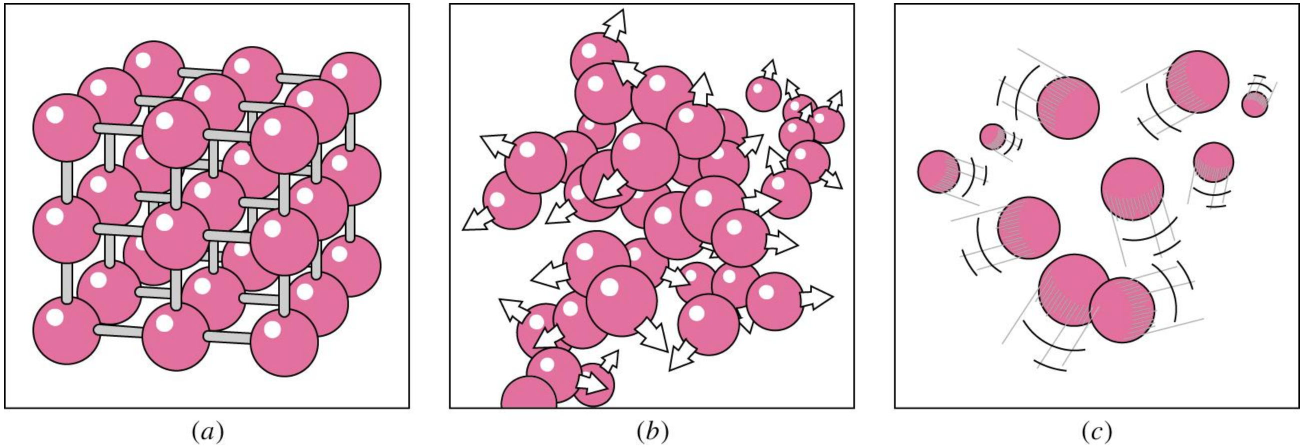
- Stress is defined as the force per unit area.
- Normal component: normal stress
  - In a fluid at rest, the normal stress is called **pressure**
- Tangential component: shear stress

What is a fluid?



- A liquid takes the shape of the container it is in and forms a free surface in the presence of gravity
- A gas expands until it encounters the walls of the container and fills the entire available space. Gases cannot form a free surface
- Gas and vapor are often used as synonymous words

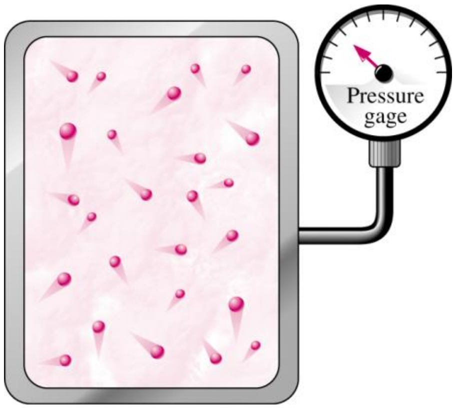
What is a fluid?



solid   liquid   gas

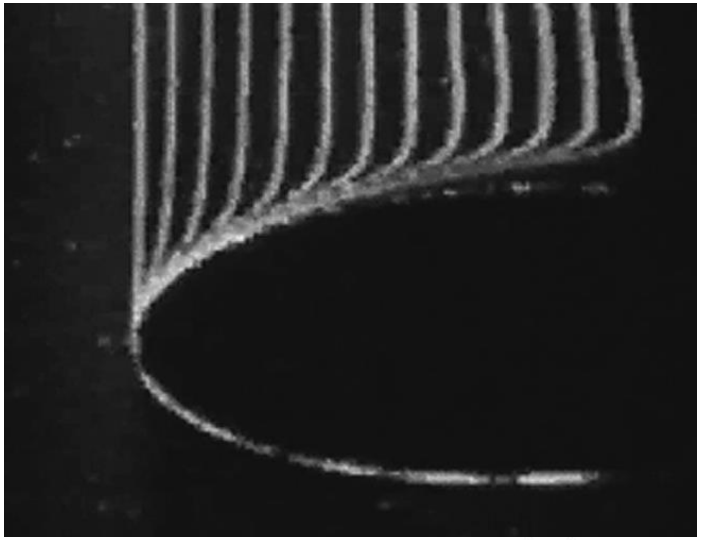
strong → *weak*

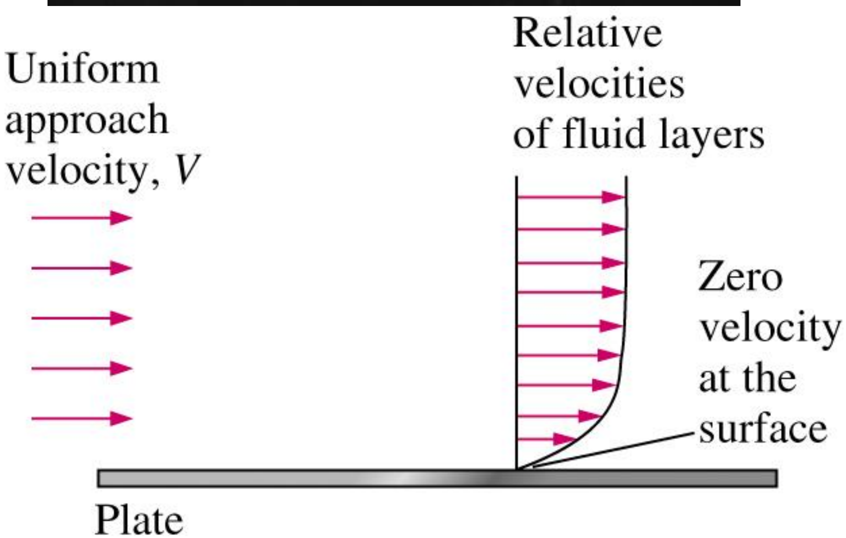
intermolecular bonds



Pressure can be measured on a macroscopic scale …

No-slip condition





- No-slip condition: A fluid in direct contact with a solid ``sticks'' to the surface due to viscous effects
- Responsible for generation of wall shear stress τ_w, surface drag $D = \int \tau_w \, dA$, and the development of the boundary layer
- The fluid property responsible for the no-slip condition is **viscosity**
- Important boundary condition in formulating initial boundary value problem (IBVP) for analytical and computational fluid dynamics analysis

Classification of Flows

- We classify flows as a tool in making simplifying assumptions to the governing partial-differential equations, which are known as the Navier-Stokes equations (for Newtonian fluids)
  - Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$$

  - Conservation of Momentum

$$\rho \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\, \mathbf{U} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{U}$$

Classification of Flows

- We classify flows as a tool in making simplifying assumptions to the governing partial-differential equations, which are known as the Navier-Stokes equations (for Newtonian fluids)
  - Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{U}) = 0$$

  - Conservation of Momentum

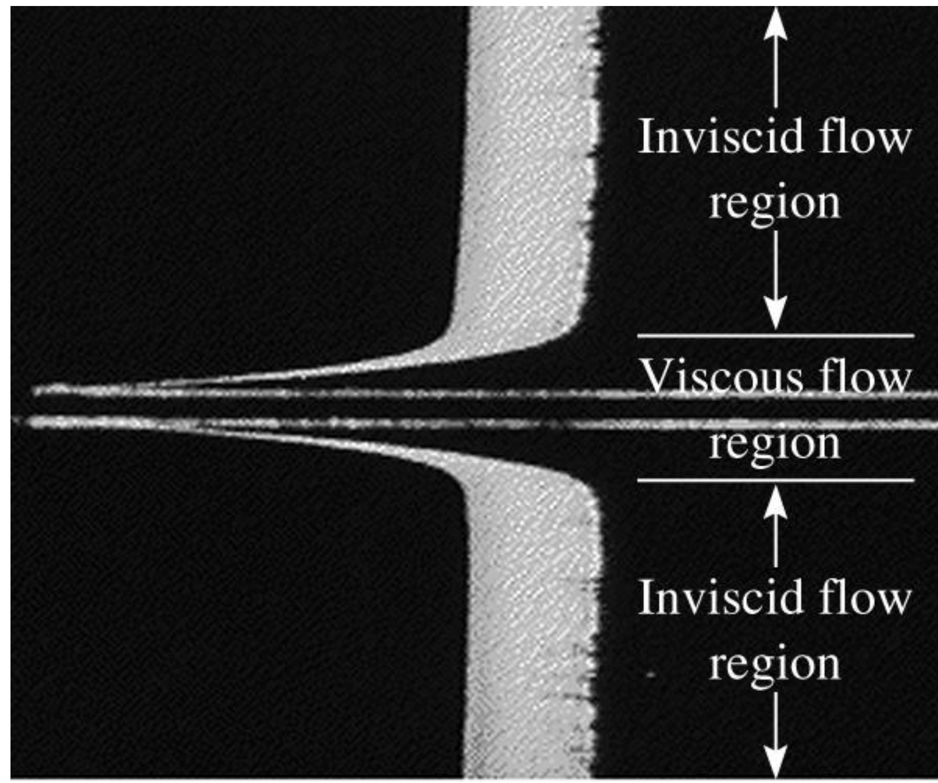
$$\rho \left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right] = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{U}$$

Viscous vs. Inviscid Regions of Flow

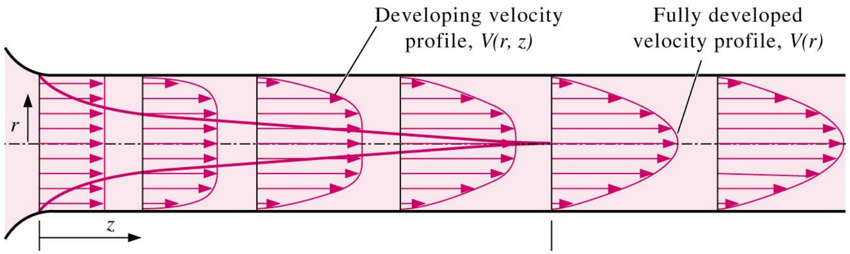
- Regions where frictional effects are significant are called viscous regions. They are usually close to solid surfaces.
- Regions where frictional forces are small compared to inertial or pressure forces are called inviscid

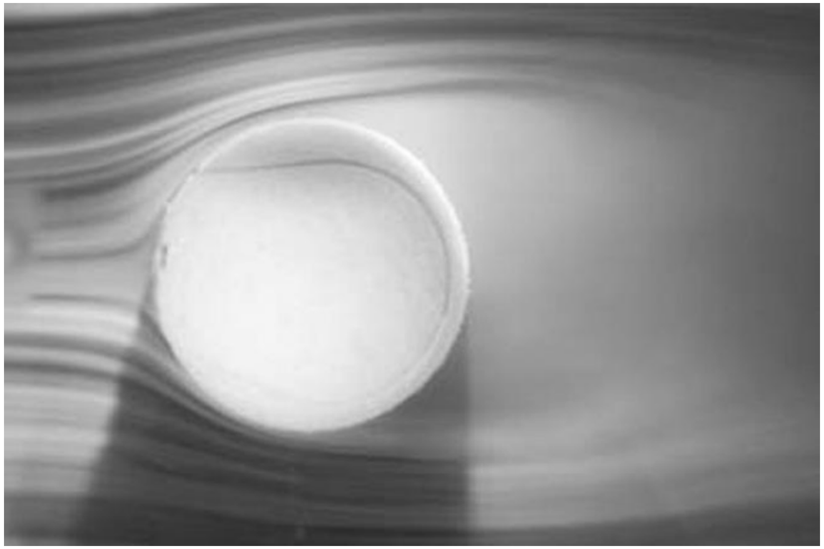
For inviscid flows:

$$\rho\left[\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\,\mathbf{U}\right] = -\nabla p + \rho\mathbf{g} + \cancelto{0}{\mu\nabla^2\mathbf{U}}$$



Internal vs. External Flow





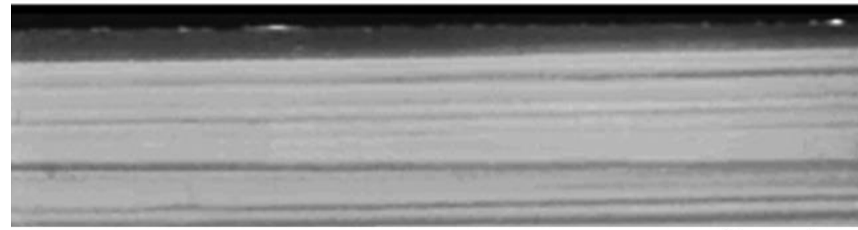
- Internal flows are dominated by the influence of viscosity throughout the flowfield
- For external flows, viscous effects are limited to the boundary layer and wake.

Compressible vs. Incompressible Flow

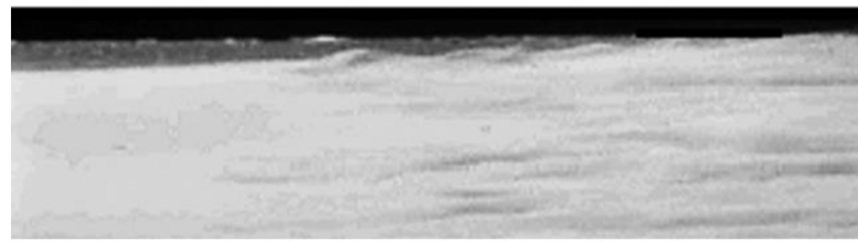
- A flow is classified as incompressible if the density remains nearly constant.
- Liquid flows are typically incompressible.
- Gas flows are often compressible, especially for high speeds.
- Mach number, **Ma = V/c** is a good indicator of whether or not compressibility effects are important.
  - Ma < 0.3 : Incompressible
  - Ma < 1 : Subsonic
  - Ma = 1 : Sonic
  - Ma > 1 : Supersonic
  - Ma >> 1 : Hypersonic

Laminar vs. Turbulent Flow

- Laminar: highly ordered fluid motion with smooth streamlines.
- Turbulent: highly disordered fluid motion characterized by velocity fluctuations and eddies.
- Transitional: a flow that contains both laminar and turbulent regions
- The Reynolds number, **$Re = \rho UL/\mu$** is the key parameter in determining whether or not a flow is laminar or turbulent.



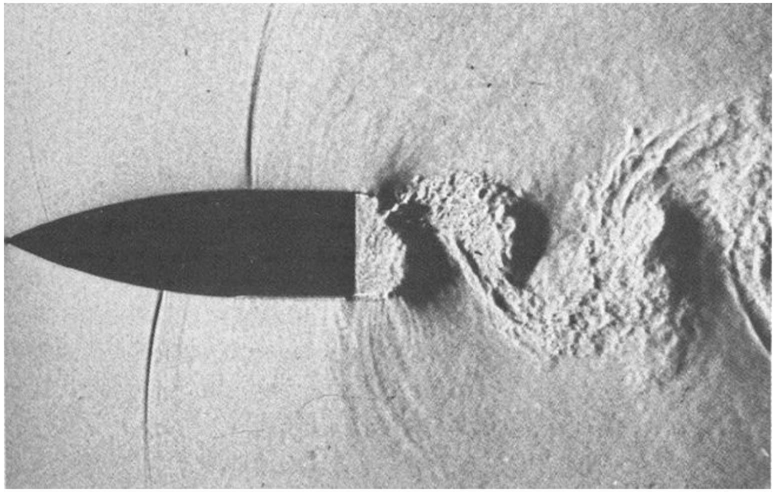
Laminar

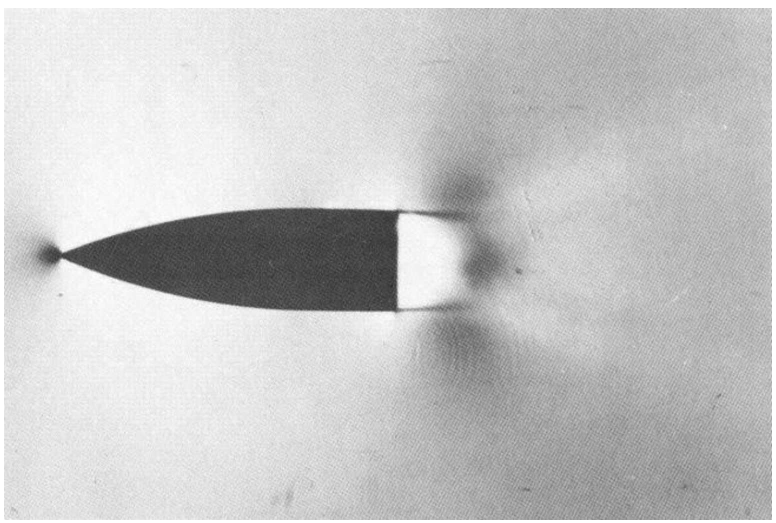


Transitional

Turbulent

Steady vs. Unsteady Flow

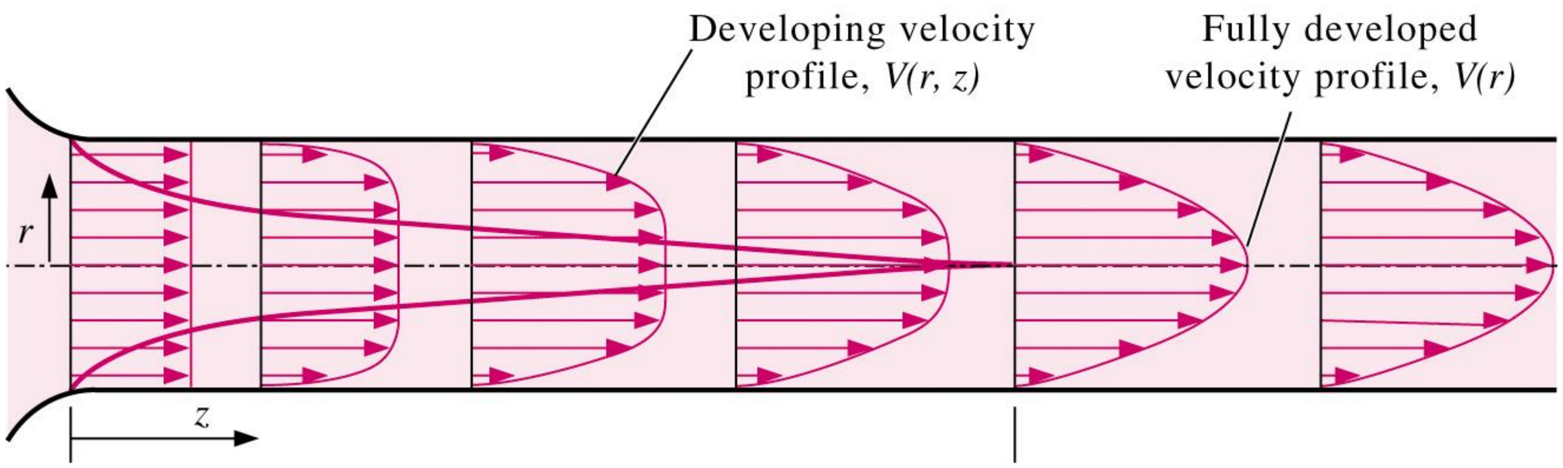




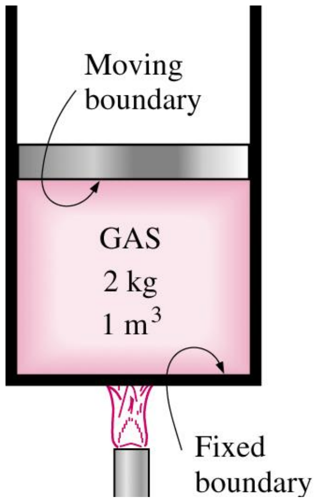
- Steady implies no change at a point with time. Transient terms in N-S equations are zero
- Unsteady is the opposite of steady. $\frac{\partial \mathbf{U}}{\partial t} = \frac{\partial \rho}{\partial t} = 0$
  - Transient usually describes a starting, or developing flow.
  - Periodic refers to a flow which oscillates about a mean.
- Unsteady flows may appear steady if "time-averaged"

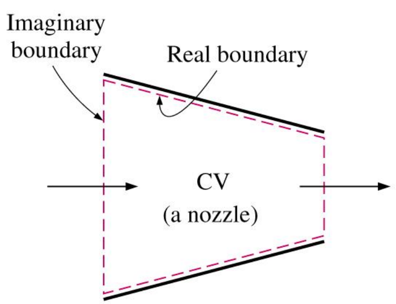
One-, Two-, and Three-Dimensional Flows

- N-S equations are 3D vector equations.
- Velocity vector, $\boldsymbol{U}(x,y,z,t) = [U_x(x,y,z,t), U_y(x,y,z,t), U_z(x,y,z,t)]$
- Lower dimensional flows reduce complexity of analytical and computational solution
- Change in coordinate system (cylindrical, spherical, etc.) may facilitate reduction in order.
- Example: for fully-developed pipe flow, velocity *V(r)* is a function of radius *r* and pressure *p(z)* is a function of distance *z* along the pipe.

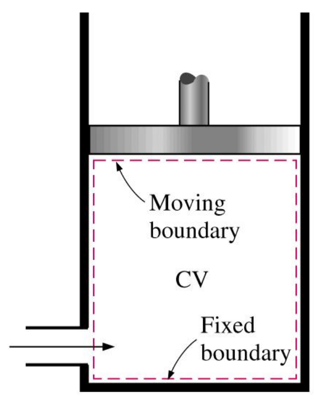


System and Control Volume





(*a*) A control volume (CV) with real and imaginary boundaries



(*b*) A control volume (CV) with fixed and moving boundaries

- A **system** is defined as a quantity of matter or a region in space chosen for study.
- A closed system consists of a fixed amount of mass.
- An open system, or **control volume**, is a properly selected region in space.

Dimensions and Units

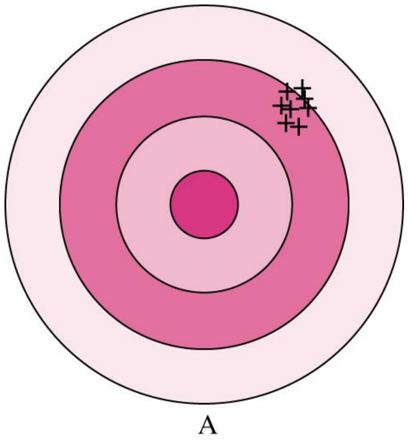
- Any physical quantity can be characterized by **dimensions**.
- The magnitudes assigned to dimensions are called **units**.
- Primary dimensions include: mass *m*, length *L*, time *t*, and temperature *T*.
- Secondary dimensions can be expressed in terms of primary dimensions and include: velocity *V*, energy *E*, and volume *V*.
- Unit systems include English system and the metric SI (International System). We'll use only the SI system.
- **Dimensional homogeneity** is a valuable tool in checking for errors. Make sure every term in an equation has the same units.

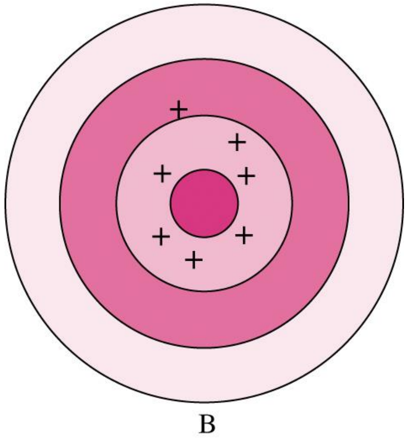
Accuracy, Precision, and Significant Digits

Engineers must be aware of three principals that govern the proper use of numbers.

1. ***Accuracy error :*** Value of one reading minus the true value. Closeness of the average reading to the true value. Generally associated with repeatable, fixed errors.
2. ***Precision error :*** Value of one reading minus the average of readings. Is a measure of the fineness of resolution and repeatability of the instrument. Generally associated with random errors.
3. ***Significant digits :*** Digits that are relevant and meaningful. When performing calculations, the final result is only as precise as the least precise parameter in the problem. When the number of significant digits is unknown, the accepted standard is 3. Use 3 in all homework and exams.

Example of Accuracy and Precision



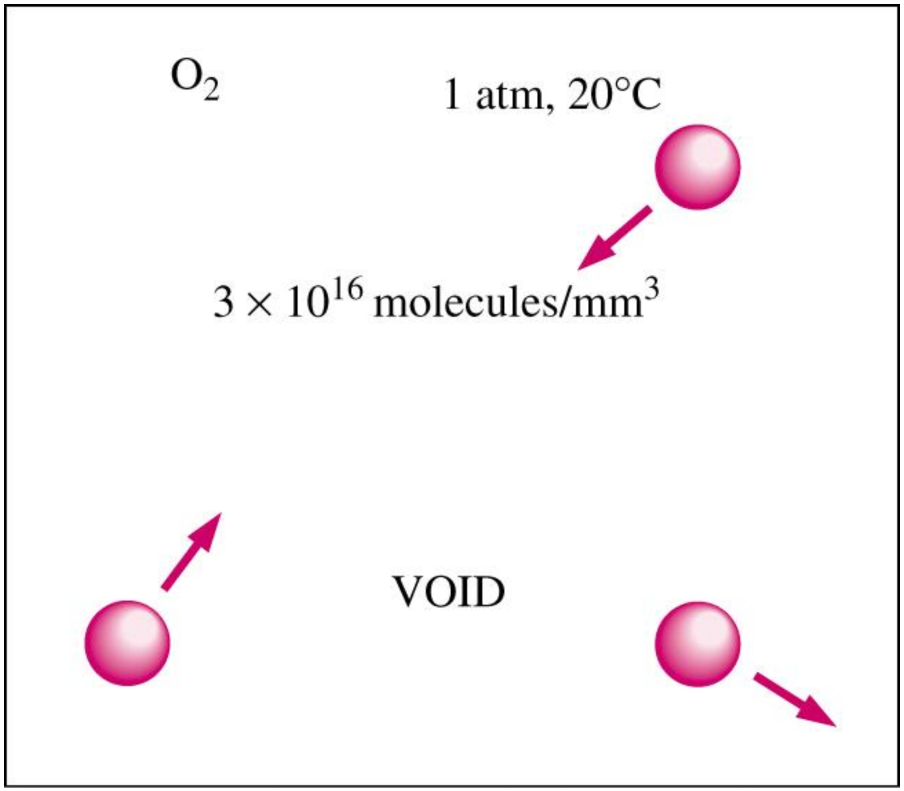


Shooter A is more precise but less accurate, while shooter B is more accurate, but less precise.

Physical characteristics

- Any characteristic of a system is called a **property.**
  - Familiar: pressure P, temperature T, volume V, and mass m.
  - Less familiar: viscosity, thermal conductivity, modulus of elasticity, thermal expansion coefficient, vapor pressure, surface tension.
- *Intensive* properties are independent of the mass of the system. Examples: temperature, pressure, and density.
- *Extensive* properties are those whose value depends on the size of the system. Examples: Total mass, total volume, and total momentum.
- Extensive properties per unit mass are called **specific properties**. Examples include specific volume $v = V/m$ and specific total energy $e=E/m$.

Continuum



- Atoms are widely spaced in the gas phase.
- However, we can disregard the atomic nature of a substance.
- View it as a continuous, homogeneous matter with no holes, that is, a **continuum**.
- This allows us to treat properties as smoothly varying quantities.
- Continuum is valid as long as size of the system is large in comparison to distance between molecules.

Density and Specific Gravity

- Density is defined as the *mass per unit volume* $\rho = m/V$. Density has units of kg/m^3
- Specific volume is defined as $v = 1/\rho = V/m$.
- For a gas, density depends on temperature and pressure.
- **Specific gravity**, or relative density is defined as *the ratio of the density of a substance to the density of some standard substance at a specified temperature* (usually water at 4°C), i.e., $SG=\rho/\rho_{H_20}$. SG is a dimensionless quantity.
- The **specific weight** is defined as the weight per unit volume, i.e., $\gamma_s = \rho g$ where g is the gravitational acceleration. γ_s has units of N/m^3.

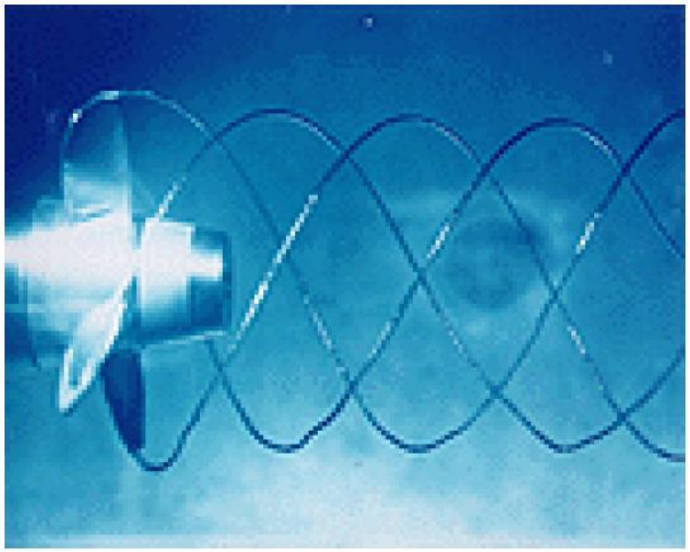
Density of Ideal Gases

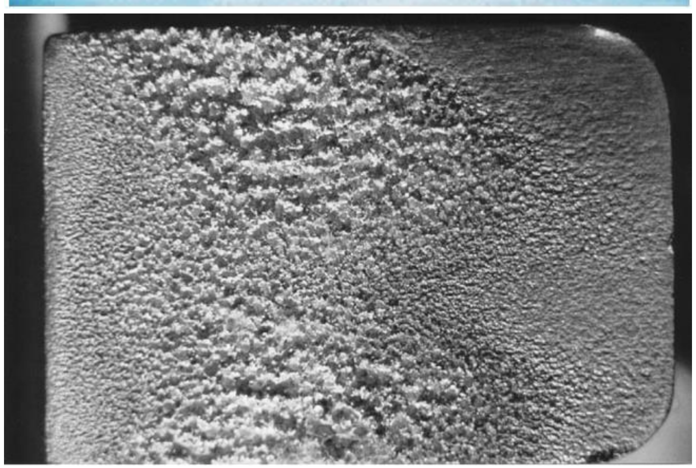
- **Equation of State**: equation for the relationship between pressure, temperature, and density.
- The simplest and best-known equation of state is the ideal-gas equation.

$$P\,v = R\,T \qquad \text{or} \qquad P = \rho\,R\,T$$

- Ideal-gas equation holds for most gases.
- However, dense gases such as water vapor and refrigerant vapor should not be treated as ideal gases.

Vapor Pressure and Cavitation





- **Vapor Pressure** P_v of a pure substance is defined as *the pressure exerted by its vapor in phase equilibrium with its liquid at a given temperature*
- If P drops below P_v, liquid is locally vaporized, creating cavities of vapor.
- Vapor cavities collapse when local P rises above P_v.
- Collapse of cavities is a violent process which can damage machinery.
- Cavitation is noisy, and can cause structural vibrations.

Energy and Specific Heats

- Total energy E is comprised of numerous forms: thermal, mechanical, kinetic, potential, electrical, magnetic, chemical, and nuclear.
- Units of energy are *joule* (J) or *British thermal unit* (BTU).
- Microscopic energy
  - Internal energy u is for a non-flowing fluid and is due to molecular activity.
  - Enthalpy $h=u+Pv$ is for a flowing fluid and includes flow energy (Pv).
- Macroscopic energy
  - Kinetic energy $ke=V^2/2$
  - Potential energy $pe=gz$
- In the absence of electrical, magnetic, chemical, and nuclear energy, the total energy is $e_{flowing}=h+V^2/2+gz$.

Coefficient of Compressibility

- How does fluid volume change with *P* and *T*?
- Fluids expand as *T* ↑ or *P* ↓ ; fluids contract as *T* ↓ or *P* ↑
- Need fluid properties that relate volume changes to changes in *P* and *T*.
  - Coefficient of compressibility or bulk modulus of elasticity

$$\kappa = -v\left(\frac{\partial P}{\partial v}\right)_T = \rho\left(\frac{\partial P}{\partial \rho}\right)_T \qquad \kappa_{ideal\ gas} = P$$

$\alpha = 1/\kappa$ = coefficient of isothermal compressibility

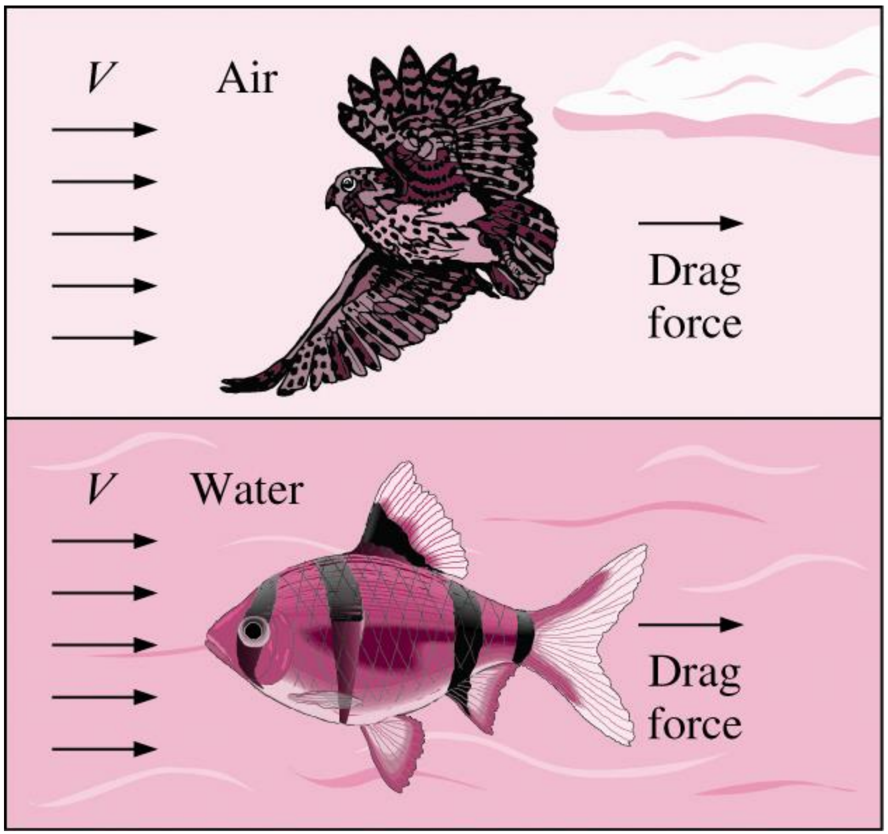
  - Coefficient of volume expansion

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P$$

- Combined effects of *P* and *T* can be written as

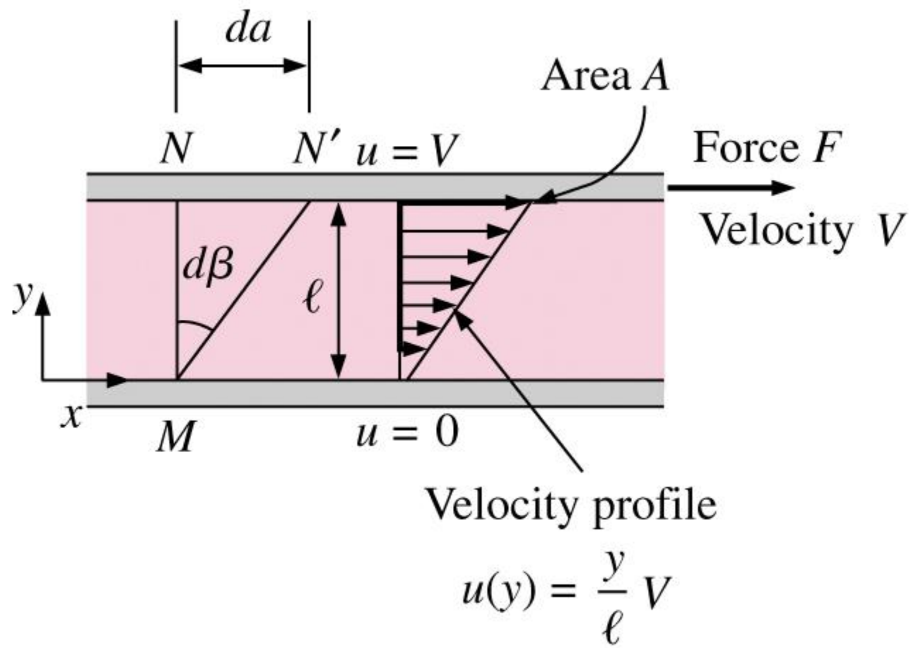
$$dv = \left(\frac{\partial v}{\partial T}\right)_P dT + \left(\frac{\partial v}{\partial P}\right)_T dP$$

Viscosity



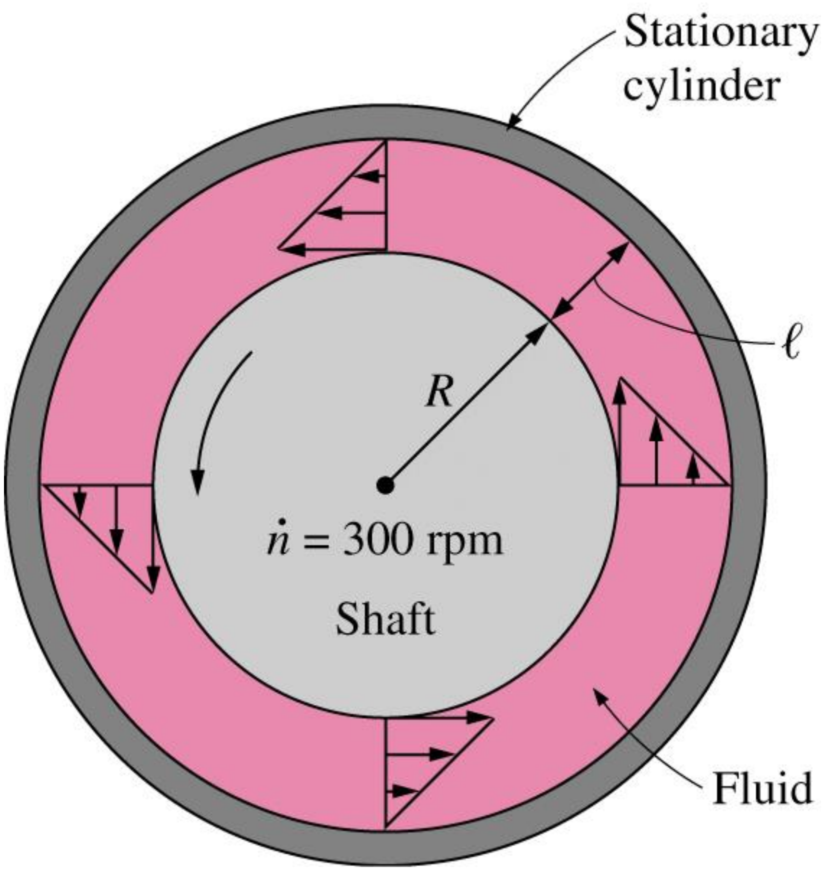
- **Viscosity** is a property that represents the internal resistance of a fluid to motion.
- The force a flowing fluid exerts on a body in the flow direction is called the **drag force**, and the magnitude of this force depends, in part, on viscosity.

Viscosity



- To obtain a relation for viscosity, consider a fluid layer between two very large parallel plates separated by a distance ℓ
- Definition of shear stress is $\tau = F/A$.
- Using the no-slip condition, $u(0) = 0$ and $u(\ell) = V$, the velocity profile and gradient are $u(y)= Vy/\ell$ and $du/dy=V/\ell$
- Shear stress for Newtonian fluid: $\tau = \mu \, du/dy$
- μ is the **dynamic viscosity** and has units of *kg/m·s*, *Pa·s*, or **poise** (1 poise = 0.1 Pa S).
- The viscosity of water at 20°C is 1 cP (cP = centiPoise)
- The **kinematic viscosity** of water at 20°C is 1 cSt (cSt = centiStoke) (1 St = 1 cm^2/s)

Viscometry

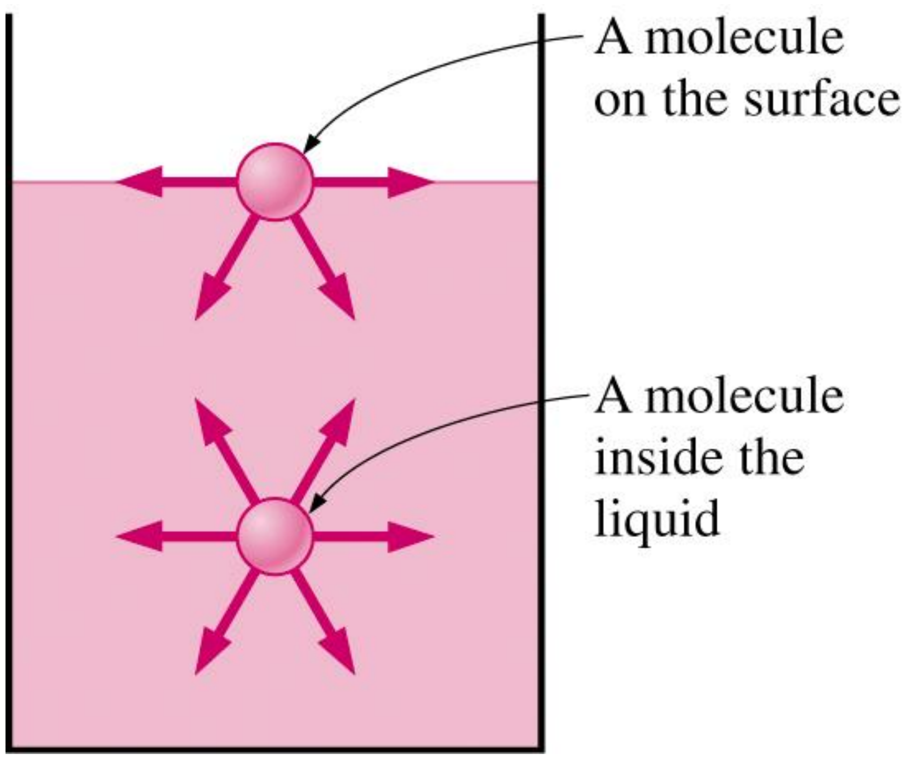


- How is viscosity measured? A rotating viscometer.
  - Two concentric cylinders with a fluid in the small gap ℓ.
  - Inner cylinder is rotating, outer one is fixed.
- Use definition of shear force:

$$F = \tau A = \mu A \frac{du}{dy}$$

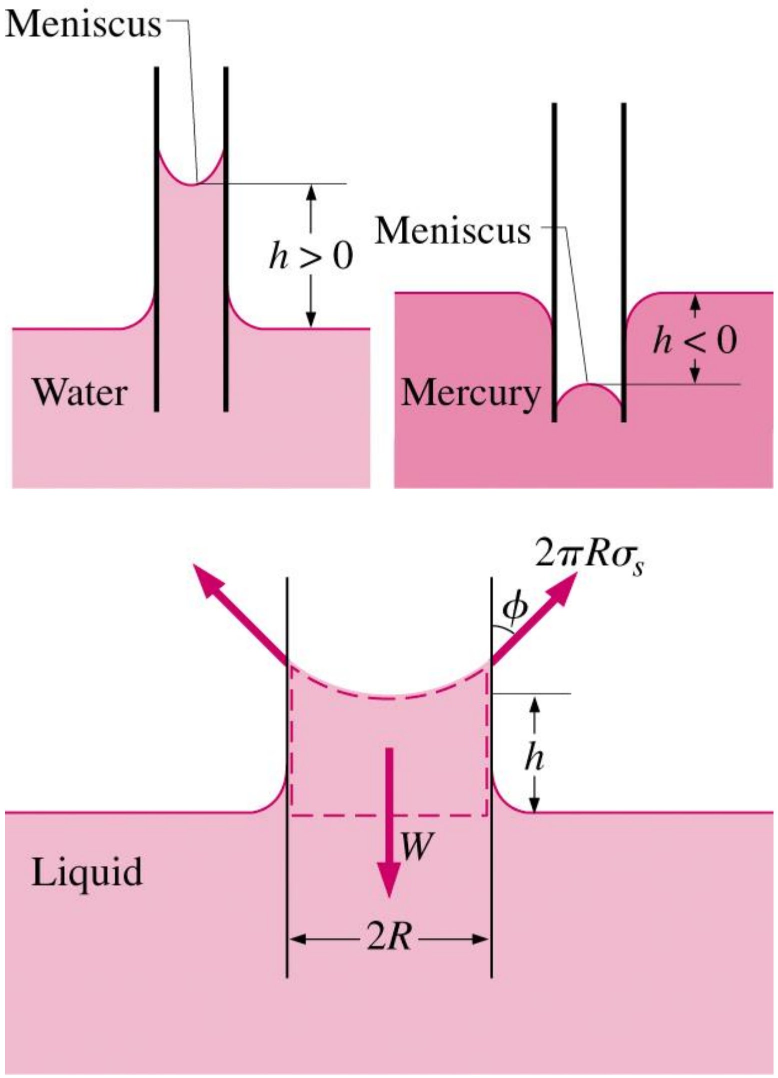
- If ℓ/R << 1, then cylinders can be modeled as flat plates.
- Torque $T = FR$, and tangential velocity $V=\omega R$
- Wetted surface area $A=2\pi RL$.
- Measure T and ω to compute μ

Surface Tension



- Liquid droplets behave like small spherical balloons filled with liquid, and the surface of the liquid acts like a stretched elastic membrane under tension.
- The pulling force that causes this is
  - due to the attractive forces between molecules
  - called **surface tension** σ_s.
- Attractive force on surface molecule is not symmetric.
- Repulsive forces from interior molecules causes the liquid to minimize its surface area and attain a spherical shape.

Capillary Effect



- **Capillary effect** is the rise or fall of a liquid in a small-diameter tube.
- The curved free surface in the tube is call the **meniscus**.
- Water meniscus curves up because water is a *wetting fluid.*
- Mercury meniscus curves down because mercury is a *nonwetting fluid*.
- Force balance can describe magnitude of capillary rise.

Fluids Kinematics

Overview

- Fluid Kinematics deals with the motion of fluids without considering the forces and moments which create the motion.
- Items discussed here:
  - Material derivative and its relationship to Lagrangian and Eulerian descriptions of fluid flow.
  - Flow visualization.
  - Plotting flow data.
  - Fundamental kinematic properties of fluid motion and deformation.
  - Reynolds Transport Theorem

Lagrangian Description

- Lagrangian description of fluid flow tracks the position and velocity of individual particles.
- Based upon Newton's laws of motion.
- Difficult to use for practical flow analysis.
  - Fluids are composed of *billions* of molecules.
  - Interaction between molecules hard to describe/model.
- However, useful for specialized applications
  - Sprays, particles, bubble dynamics, rarefied gases.
  - Coupled Eulerian-Lagrangian methods.
- Named after Italian mathematician Joseph Louis Lagrange (1736-1813).

Eulerian Description

- Eulerian description of fluid flow: a **flow domain** or **control volume** is defined by which fluid flows in and out.
- We define **field variables** which are functions of space and time.
  - Pressure field, P=P(x,y,z,t)
  - Velocity field, $\vec{V} = \vec{V}(x, y, z, t)$

$$\vec{V} = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k}$$

  - Acceleration field, $\vec{a} = \vec{a}(x, y, z, t)$

$$\vec{a} = a_x(x, y, z, t)\vec{i} + a_y(x, y, z, t)\vec{j} + a_z(x, y, z, t)\vec{k}$$

  - These (and other) field variables define the **flow field**.
- Well suited for formulation of initial boundary-value problems (PDE's).
- Named after Swiss mathematician Leonhard Euler (1707-1783).

Acceleration Field

- Consider a fluid particle and Newton's second law,

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

- The acceleration of the particle is the time derivative of the particle's velocity.

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

- However, particle velocity at a point is the same as the fluid velocity, $\vec{V}_{particle} = \vec{V}\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$
- To take the time derivative of, chain rule must be used.

$$\vec{a}_{particle} = \frac{\partial\vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial\vec{V}}{\partial x}\frac{dx_{particle}}{dt} + \frac{\partial\vec{V}}{\partial y}\frac{dy_{particle}}{dt} + \frac{\partial\vec{V}}{\partial z}\frac{dz_{particle}}{dt}$$

Acceleration Field

- Since $\frac{dx_{particle}}{dt} = u, \frac{dy_{particle}}{dt} = v, \frac{dz_{particle}}{dt} = w$

$$\vec{a}_{particle} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- In vector form, the acceleration can be written as

$$\vec{a}(x, y, z, t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

- First term is called the **local acceleration** and is nonzero only for unsteady flows.
- Second term is called the **advective acceleration** and accounts for the effect of the fluid particle moving to a new location in the flow, where the velocity is different.

Material Derivative

- The total derivative operator d/dt is call the **material derivative** and is often given special notation, D/Dt.

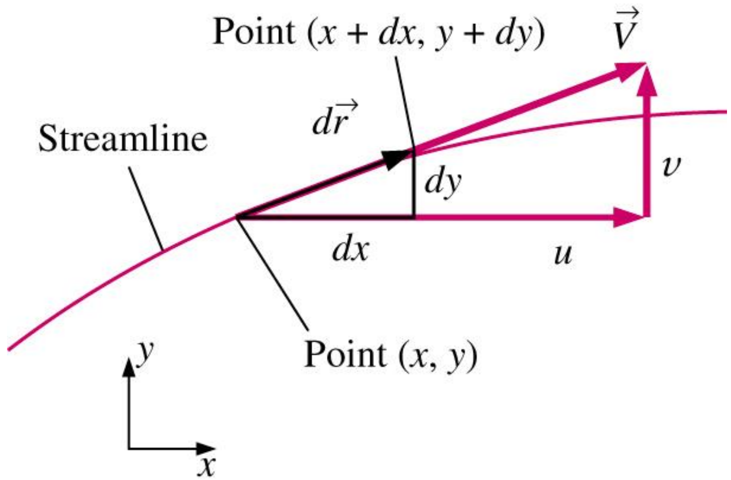
$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right)\vec{V}$$

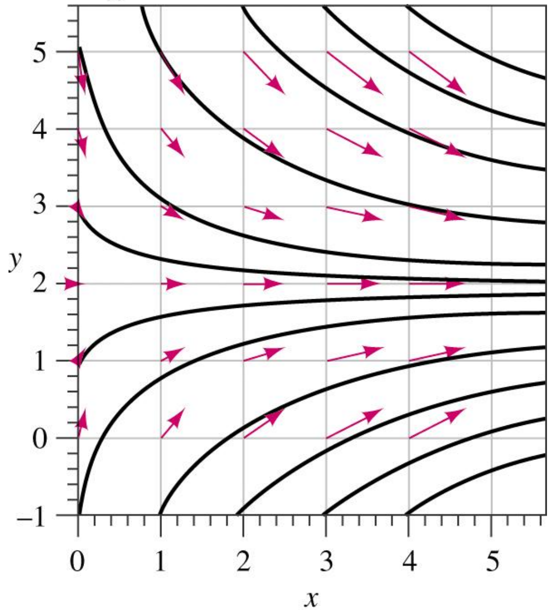
- Advective acceleration is nonlinear: source of many phenomenon and primary challenge in solving fluid flow problems.
- Provides ``transformation'' between Lagrangian and Eulerian frames.
- Other names for the material derivative include: **total, particle, Lagrangian, Eulerian,** and **substantial** derivative.

Flow Visualization

- Flow visualization is the visual examination of flow-field features.
- Important for both physical experiments and numerical (CFD) solutions.
- Numerous methods
  - Streamlines and streamtubes
  - Pathlines
  - Streaklines
  - Timelines
  - Refractive techniques
  - Surface flow techniques

Streamlines





- A **Streamline** is a curve that is everywhere tangent to the *instantaneous* local velocity vector.
- Consider an arc length

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

- $d\vec{r}$ must be parallel to the local velocity vector

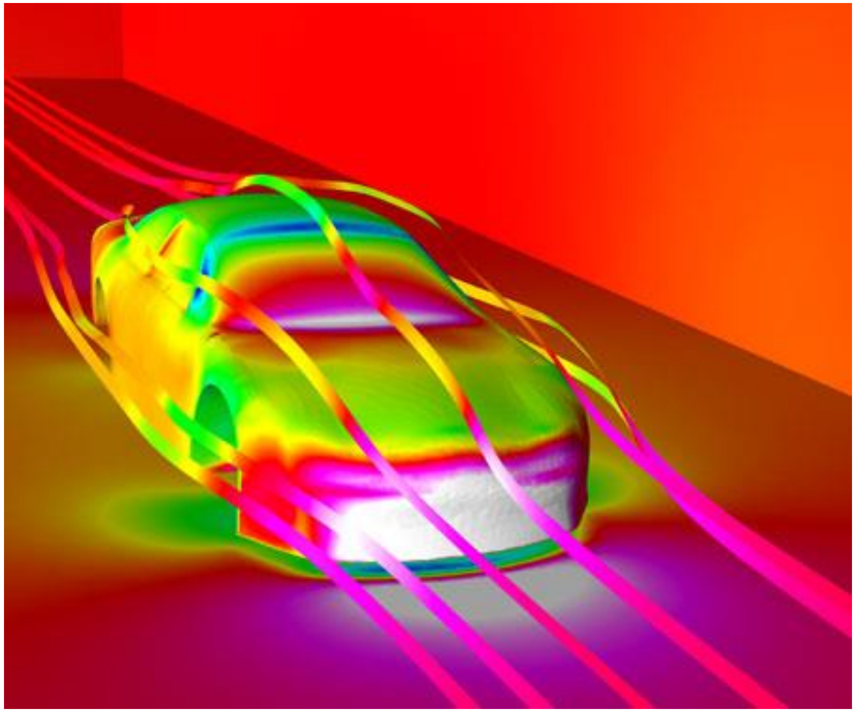
$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- Geometric arguments results in the equation for a streamline

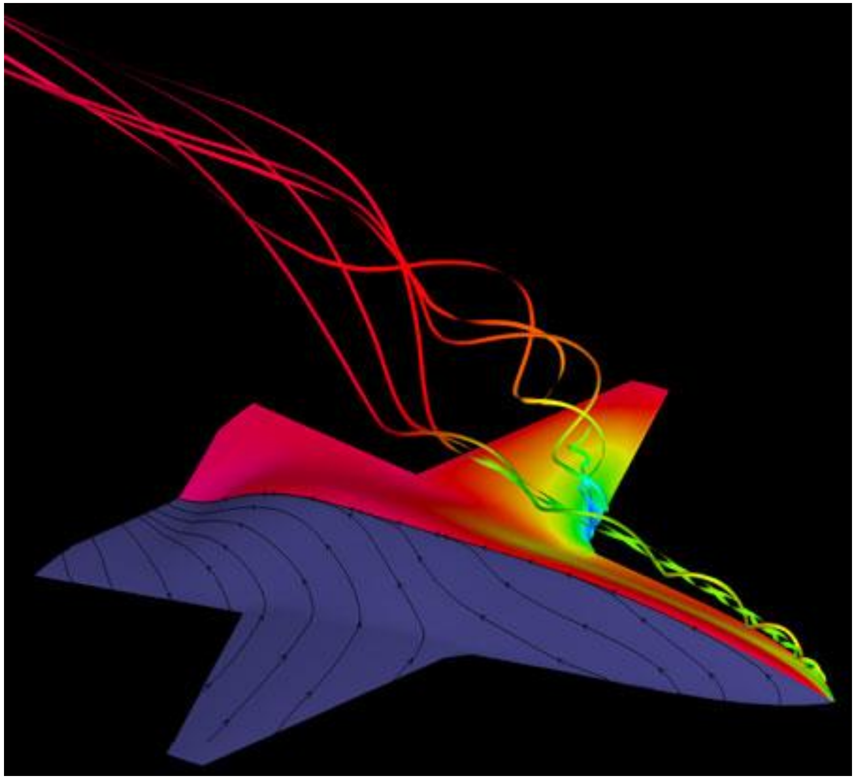
$$\frac{dr}{V} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Streamlines

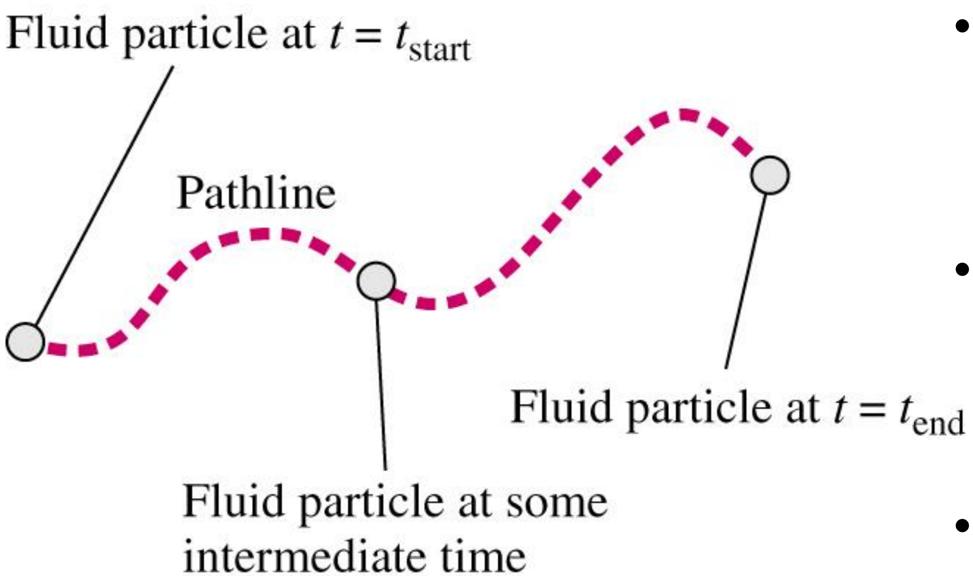
NASCAR surface pressure contours and streamlines



Airplane surface pressure contours, volume streamlines, and surface streamlines



Pathlines



- A **Pathline** is the actual path traveled by an individual fluid particle over some time period.
- Same as the fluid particle's material position vector

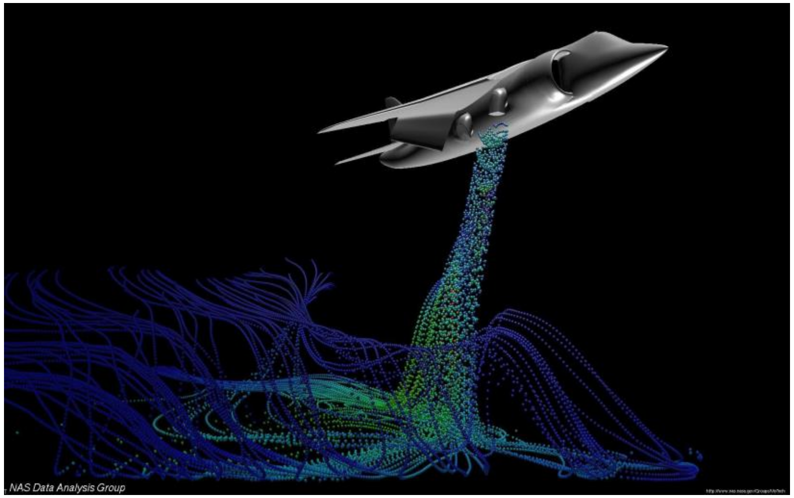
$$\left(x_{particle}(t), y_{particle}(t), z_{particle}(t)\right)$$

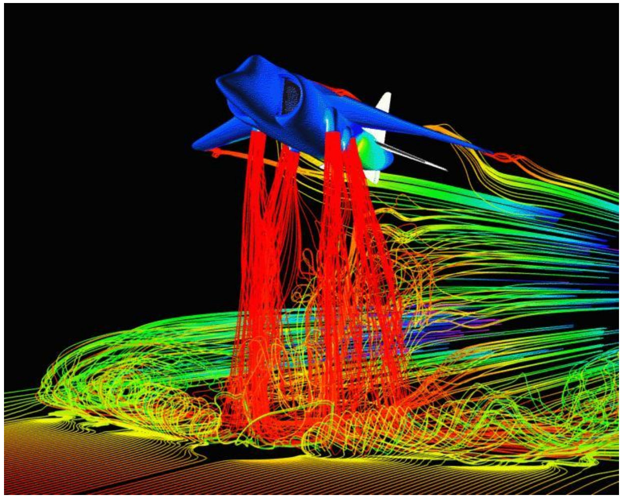
- Particle location at time t:

$$\vec{x} = \vec{x}_{start} + \int_{t_{start}}^{t} \vec{V} dt$$

- Particle Image Velocimetry (PIV) is a modern experimental technique to measure velocity field over a plane in the flow field.

Streaklines



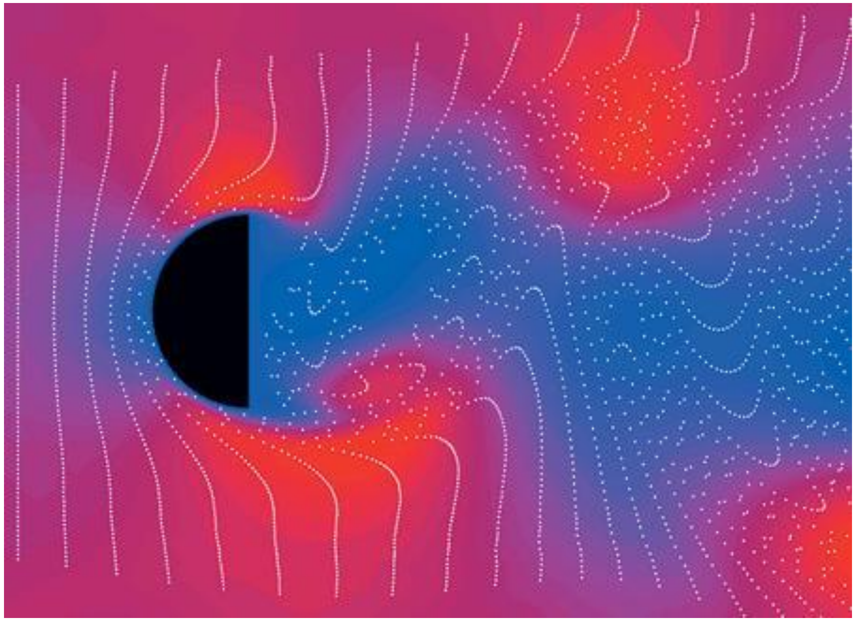


- A **Streakline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Easy to generate in experiments: dye in a water flow, or smoke in an airflow.

Comparisons

- For steady flow, streamlines, pathlines, and streaklines are identical.
- For unsteady flow, they can be very different.
  - Streamlines are an instantaneous picture of the flow field
  - Pathlines and Streaklines are flow patterns that have a time history associated with them.
  - Streakline: instantaneous snapshot of a time-integrated flow pattern.
  - Pathline: time-exposed flow path of an individual particle.

Timelines

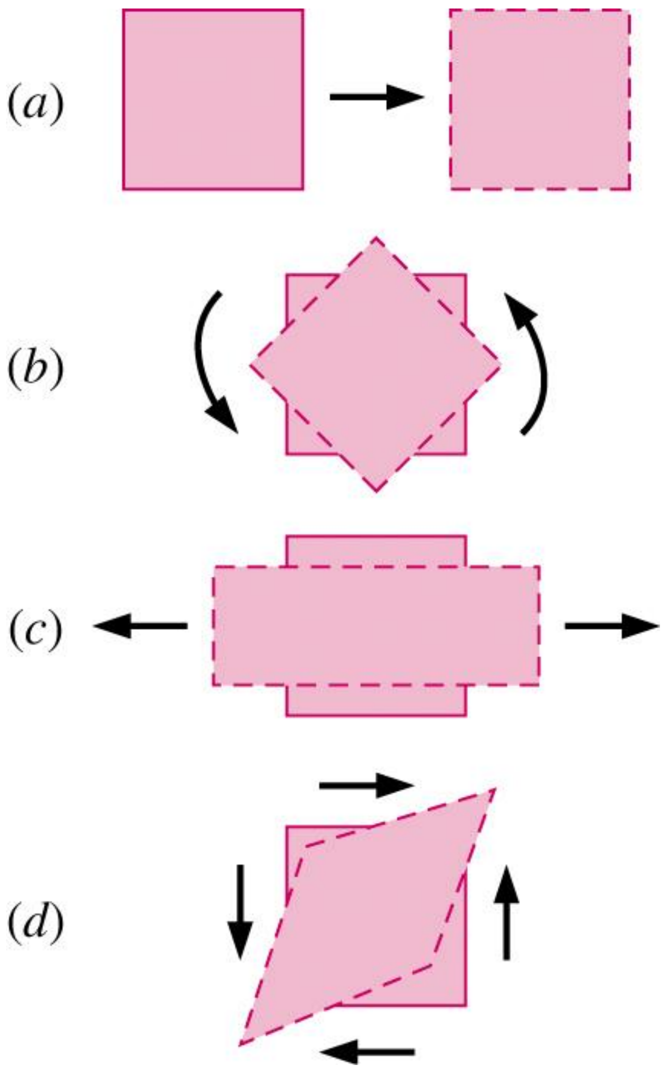


- A **Timeline** is the locus of fluid particles that have passed sequentially through a prescribed point in the flow.
- Timelines can be generated using a hydrogen bubble wire.

Plots of Data

- A **Profile plot** indicates how the value of a scalar property varies along some desired direction in the flow field.
- A **Vector plot** is an array of arrows indicating the magnitude and direction of a vector property at an instant in time.
- A **Contour plot** shows curves of constant values of a scalar property for magnitude of a vector property at an instant in time.

Kinematic Description



- In fluid mechanics, an element may undergo four fundamental types of motion.
  a) Translation
  b) Rotation
  c) Linear strain
  d) Shear strain
- Because fluids are in constant motion, motion and deformation is best described in terms of rates
  a) velocity: rate of translation
  b) angular velocity: rate of rotation
  c) linear strain rate: rate of linear strain
  d) shear strain rate: rate of shear strain

Rate of Translation and Rotation

- To be useful, these rates must be expressed in terms of velocity and derivatives of velocity
- The **rate of translation vector** is described as the velocity vector. In Cartesian coordinates:

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- **Rate of rotation** at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point. The rate of rotation vector in Cartesian coordinates:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

Linear Strain Rate

- **Linear Strain Rate** is defined as the rate of increase in length per unit length.
- In Cartesian coordinates

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

- Volumetric strain rate in Cartesian coordinates

$$\frac{1}{V}\frac{DV}{Dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Since the volume of a fluid element is constant for an incompressible flow, the volumetric strain rate must be zero.

Shear Strain Rate

- **Shear Strain Rate** at a point is defined as *half of the rate of decrease of the angle between two initially perpendicular lines that intersect at a point*.
- Shear strain rate can be expressed in Cartesian coordinates as:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Shear Strain Rate

We can combine linear strain rate and shear strain rate into one symmetric second-order tensor called the **strain-rate tensor.**

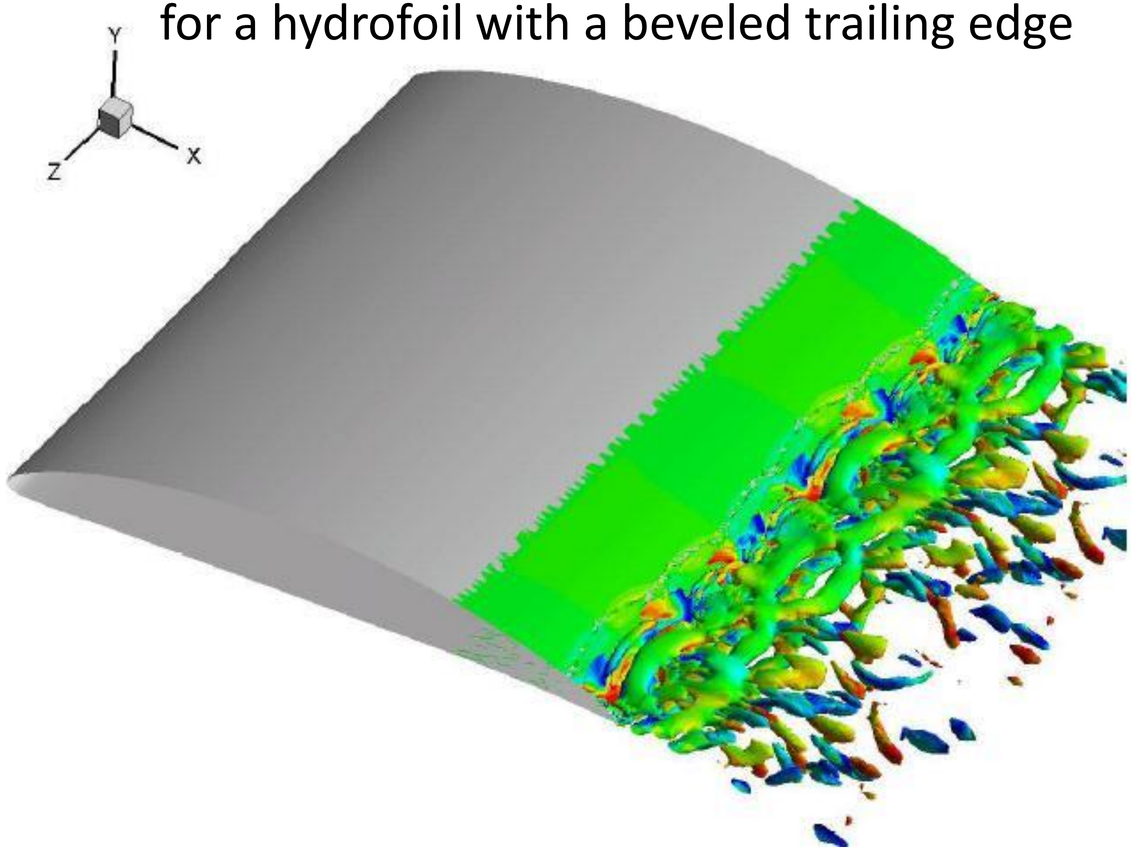
$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

Shear Strain Rate

- Purpose of our discussion of fluid element kinematics:
  - Better appreciation of the inherent complexity of fluid dynamics
  - Mathematical sophistication required to fully describe fluid motion
- Strain-rate tensor is important for numerous reasons. For example,
  - Develop relationships between fluid stress and strain rate.
  - Feature extraction and flow visualization in CFD simulations.

Shear Strain Rate

Example: Visualization of trailing-edge turbulent eddies
for a hydrofoil with a beveled trailing edge



Feature extraction method is based upon eigen-analysis of the strain-rate tensor.

Vorticity and Rotationality

- The **vorticity vector** is defined as the curl of the velocity vector
- Vorticity is equal to twice the angular velocity of a fluid particle. $\vec{\zeta} = \vec{\nabla} \times \vec{V}$ $\quad\quad \vec{\zeta} = 2\vec{\omega}$

  Cartesian coordinates

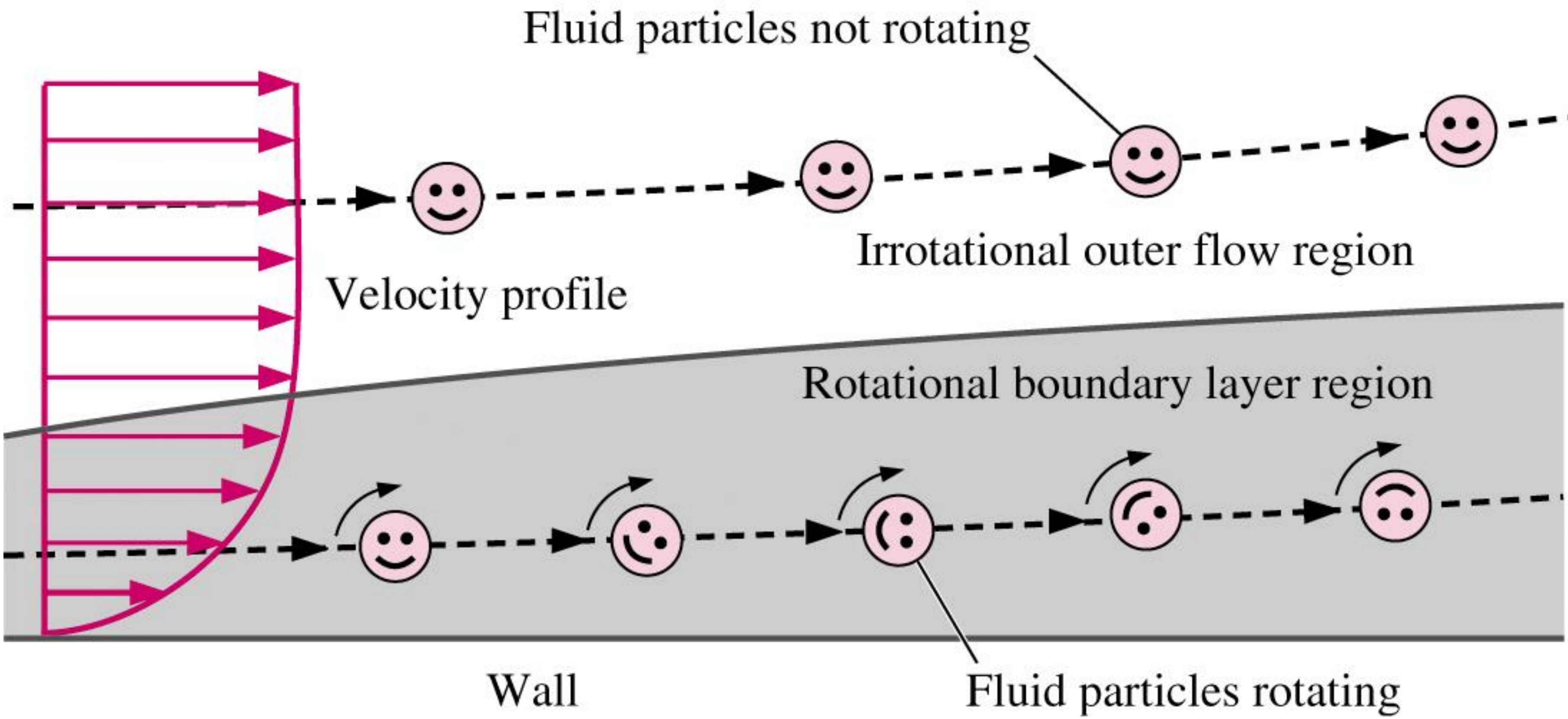
  $$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\vec{k}$$

  Cylindrical coordinates

  $$\vec{\zeta} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \right)\vec{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \right)\vec{e}_\theta + \left(\frac{\partial \left(r u_\theta \right)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right)\vec{e}_z$$

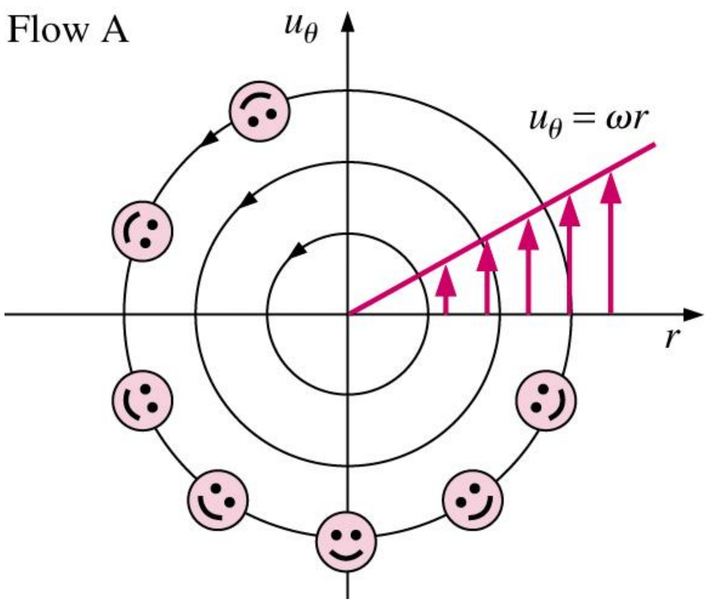
- In regions where $\zeta = 0$, the flow is called **irrotational.**
- Elsewhere, the flow is called **rotational.**

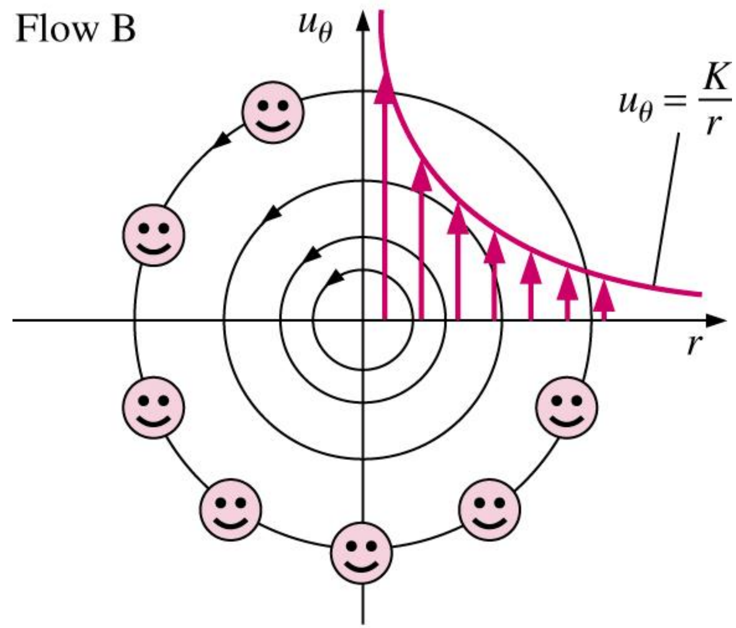
Vorticity and Rotationality



Comparison of Two Circular Flows

Special case: consider two flows with circular streamlines





(*b*)

$$u_r = 0, u_\theta = \omega r$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial\left(\omega r^2\right)}{\partial r} - 0\right)\vec{e}_z = 2\omega\vec{e}_z$$

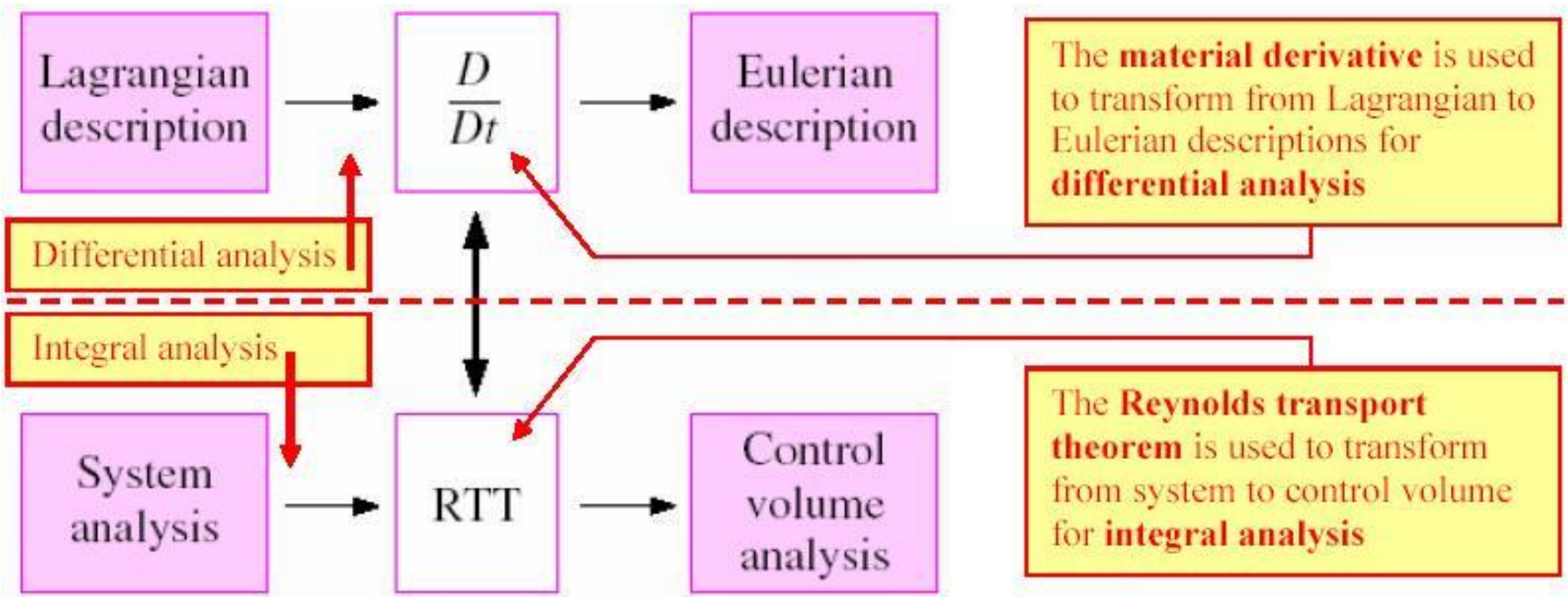
$$u_r = 0, u_\theta = \frac{K}{r}$$

$$\vec{\zeta} = \frac{1}{r}\left(\frac{\partial\left(r u_\theta\right)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)\vec{e}_z = \frac{1}{r}\left(\frac{\partial\left(K\right)}{\partial r} - 0\right)\vec{e}_z = 0\vec{e}_z$$

Reynolds—Transport Theorem (RTT)

- A **system** is a quantity of matter of fixed identity. *No mass can cross a system boundary.*
- A **control volume** is a region in space chosen for study. Mass can cross a control surface.
- The fundamental conservation laws (conservation of mass, energy, and momentum) apply directly to systems.
- However, in most fluid mechanics problems, control volume analysis is preferred over system analysis (for the same reason that the Eulerian description is usually preferred over the Lagrangian description).
- Therefore, we need to transform the conservation laws from a system to a control volume. This is accomplished with the Reynolds transport theorem (RTT).

Reynolds—Transport Theorem (RTT)



There is a direct analogy between the transformation from Lagrangian to Eulerian descriptions (for differential analysis using infinitesimally small fluid elements) and the transformation from systems to control volumes (for integral analysis using large, finite flow fields).

Reynolds—Transport Theorem (RTT)

- Material derivative (differential analysis):

$$\frac{Db}{Dt} = \frac{\partial b}{\partial t} + \left(\vec{V} \cdot \vec{\nabla}\right) b$$

- General RTT, nonfixed CV (integral analysis):

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}\left(\rho b\right) d\mathcal{V} + \int_{CS} \rho b \vec{V} \cdot \vec{n}\, dA$$

	Mass	**Momentum**	**Energy**	**Angular momentum**
B, Extensive properties	m	$m\vec{V}$	E	$\vec{H}$
b, Intensive properties	1	$\vec{V}$	e	$\left(\vec{r} \times \vec{V}\right)$

- Typically, RTT is applied to conservation of mass, energy, linear momentum and angular momentum.

Reynolds—Transport Theorem (RTT)

- Interpretation of the RTT:
  - Time rate of change of the property B of the system is equal to (Term 1) + (Term 2)
  - Term 1: the time rate of change of B of the control volume
  - Term 2: the net flux of B out of the control volume by mass crossing the control surface

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}\left(\rho b\right) d\mathcal{V} + \int_{CS} \rho b \vec{V} \cdot \vec{n} \, dA$$

RTT Special Cases

For **moving** and/or **deforming** control volumes,

$$\frac{dB_{sys}}{dt} = \int_{CV} \frac{\partial}{\partial t}(\rho b)\, d\forall + \int_{CS} \rho b \vec{V}_r \cdot \vec{n}\, dA$$

- Where the absolute velocity *V* in the second term is replaced by the **relative velocity** $V_r = V - V_{CS}$
- V_r is the fluid velocity expressed relative to a coordinate system moving **with** the control volume.

Balance equations in differential form

Conservation of Mass

- From Reynolds Transport Theorem (RTT)

$$0 = \int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \int_{CS} \rho \left(\vec{V} \cdot \vec{n} \right)\, dA$$

- We'll examine two methods to derive differential form of conservation of mass
  - Divergence (Gauss) Theorem
  - Differential CV and Taylor series expansions

Conservation of Mass

Divergence Theorem

- Divergence theorem allows us to transform a volume integral of the divergence of a vector into an area integral over the surface that defines the volume.

$$\int_{\mathcal{V}} \nabla \cdot \vec{G}\, d\mathcal{V} = \oint_{A} \vec{G} \cdot \vec{n}\, dA$$

Conservation of Mass

Divergence Theorem

- Rewrite conservation of mass

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} + \oint_{A} \rho\left(\vec{V}\cdot\vec{n}\right)\,\mathrm{d}A = 0$$

- Using divergence theorem, replace area integral with volume integral and collect terms

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} + \int_{\mathcal{V}} \nabla\cdot\rho\vec{V}\,\mathrm{d}\mathcal{V} = 0 \quad \Longrightarrow \quad \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right)\right]\mathrm{d}\mathcal{V} = 0$$

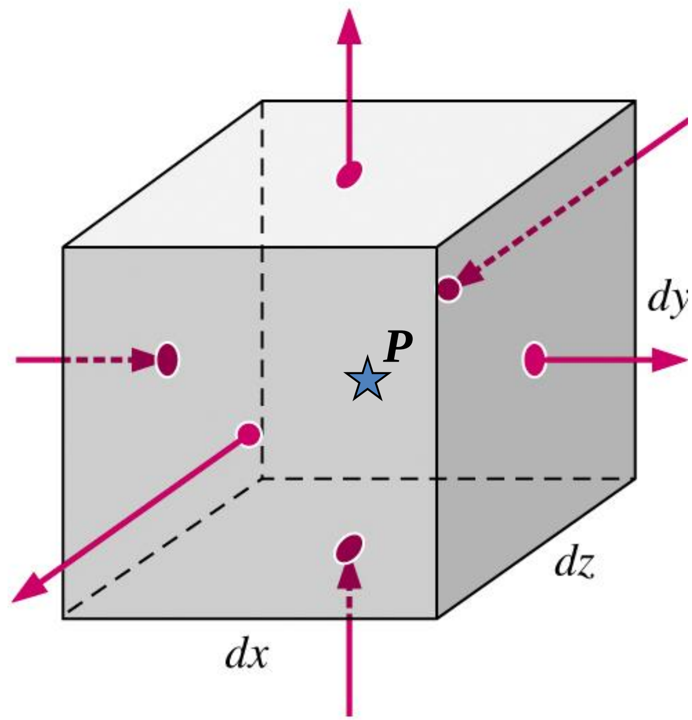
- Integral holds for ***ANY*** CV, therefore:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0}$$

Conservation of Mass

Differential CV and Taylor series

- First, define an infinitesimal control volume *dx dy dz* around a central point ***P***
- Next, we approximate the mass flow rate into or out of each of the 6 faces using Taylor series expansions around the center point, e.g., at the right face

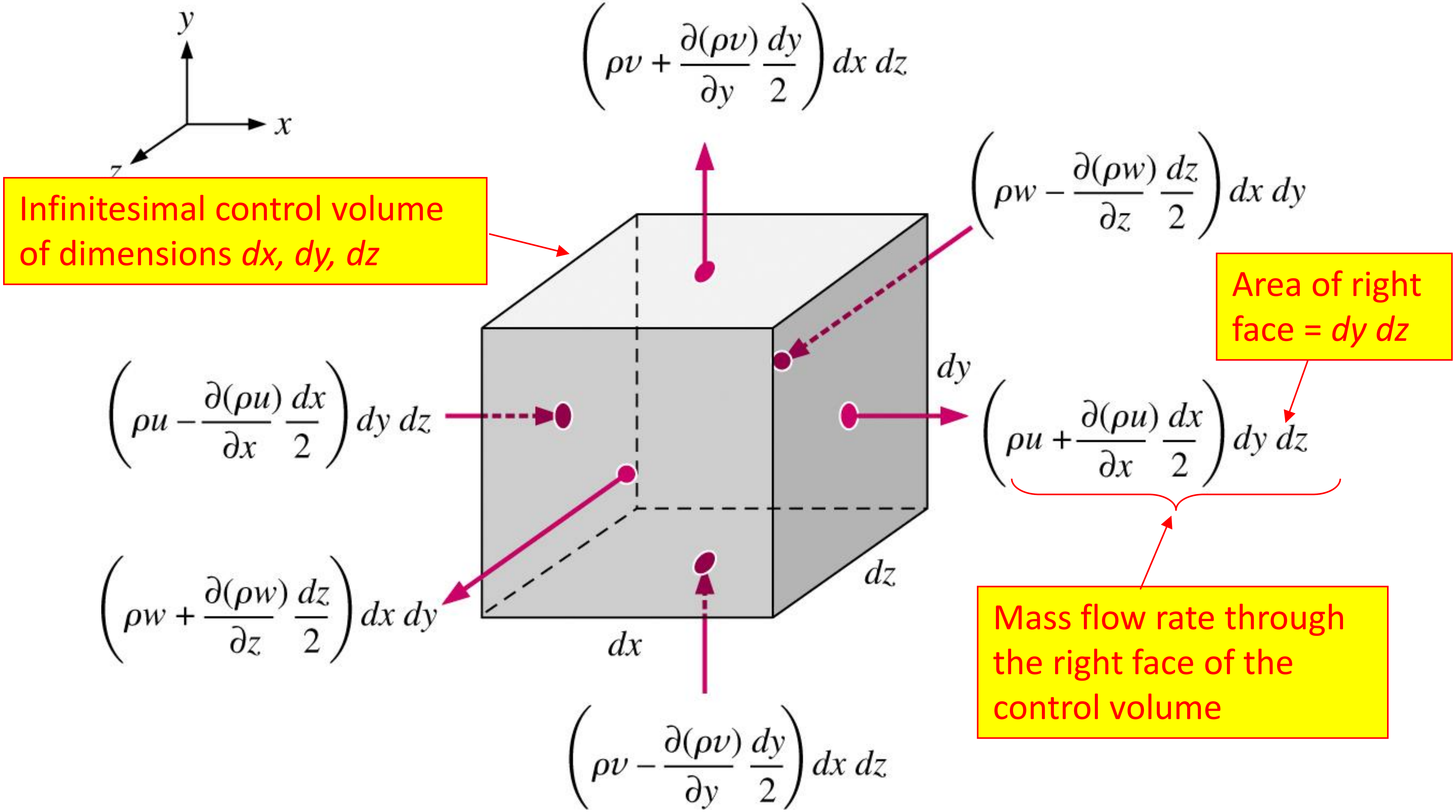


Ignore terms higher than order *dx*

$$(\rho u)_{\text{center of right face}} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2} + \cancel{\frac{1}{2!}\frac{\partial^2(\rho u)}{\partial x^2}}\left(\frac{dx}{2}\right)^2 + \ldots$$

Conservation of Mass

Differential CV and Taylor series



Conservation of Mass

Differential CV and Taylor series

- Now, sum up the mass flow rates into and out of the 6 faces of the CV

Net mass flow rate into CV:

$$\sum_{in} \dot{m} \approx \left(\rho u - \frac{\partial\left(\rho u\right)}{\partial x}\frac{dx}{2}\right) dy\, dz + \left(\rho v - \frac{\partial\left(\rho v\right)}{\partial y}\frac{dy}{2}\right) dx\, dz + \left(\rho w - \frac{\partial\left(\rho w\right)}{\partial z}\frac{dx}{2}\right) dx\, dy$$

Net mass flow rate out of CV:

$$\sum_{out} \dot{m} \approx \left(\rho u + \frac{\partial\left(\rho u\right)}{\partial x}\frac{dx}{2}\right) dy\, dz + \left(\rho v + \frac{\partial\left(\rho v\right)}{\partial y}\frac{dy}{2}\right) dx\, dz + \left(\rho w + \frac{\partial\left(\rho w\right)}{\partial z}\frac{dx}{2}\right) dx\, dy$$

- Plug into integral conservation of mass equation

$$\int_{CV} \frac{\partial \rho}{\partial t}\,\mathrm{d}\mathcal{V} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

Conservation of Mass

Differential CV and Taylor series

- After substitution,

$$\frac{\partial \rho}{\partial t}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = -\frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial(\rho v)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial(\rho w)}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

- Dividing through by volume *dxdydz*

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial x} + \frac{\partial(\rho w)}{\partial x} = 0}$$

Or, if we apply the definition of the divergence of a vector

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0}$$

Conservation of Mass

Alternative form

- Use product rule on divergence term

$$\frac{\partial \rho}{\partial t} + \boxed{\vec{\nabla} \cdot \left(\rho \vec{V}\right)} = \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{} + \rho \nabla \cdot \vec{V} = 0$$

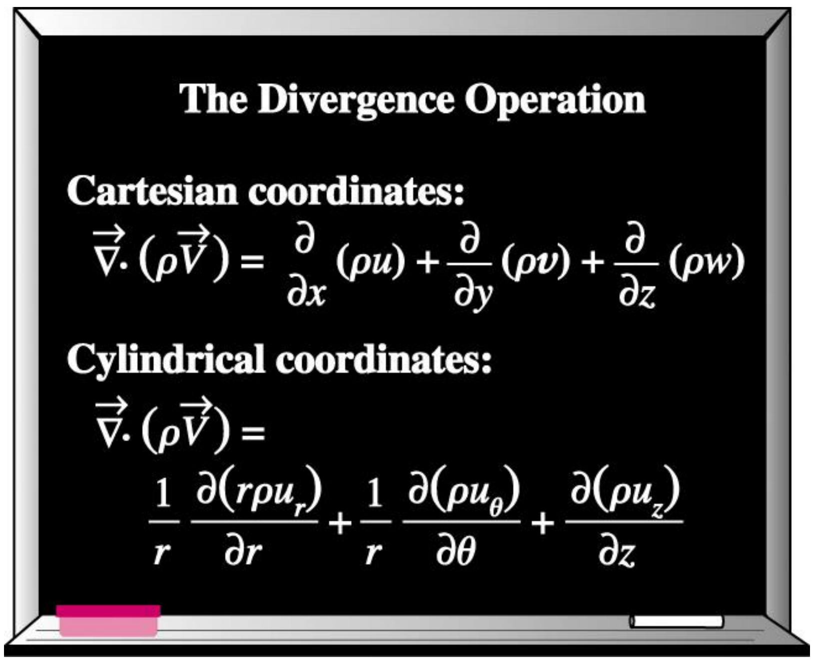
$$\boxed{\frac{D\rho}{Dt}} + \rho \nabla \cdot \vec{V} = 0$$

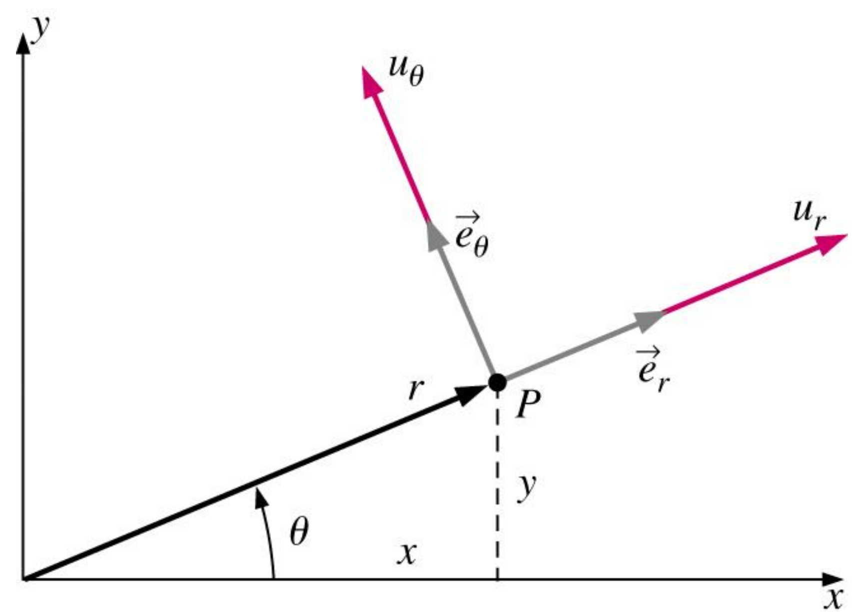
$$\boxed{\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \vec{V} = 0}$$

Conservation of Mass

Cylindrical coordinates

- There are many problems which are simpler to solve if the equations are written in cylindrical-polar coordinates
- Easiest way to convert from Cartesian is to use vector form and definition of divergence operator in cylindrical coordinates





Conservation of Mass

Cylindrical coordinates

$$\vec{\nabla} = \frac{1}{r}\frac{\partial(r)}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{e}_\theta + \frac{\partial}{\partial z}\hat{e}_z$$

$$\vec{V} = U_r\hat{e}_r + U_\theta\hat{e}_\theta + U_z\hat{e}_z$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\right) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0}$$

Conservation of Mass

Special Cases

- Steady compressible flow

$$\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0$$

$$\boxed{\vec{\nabla} \cdot \left(\rho \vec{V} \right) = 0}$$

Cartesian

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial x} + \frac{\partial(\rho w)}{\partial x} = 0$$

Cylindrical

$$\frac{1}{r}\frac{\partial(r\rho U_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho U_\theta)}{\partial \theta} + \frac{\partial(\rho U_z)}{\partial z} = 0$$

Conservation of Mass

Special Cases

- Incompressible flow

$$\frac{\partial \rho}{\partial t} = 0$$ and ρ = constant

$$\boxed{\vec{\nabla} \cdot \vec{V} = 0}$$

Cartesian
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} = 0$$

Cylindrical
$$\frac{1}{r}\frac{\partial (rU_r)}{\partial r} + \frac{1}{r}\frac{\partial (U_\theta)}{\partial \theta} + \frac{\partial (U_z)}{\partial z} = 0$$

Conservation of Mass

- In general, continuity equation cannot be used by itself to solve for flow field, however it can be used to

1. Determine if velocity field is incompressible
2. Find missing velocity component

The Stream Function

- Consider the continuity equation for an incompressible 2D flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

- Substituting the clever transformation

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

- Gives

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

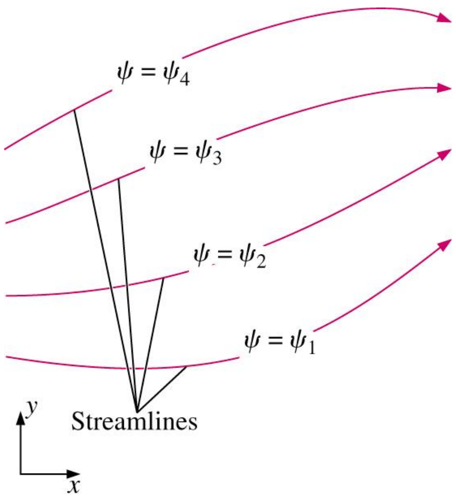
This is true for any smooth function $\psi(x,y)$

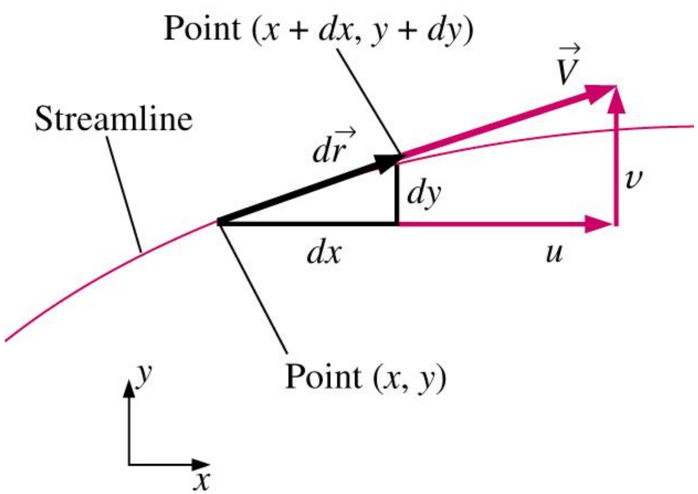
The Stream Function

- Why do this?
  - Single variable ψ replaces (*u,v*). Once ψ is known, (*u,v*) can be computed.
  - Physical significance
    1. Curves of constant ψ are streamlines of the flow
    2. Difference in ψ between streamlines is equal to volume flow rate between streamlines

The Stream Function

Physical Significance





Recall that along a streamline

$$\frac{dy}{dx} = \frac{v}{u} \Rightarrow -v\,dx + u\,dy = 0$$

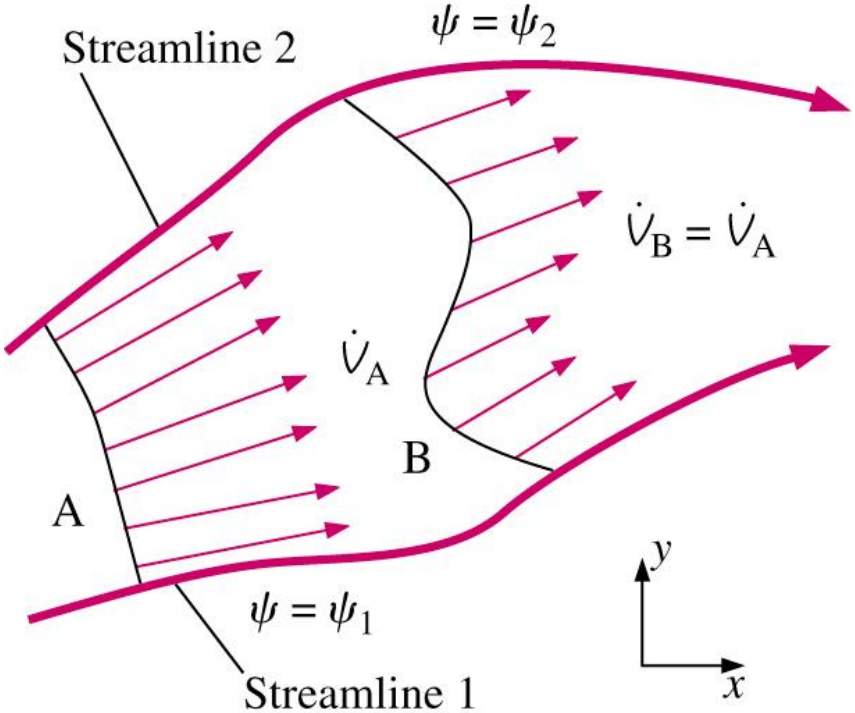
$$\frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy = 0$$

$$d\psi = 0$$

$\therefore$ Change in ψ along streamline is zero

The Stream Function

Physical Significance



Difference in ψ between streamlines is equal to volume flow rate between streamlines

$$\boxed{\dot{V}_A = \dot{V}_B = \psi_2 - \psi_1}$$

(by definition, *no flow can cross a streamline*)

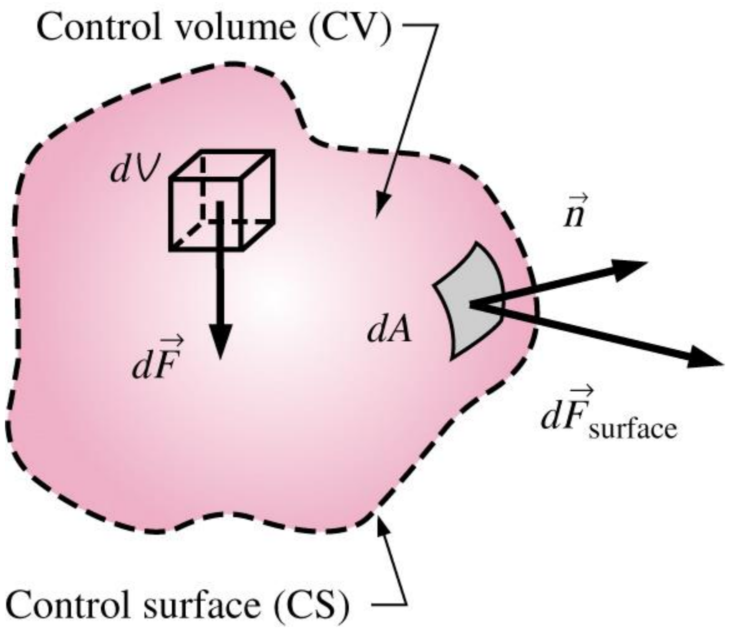
Newton’s Laws

- Newton’s laws are relations between motions of bodies and the forces acting on them.
  - **First law**: a body at rest remains at rest, and a body in motion remains in motion at the same velocity in a straight path when the net force acting on it is zero.
  - **Second law**: the acceleration of a body is proportional to the net force acting on it and is inversely proportional to its mass.

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d\left(m\vec{V}\right)}{dt}$$

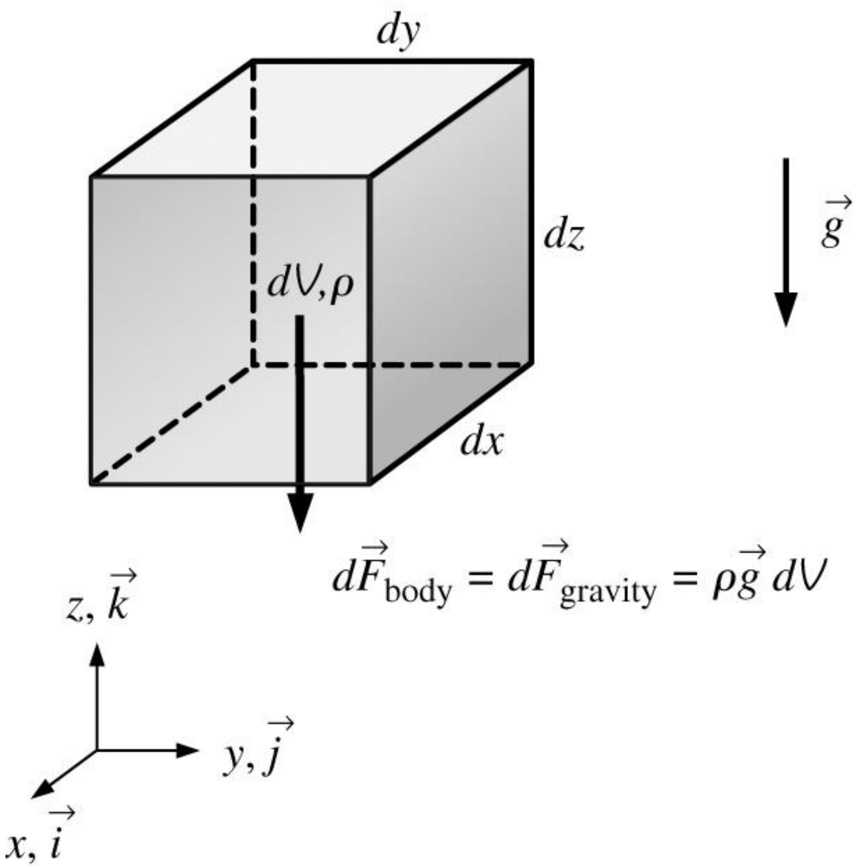
  - **Third law**: when a body exerts a force on a second body, the second body exerts an equal and opposite force on the first.

- Forces acting on CV consist of **body forces** that act throughout the entire body of the CV (such as gravity, electric, and magnetic forces) and **surface forces** that act on the control surface (such as pressure and viscous forces, and reaction forces at points of contact).



- Body forces act on each volumetric portion *dV* of the CV.
- Surface forces act on each portion *dA* of the CS.

Body Forces



- The most common body force is gravity, which exerts a downward force on every differential element of the CV
- The different body force

$$d\vec{F}_{body} = d\vec{F}_{gravity} = \rho\vec{g}d\mathcal{V}$$

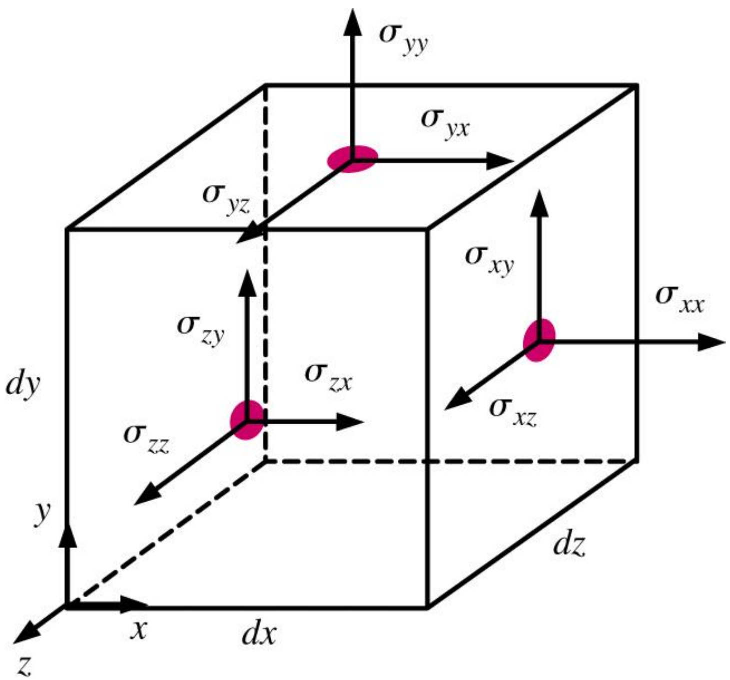
- Typical convention is that $\vec{g}$ acts in the negative *z*-direction,

$$\vec{g} = -g\vec{k}$$

- Total body force acting on CV

$$\sum \vec{F}_{body} = \int_{CV} \rho\vec{g}d\mathcal{V} = m_{CV}\vec{g}$$

Surface Forces



- Surface forces are not as simple to analyze since they include both normal and tangential components
- Diagonal components σ_{xx}, σ_{yy}, σ_{zz} are called **normal stresses** and are due to pressure and viscous stresses
- Off-diagonal components σ_{xy}, σ_{xz}, etc., are called **shear stresses** and are due solely to viscous stresses
- Total surface force acting on CS

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

$$\sum \vec{F}_{surface} = \int_{CS} \sigma_{ij} \cdot \vec{n} dA$$

Conservation of Linear Momentum

- Recall CV form from Chap. 6

$$\sum \vec{F} = \underbrace{\int_{CV} \rho g \, \mathrm{d}\mathcal{V}}_{\text{Body Force}} + \underbrace{\boxed{\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A}}_{\text{Surface Force}} = \int_{CV} \frac{\partial}{\partial t}\left(\rho \vec{V}\right) \mathrm{d}\mathcal{V} + \boxed{\int_{CS} \left(\rho \vec{V}\right) \vec{V} \cdot \vec{n} \, \mathrm{d}A}$$

σ_{ij} = stress tensor

- Using the divergence theorem to convert area integrals

$$\int_{CS} \sigma_{ij} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \sigma_{ij} \, \mathrm{d}\mathcal{V}$$

$$\int_{CS} \left(\rho \vec{V}\right) \vec{V} \cdot \vec{n} \, \mathrm{d}A = \int_{CV} \nabla \cdot \left(\rho \vec{V} \vec{V}\right) \mathrm{d}\mathcal{V}$$

Conservation of Linear Momentum

- Substituting volume integrals gives,

$$\int_{CV} \left[\frac{\partial}{\partial t} \left(\rho \vec{V} \right) + \nabla \cdot \left(\rho \vec{V} \vec{V} \right) - \rho \vec{g} - \nabla \cdot \sigma_{ij} \right] \mathrm{d}\mathcal{V} = 0$$

- Recognizing that this holds for **<u>any</u>** CV, the integral may be dropped

$$\boxed{\frac{\partial}{\partial t} \left(\rho \vec{V} \right) + \nabla \cdot \left(\rho \vec{V} \vec{V} \right) = \rho \vec{g} + \nabla \cdot \sigma_{ij}}$$

<u>This is Cauchy's Equation</u>

Can also be derived using infinitesimal CV and Newton's 2nd law

Conservation of Linear Momentum

- Alternate form of the Cauchy Equation can be derived by introducing

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t}=\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\frac{\partial\rho}{\partial t}$$ (Chain Rule)

$$\nabla\cdot\left(\rho\vec{V}\vec{V}\right)=\vec{V}\nabla\cdot\left(\rho\vec{V}\right)+\rho\left(\vec{V}\cdot\nabla\right)\vec{V}$$

- Inserting these into Cauchy Equation and rearranging gives

$$\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\nabla\right)\vec{V}\right]=\rho\vec{g}+\nabla\cdot\sigma_{ij}$$

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}+\nabla\cdot\sigma_{ij}$$

Conservation of Linear Momentum

- Unfortunately, this equation is not very useful
  - 10 unknowns
    - Stress tensor, σ_{ij} : 6 independent components
    - Density ρ
    - Velocity, $\vec{V}$: 3 independent components
  - 4 equations (continuity + momentum)
  - 6 more equations required to close problem!

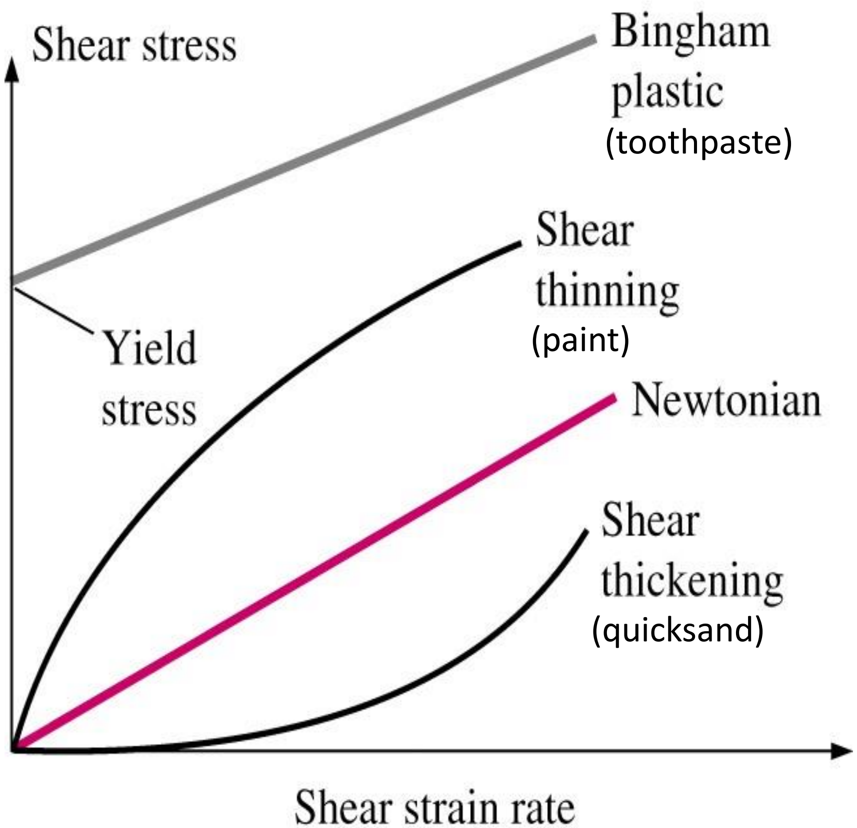
Navier-Stokes Equation

- First step is to separate σ_{ij} into pressure and viscous stresses

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \underbrace{\begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}}_{\text{Viscous (Deviatoric) Stress Tensor}}$$

- Situation not yet improved
  - 6 unknowns in σ_{ij} $\Rightarrow$ 6 unknowns in τ_{ij} + 1 in P, *which means that we've added 1!*

Navier-Stokes Equation



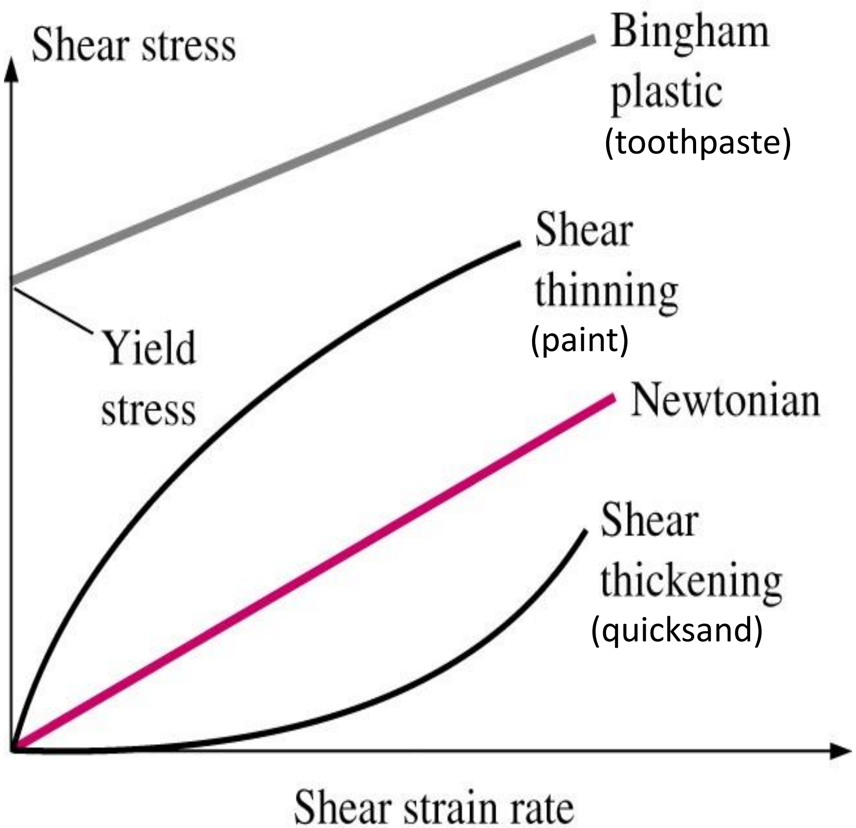
Newtonian fluid includes <u>most</u> common fluids: air, other gases, water, gasoline

- Reduction in the number of variables is achieved by relating shear stress to strain-rate tensor.
- For Newtonian fluid with constant properties (with $\boxed{\vec{\nabla} \cdot \vec{V} = 0}$)

$$\boxed{\tau_{ij} = 2\mu\epsilon_{ij}}$$

Newtonian closure is analogous to Hooke's Law for elastic solids

An excursus in rheology



"Rheology is the study of the flow of materials that behave in an interesting or unusual manner. Oil and water flow in familiar, normal ways, whereas mayonnaise, peanut butter, chocolate, bread dough, and silly putty flow in complex and unusual ways. In rheology, we study the flows of unusual materials."

"… all normal or Newtonian fluids (air, water, oil, honey) follow the same scientific laws. On the other hand, there are also fluids that do not follow the Newtonian flow laws. These non-Newtonian fluids, for example mayo, paint, molten plastics, foams, clays, and many other fluids, behave in a wide variety of ways. The science of studying these types of unusual materials is called rheology"

Faith Morrison, "The News and Information Publication of The Society of Rheology", Vol 73(1) Jan 2004, pp 8-10

Examples of Complex Fluids

- Foods
  - Emulsions (mayonaisse, ice cream)
  - Foams (ice cream, whipped cream)
  - Suspensions (mustard, chocolate)
  - Gels (cheese)
- Biofluids
  - Suspension (blood)
  - Gel (mucin)
  - Solutions (spittle)
- Personal Care Products
  - Suspensions (nail polish, face scrubs)
  - Solutions/Gels (shampoos, conditioners)
  - Foams (shaving cream)
- Electronic and Optical Materials
  - Liquid Crystals (Monitor displays)
  - Melts (soldering paste)
- Pharmaceuticals
  - Gels (creams, particle precursors)
  - Emulsions (creams)
  - Aerosols (nasal sprays)
- Polymers

Rheology's Goals

1. Establishing the relationship between applied forces and geometrical effects induced by these forces at a point (in a fluid).

   – The mathematical form of this relationship is called the rheological equation of state, or **the constitutive equation.**
   – The constitutive equations are used to solve macroscopic problems related to continuum mechanics of these materials.
   – Any equation is just a model of physical reality.

Rheology's Goals

1. Establishing the relationship between rheological properties of material and its molecular structure (composition).

   – Related to:
     - Estimating quality of materials
     - Understanding laws of molecular movements
     - Intermolecular interactions

   – Interested in what happens inside a point during deformation of the medium.

What happens inside a point?

(Material) Structure

- More or less well-organized and regularly spaced shapes
- Arrangements, organization or intermolecular interactions
- Structured Materials – properties change due to the influence of applied of applied forces on the structure of matter
- Rheology sometimes is referred to as mechanical spectroscopy.
- “Structure Mechanisms” are usually proposed, analogous to reaction mechanisms in reaction kinetics
- Structural probes are used to support rheological studies and proposed mechanisms.

Do Newtonian fluids suffer structural changes?

Rheological analysis is based on the use of continuum theories

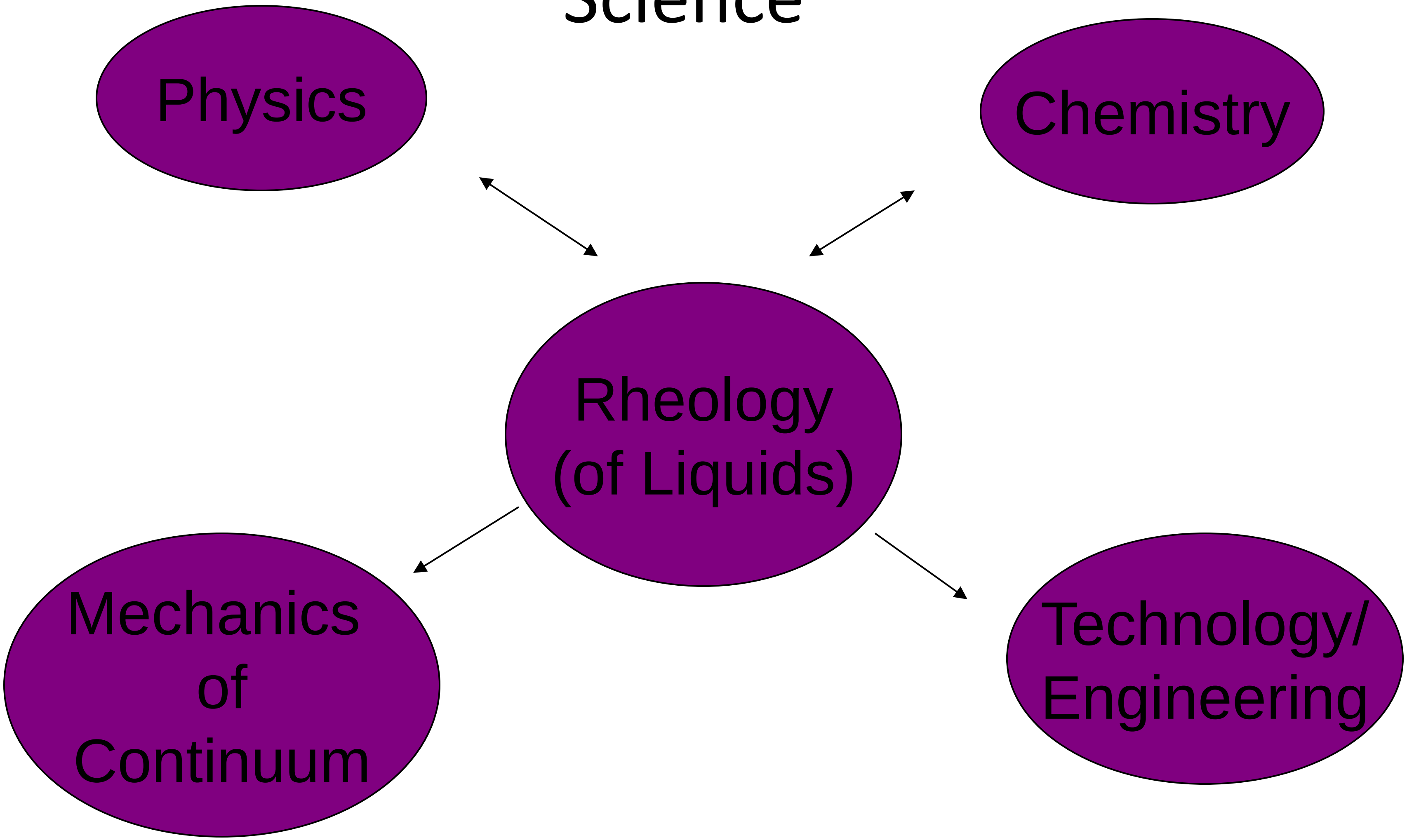
meaning that:

- There is no discontinuity in transition from one geometrical point to another, and the mathematical analysis of infinitesimal quantities can be used; discontinuities appear only at boundaries

- Properties of materials may change in space (due to gradients) but such changes occur gradually
  - changes are reflected in space dependencies of material properties entering equations of continuum theories

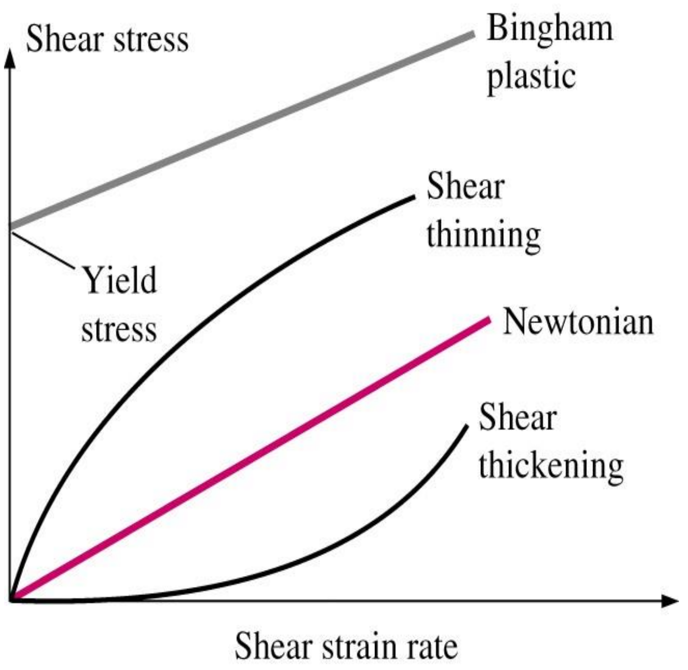
- Continuity theories may include an idea of anisotropy of properties of material along different directions.

Rheology as an Interdisciplinary Science



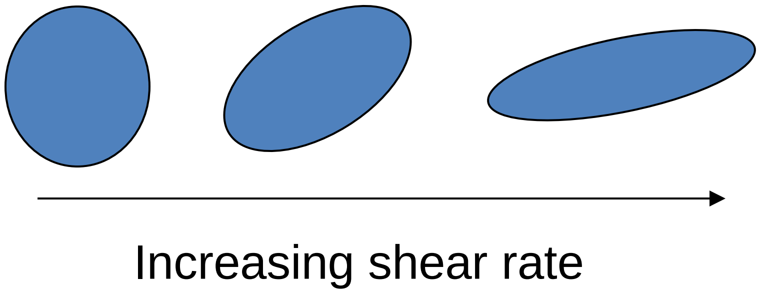
Common Non-Newtonian Behavior

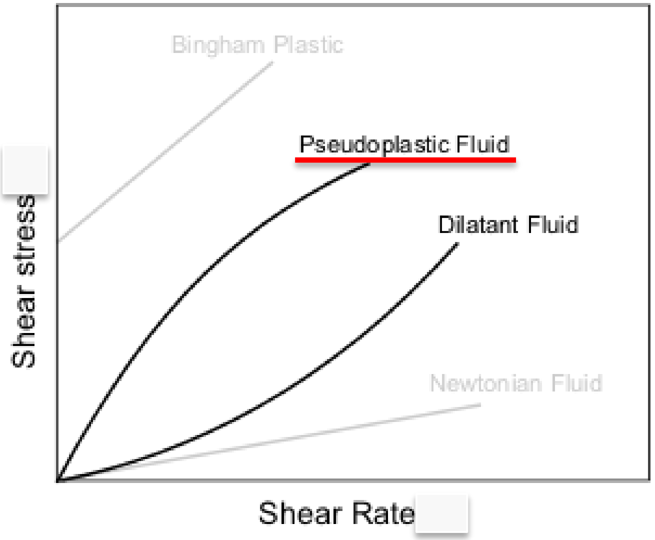
- shear thinning (pseudoplastic)
- shear thickening (dilatant)
- yield stress
- viscoelastic effects
  - Weissenberg effect
  - Fluid memory
  - Die swell



Shear Thinning

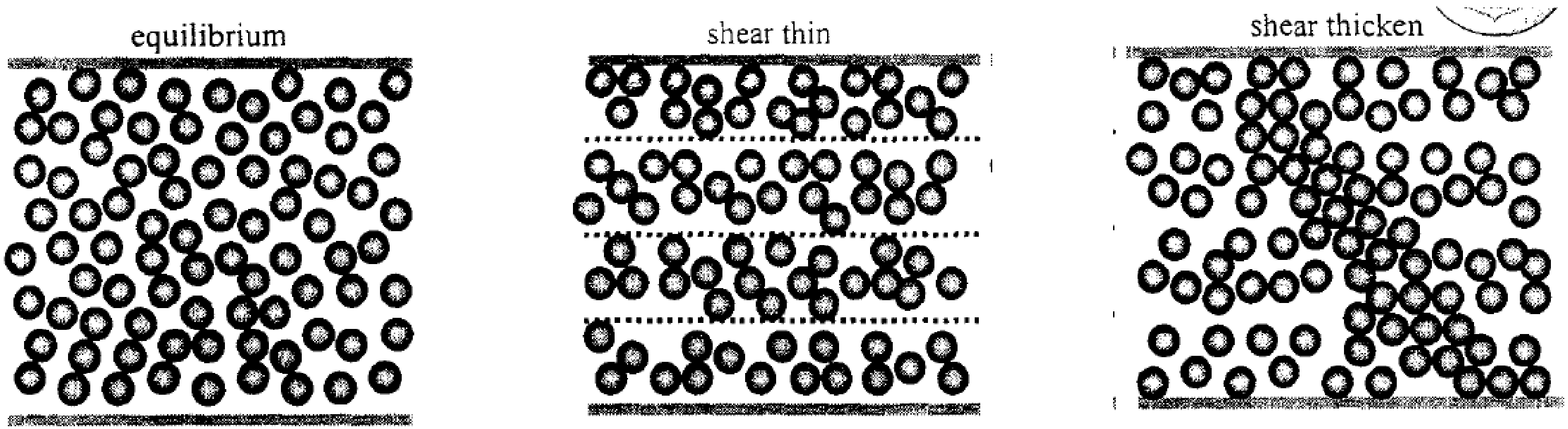
- shear thinning – tendency of some materials to **decrease in viscosity** when driven to flow at **high shear rates**, such as by higher pressure drops





Shear Thickening

- shear thickening – tendency of some materials to **increase in viscosity** when driven to flow at **high shear rates**



Phenomenological Modeling of Shear Thinning and Thickening

- Generalized Newtonian Equation: $\tau = \eta(\dot{\gamma})\dot{\gamma}$
- Power Law Model:

$$\boxed{\eta = m\dot{\gamma}^{n-1}}$$

  - m = μ ; n = 1 Newtonian
  - m n > 1 shear thickening, dilatant
  - m n < 1 shear thinning, pseudoplastic

- Advantages: simple, success at predicting Q vs ΔP
- Disadvantages: does not describe Newtonian Plateau at small shear rates

Modeling of Shear Thinning and Thickening

- Carreau-Yasuda Model

$$\boxed{\frac{\eta(\dot{\gamma})-\eta_\infty}{\eta_0-\eta_\infty}=[1+(\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}}$$

a – affects the shape of the transition region

λ – time constant determines where it changes from constant to power law

n – describes the slope of the power law

η_0, η_∞ - describe plateau viscosities

- Advantages: fits most data
- Disadvantages: contains 5 parameters, do not give molecular insight into polymer behavior

Yield Stress

- Tendency of a material to flow only when stresses are above a threshold stress

- Bingham Model:

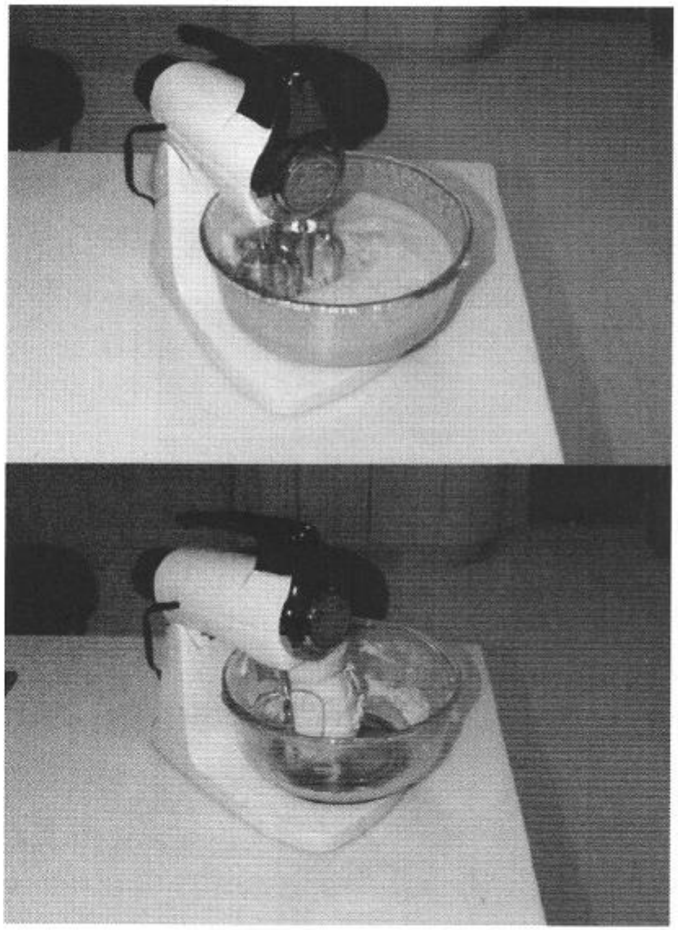
$$\eta(\dot{\gamma}) = \begin{cases} \infty & \tau \leq \tau_y \\ \mu_0 + \dfrac{\tau_y}{\dot{\gamma}} & \tau \geq \tau_y \end{cases}$$

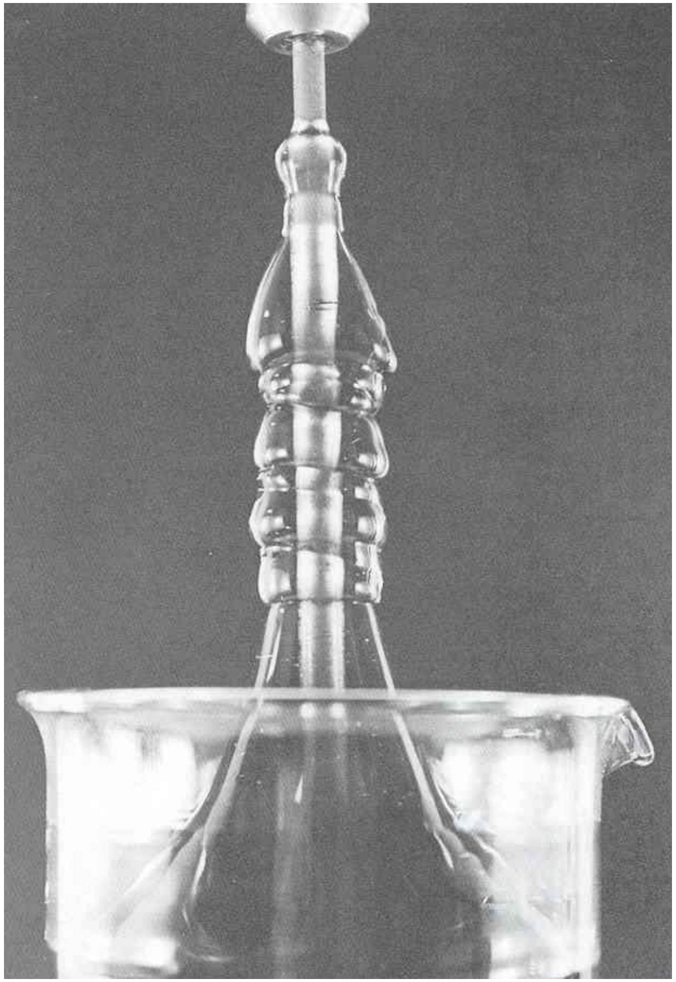
τ_y = yield stress, always positive

μ_0 = viscosity at higher shear rates

Elastic and Viscoelastic Effects

- Weissenberg Effect (rod climbing effect)
  - does not flow outward when stirred at high speeds





Elastic and Viscoelastic Effects

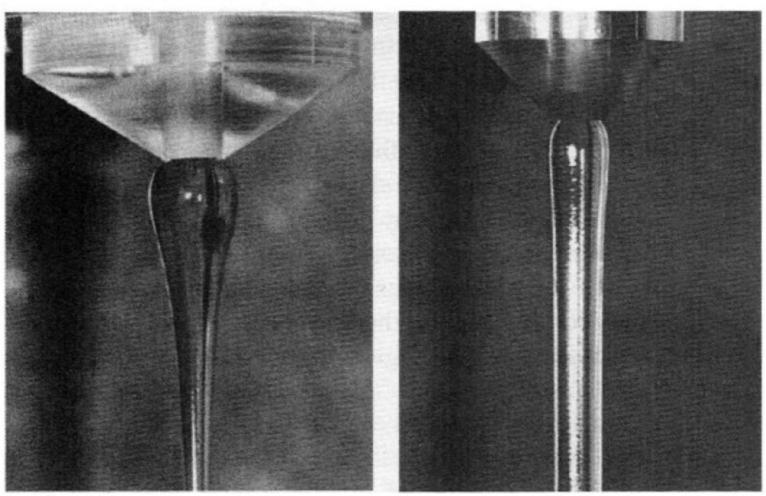
- Fluid Memory
  - Conserve their shape over time periods or seconds or minutes
  - Elastic like rubber
  - Can bounce or partially retract
  - Example: clay (plastilina)

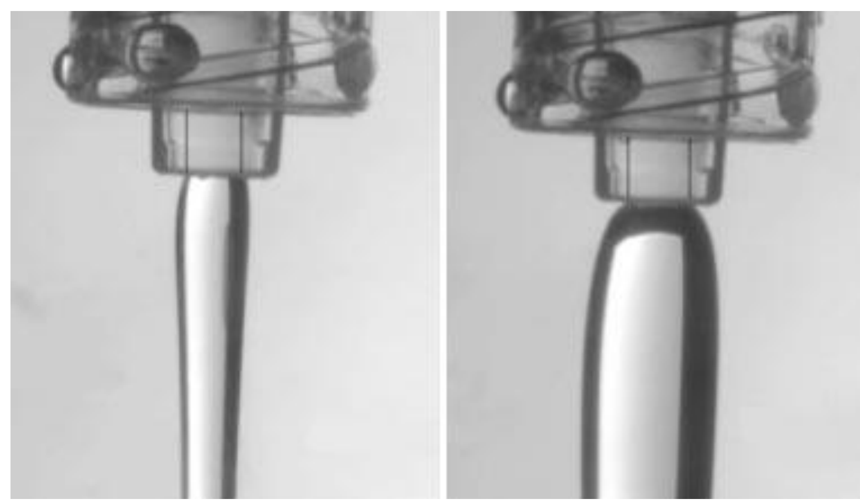
Elastic and Viscoelastic Effects

- Viscoelastic fluids subjected to a stress deform
  - when the stress is removed, it does not instantly vanish
  - internal structure of material can sustain stress for some time
  - this time is known as the relaxation time, varies with materials
  - due to the internal stress, the fluid will deform on its own, even when external stresses are removed
  - important for processing of polymer melts, casting, etc..

Elastic and Viscoelastic Effects – Die Swell

- as a polymer exits a die, the diameter of liquid stream increases by up to an order of magnitude
- caused by relaxation of extended polymer coils, as stress is reduced from high flow producing stresses present within the die to low stresses, associated with the extruded stream moving through ambient air





BACK TO GOOD OLD "SIMPLE" NEWTONIAN FLUIDS, IN THE INCOMPRESSIBLE FLOW LIMIT

Navier-Stokes Equation

- Substituting Newtonian closure into stress tensor gives

$$\overset{=}{\sigma}_{ij} = p\delta_{ij} + 2\mu\epsilon_{ij}$$

- Using the definition of ε_{ij}

$$\sigma_{ij} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} + \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z} + \frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix}$$

Navier-Stokes Equation

- Substituting σ_{ij} into Cauchy's equation gives the Navier-Stokes equations

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

$$\nabla \cdot \vec{V} = 0$$

Incompressible NSE written in vector form

- This results in a *closed* system of equations!
  - 4 equations (continuity and momentum equations)
  - 4 unknowns (U, V, W, p)

Navier-Stokes Equation

- In addition to vector form, incompressible N-S equation can be written in several other forms
  - Cartesian coordinates
  - Cylindrical coordinates
  - Tensor notation

Navier-Stokes Equation

Cartesian Coordinates

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0$$

X-momentum

$$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

Y-momentum

$$\rho\left(\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

Z-momentum

$$\rho\left(\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}\right)$$

Navier-Stokes Equation

Tensor and Vector Notation

Tensor and Vector notation offer a more compact form of the equations.

Continuity

Tensor notation

$$\boxed{\frac{\partial U_i}{\partial x_i} = 0}$$

Vector notation

$$\boxed{\nabla \cdot \vec{V} = 0}$$

Conservation of Momentum

Tensor notation

$$\boxed{\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = -\frac{\partial P}{\partial x_i} + \rho g_{x_i} + \mu \left(\frac{\partial^2 U_i}{\partial x_j \partial x_j} \right)}$$

Vector notation

$$\boxed{\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}}$$

Repeated indices are summed over j

($x_1 = x$, $x_2 = y$, $x_3 = z$, $U_1 = U$, $U_2 = V$, $U_3 = W$)

Differential Analysis of Fluid Flow Problems

- Now that we have a set of governing partial differential equations, there are 2 problems we can solve
  1. Calculate pressure (*P*) for a known velocity field
  2. Calculate velocity (*U, V, W*) and pressure (*P*) for known geometry, boundary conditions (BC), and initial conditions (IC)

Exact Solutions of the NSE

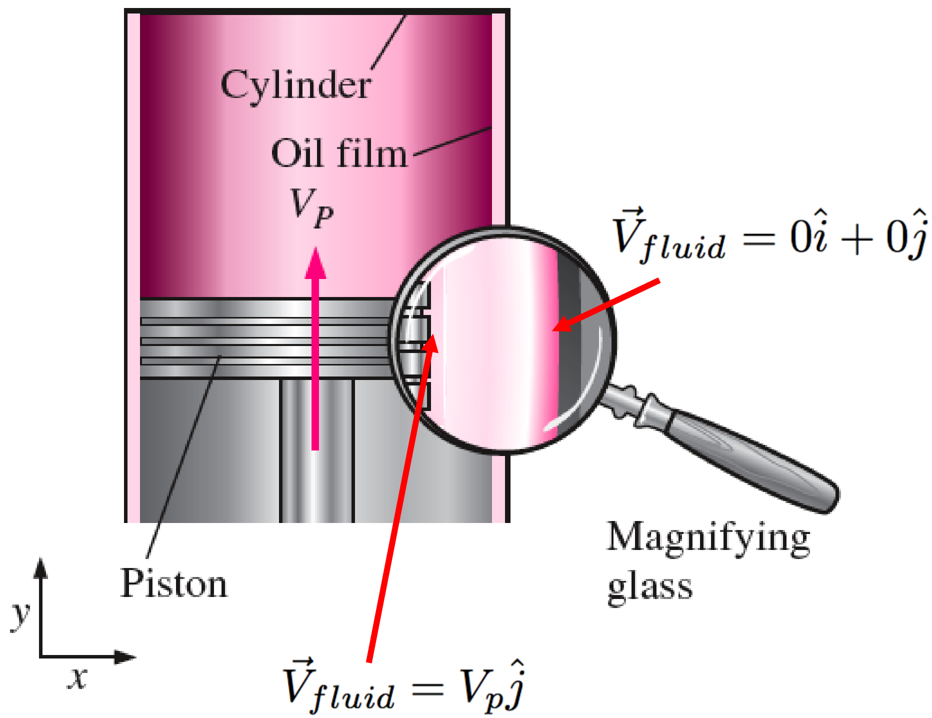
- There are about 80 known exact solutions to the NSE
- The can be classified as:
  - Linear solutions where the convective $\left(\vec{V} \cdot \nabla\right)\vec{V}$ term is zero
  - Nonlinear solutions where convective term is not zero

- Solutions can also be classified by type or geometry
  1. Couette shear flows
  2. Steady duct/pipe flows
  3. Unsteady duct/pipe flows
  4. Flows with moving boundaries
  5. Similarity solutions
  6. Asymptotic suction flows
  7. Wind-driven Ekman flows

Boundary conditions

- Boundary conditions are critical to exact, approximate, and computational solutions.
  - BC's used in analytical solutions are discussed here
    - No-slip boundary condition
    - Interface boundary condition
  - These are used in CFD as well, plus there are some BC's which arise due to specific issues in CFD modeling.
    - Inflow and outflow boundary conditions
    - Symmetry and periodic boundary conditions

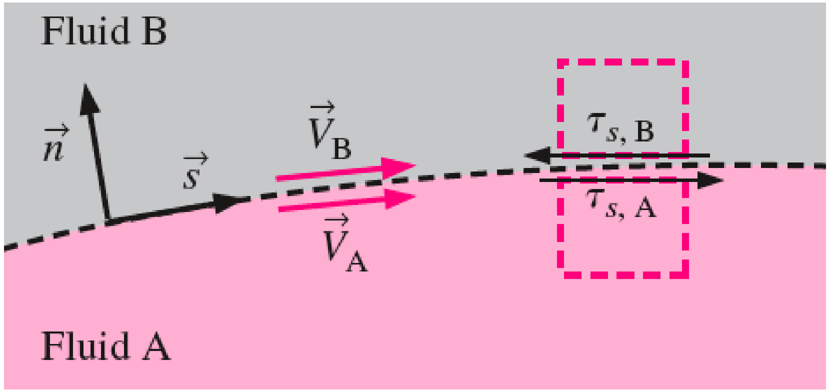
No-slip boundary condition



- For a fluid in contact with a solid wall, the velocity of the fluid must equal that of the wall

$$\vec{V}_{fluid} = \vec{V}_{wall}$$

Interface boundary condition



- When two fluids meet at an interface, the velocity and shear stress must be the same on both sides

$$\vec{V}_A = \vec{V}_B \qquad \tau_{s,A} = \tau_{s,B}$$

- If surface tension effects are negligible and the surface is nearly flat

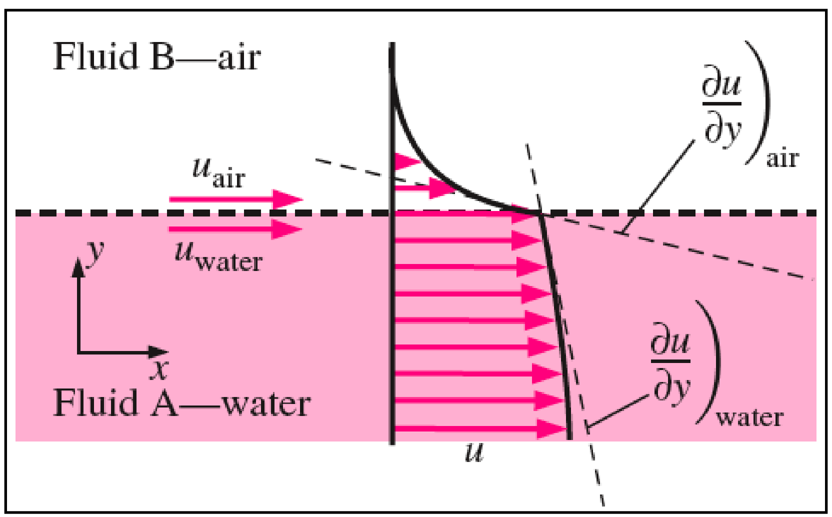
$$P_A = P_B$$

Interface boundary condition

- Degenerate case of the interface BC occurs at the free surface of a liquid.
- Same conditions hold

$$u_{air} = u_{water}$$

$$\tau_{s,water} = \mu_{water}\left(\frac{\partial u}{\partial y}\right)_{water} = \tau_{s,air} = \mu_{air}\left(\frac{\partial u}{\partial y}\right)_{air}$$



- Since $\mu_{air} << \mu_{water}$,

$$\left(\frac{\partial u}{\partial y}\right)_{water} \approx 0$$

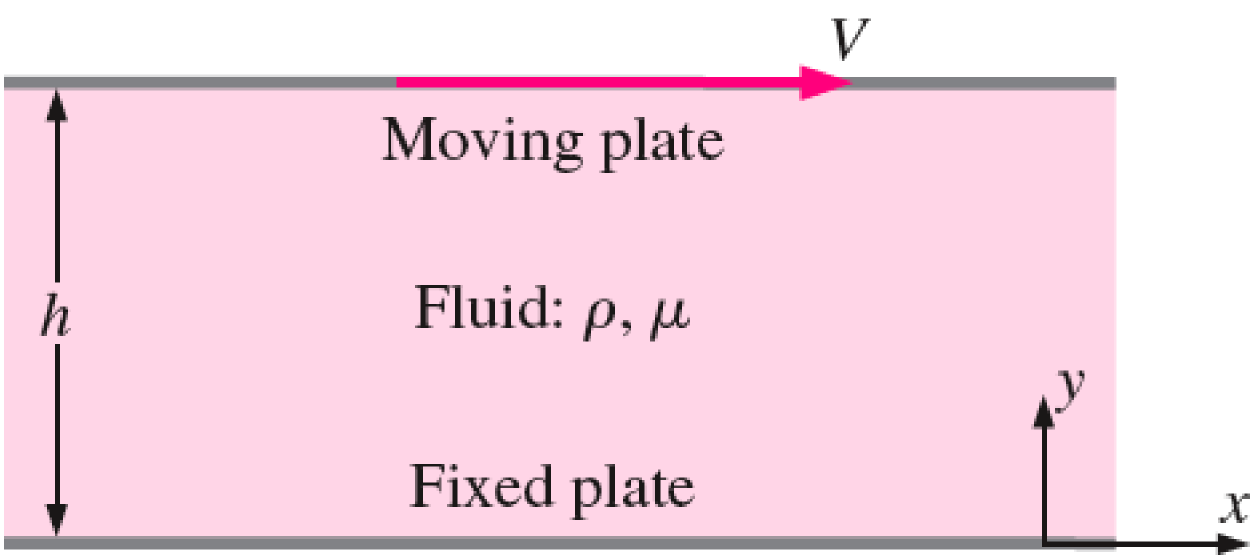
- As with general interfaces, if surface tension effects are negligible and the surface is nearly flat $P_{water} = P_{air}$

Example exact solution (Ex. 9-15)

Fully Developed Couette Flow

- For the given geometry and BC's, calculate the velocity and pressure fields, and estimate the shear force per unit area acting on the bottom plate

- Step 1



Example exact solution

Fully Developed Couette Flow

- Step 2: Assumptions and BC's
  - Assumptions
    1. Plates are infinite in x and z
    2. Flow is steady, $\partial/\partial t = 0$
    3. Parallel flow, V=0
    4. Incompressible, Newtonian, laminar, constant properties
    5. No pressure gradient
    6. 2D, W=0, $\partial/\partial z = 0$
    7. Gravity acts in the -z direction, $\vec{g} = -g\vec{k}, g_z = -g$
  - Boundary conditions
    1. Bottom plate (y=0) : u=0, v=0, w=0
    2. Top plate (y=h) : u=V, v=0, w=0

Example exact solution

Fully Developed Couette Flow

- Step 3: Simplify

Note: these numbers refer to the assumptions on the previous slide

<u>Continuity</u>

$$\frac{\partial U}{\partial x} + \overset{3}{\cancel{\frac{\partial V}{\partial y}}} + \overset{6}{\cancel{\frac{\partial W}{\partial z}}} = 0$$

$$\boxed{\frac{\partial U}{\partial x} = 0}$$

This means the flow is *"fully developed"* or not changing in the direction of flow

<u>X-momentum</u>

$$\rho\left(\overset{2}{\cancel{\frac{\partial U}{\partial t}}} + \overset{\text{Cont.}}{\cancel{U\frac{\partial U}{\partial x}}} + \overset{3}{\cancel{V\frac{\partial U}{\partial y}}} + \overset{6}{\cancel{W\frac{\partial U}{\partial z}}}\right) = -\overset{5}{\cancel{\frac{\partial P}{\partial x}}} + \overset{7}{\cancel{\rho g_x}} + \mu\left(\overset{\text{Cont.}}{\cancel{\frac{\partial^2 U}{\partial x^2}}} + \frac{\partial^2 U}{\partial y^2} + \overset{6}{\cancel{\frac{\partial^2 U}{\partial z^2}}}\right)$$

$$\boxed{\frac{\partial^2 U}{\partial x^2} = 0}$$

Example exact solution

Fully Developed Couette Flow

- Step 3: Simplify, cont.

Y-momentum

$$\rho\left(\cancelto{\boxed{2,3}}{\frac{\partial V}{\partial t}} + U\cancelto{\boxed{3}}{\frac{\partial V}{\partial x}} + V\cancelto{\boxed{3}}{\frac{\partial V}{\partial y}} + W\cancelto{\boxed{3,6}}{\frac{\partial V}{\partial z}}\right) = -\frac{\partial P}{\partial y} + \cancelto{\boxed{7}}{\rho g_y} + \mu\left(\cancelto{\boxed{3}}{\frac{\partial^2 V}{\partial x^2}} + \cancelto{\boxed{3}}{\frac{\partial^2 V}{\partial y^2}} + \cancelto{\boxed{3}}{\frac{\partial^2 V}{\partial z^2}}\right)$$

$$\frac{\partial p}{\partial y} = 0 \Longrightarrow \boxed{p = p(z)}$$

Z-momentum

$$\rho\left(\cancelto{\boxed{2,6}}{\frac{\partial W}{\partial t}} + U\cancelto{\boxed{6}}{\frac{\partial W}{\partial x}} + V\cancelto{\boxed{6}}{\frac{\partial W}{\partial y}} + W\cancelto{\boxed{6}}{\frac{\partial W}{\partial z}}\right) = -\frac{\partial P}{\partial z} + \cancelto{\boxed{7}}{\rho g_z} + \mu\left(\cancelto{\boxed{6}}{\frac{\partial^2 W}{\partial x^2}} + \cancelto{\boxed{6}}{\frac{\partial^2 W}{\partial y^2}} + \cancelto{\boxed{6}}{\frac{\partial^2 W}{\partial z^2}}\right)$$

$$\frac{\partial p}{\partial z} = \rho g_z \Longrightarrow \boxed{\frac{dp}{dz} = -\rho g}$$

Example exact solution

Fully Developed Couette Flow

- Step 4: Integrate

<u>X-momentum</u>

$$\frac{d^2u}{dy^2} = 0 \xrightarrow{\text{integrate}} \frac{du}{dy} = C_1 \xrightarrow{\text{integrate}} \boxed{u(y) = C_1 y + C_2}$$

<u>Z-momentum</u>

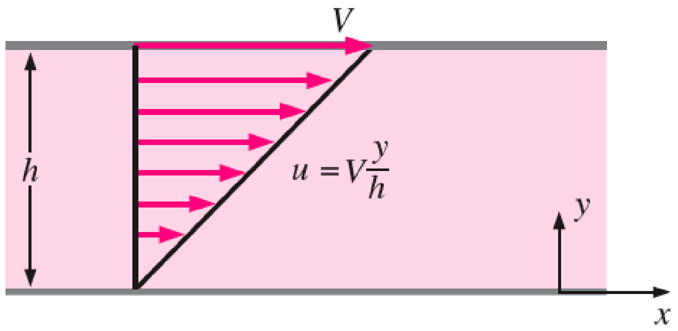
$$\frac{dp}{dz} = -\rho g \xrightarrow{\text{integrate}} \boxed{p = -\rho g z + C_3}$$

Example exact solution

Fully Developed Couette Flow

- Step 5: Apply BC's
  - y=0, $u=0=C_1(0) + C_2 \Rightarrow \underline{C_2 = 0}$
  - y=h, $u=V=C_1 h \Rightarrow \underline{C_1 = V/h}$
  - This gives

$$\boxed{u(y) = V\frac{y}{h}}$$



  - For pressure, no explicit BC, therefore C_3 can remain an arbitrary constant (recall only ∇P appears in NSE).
    - Let $p = p_0$ at $z = 0$ (C_3 renamed p_0)

$$\boxed{p(z) = p_0 - \rho g z}$$

1. Hydrostatic pressure
2. Pressure acts independently of flow

Example exact solution

Fully Developed Couette Flow

- Step 6: Verify solution by back-substituting into differential equations
  - Given the solution (u,v,w)=(Vy/h, 0, 0)

$$\frac{\partial u}{\partial x} = 0, \frac{\partial v}{\partial y} = 0, \frac{\partial w}{\partial z} = 0$$

  - Continuity is satisfied

0 + 0 + 0 = 0

  - $$\rho\left(\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)$$

$$\rho\left(0 + V\frac{y}{h}\cdot 0 + 0\cdot V/h + 0\cdot 0\right) = -0 + \rho\cdot 0 + \mu\left(0 + 0 + 0\right)$$

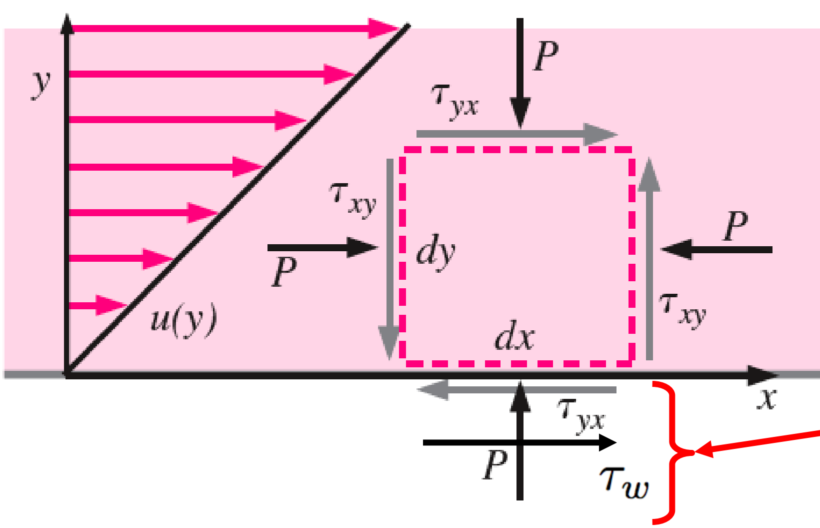
$$0 = 0$$

Example exact solution (Ex. 9-15)

Fully Developed Couette Flow

- Finally, calculate shear force on bottom plate

$$\tau_{ij} = \begin{pmatrix} 2\mu\frac{\partial U}{\partial x} & \mu\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right) & \mu\left(\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}\right) \\ \mu\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right) & 2\mu\frac{\partial V}{\partial y} & \mu\left(\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}\right) \\ \mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right) & \mu\left(\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\right) & 2\mu\frac{\partial W}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & \mu\frac{V}{h} & 0 \\ \mu\frac{V}{h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



Shear force per unit area acting on the wall

$$\boxed{\frac{\vec{F}}{A} = \tau_w = \mu\frac{V}{h}\hat{i}}$$

Note that τ_w is equal and opposite to the shear stress acting on the fluid τ_{yx} (Newton's third law).

Adimensionalization of equations and boundary conditions

[appearance of (at least 52!) **dimensionless numbers**]

A

- Archimedes number
- Atwood number

B

- Bejan number
- Biot number
- Brinkman number

C

- Capillary number
- Cauchy number

D

- Damköhler numbers
- Darcy number
- Darcy–Weisbach equation
- Deborah number
- Drag coefficient

E

- Eckert number
- Ekman number
- Eötvös number
- Euler number (physics)

F

- Fanning friction factor
- Froude number

G

- Galilei number
- Görtler vortices
- Graetz number
- Grashof number

H

- Hartmann number
- *Hydraulic gradient*

I

- Iribarren number

K

- Keulegan–Carpenter number

L

- Laplace number
- Lift coefficient

M

- Mach number
- Magnetic Prandtl number
- Magnetic Reynolds number
- Marangoni number

N

- Nusselt number

O

- Ohnesorge number

P

- Péclet number
- Power number

P cont.

- Prandtl number
- Pressure coefficient

R

- Rayleigh number
- Reduced frequency
- Reynolds number
- Roshko number
- Rossby number

S

- Schmidt number
- Sherwood number
- Stanton number
- Strouhal number

T

- Turbulent Prandtl number

U

- Ursell number

W

- Weber number
- Weissenberg number
- Womersley number

- Reynolds number : $Re = \frac{\rho V L}{\mu} = \frac{V L}{\nu}$. Ratio of inertia forces to viscous forces.
- Euler number : $Eu = \frac{\Delta p}{\frac{1}{2}\rho V^2}$. Ratio of pressure forces to inertia forces.
- Froude number : $Fr = \frac{V}{\sqrt{gL}}$. Squareroot of the ratio of inertia forces to gravitational forces.
- Mach number : $Ma = \frac{V}{\sqrt{E_v/\rho}} = \frac{V}{c}$. Squareroot of the ratio of inertia forces to compressibility forces.
- Weber number : $We = \sqrt{\frac{\rho V^2 L}{\sigma}}$. Squareroot of the ratio of inertia forces to surface tension forces.
- Strouhal number : $St = \frac{\omega L}{V}$. Used for flows with oscillatory (periodic) behavior.
- Cavitation number : $Ca = \frac{p - p_v}{\frac{1}{2}\rho V^2}$. Used for possibly cavitating flows.

The compressible equations

Constitutive relation for compressible Newtonian fluids

$$\boldsymbol{\sigma} = 2\mu\Big(\dot{\boldsymbol{\varepsilon}} - \tfrac{1}{3}(\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}})\boldsymbol{I}\Big) - p\,\boldsymbol{I} = 2\mu\,\dot{\boldsymbol{\varepsilon}} - \Big(p + \tfrac{2}{3}\mu\,\mathrm{tr}\,\dot{\boldsymbol{\varepsilon}}\Big)\boldsymbol{I}\,,$$

and after using the definition for strain rate:

$$\boldsymbol{\sigma} = \mu\Big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\Big) - \Big(p + \tfrac{2}{3}\mu\,\nabla\cdot\boldsymbol{u}\Big)\boldsymbol{I} \quad \text{or} \quad \sigma_{ij} = \mu\big(u_{i|j} + u_{j|i}\big) - \Big(p + \tfrac{2}{3}\mu\,u_{k|k}\Big)\delta_{ij}\,.$$

Compressible Navier–Stokes equations of motion:

$$\varrho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \mu\,\triangle\boldsymbol{u} + \frac{\mu}{3}\nabla(\nabla\cdot\boldsymbol{u}) - \nabla p + \varrho\,\boldsymbol{g} \quad \text{or} \quad \varrho\,\frac{\mathrm{D}u_i}{\mathrm{D}t} = \mu\,u_{i|jj} + \frac{\mu}{3}u_{j|ji} - p_{|i} + \varrho\,g_i\,.$$

(+ the continuity equation:) $\dfrac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,\nabla\cdot\boldsymbol{u} = 0$ or $\dfrac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\,u_{i|i} = 0\,.$

- These equations are **incomplete** – there are only 4 relations for 5 unknown fields: ϱ, $\boldsymbol{u}$, p.
- They can be completed by a **state relationship between** ϱ **and** p.
- However, this would normally introduce also another state variable: the temperature T, and that would involve the requirement for energy balance (yet another equations). Such approach is governed by the **complete Navier–Stokes equations for compressible flow**.
- More simplified yet complete set of equations can be used to describe an **isothermal flow with small compressibility**.

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \nabla \cdot \boldsymbol{u} = \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}) = 0.$

Momentum conservation: $\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} + \varrho \boldsymbol{g}, \quad$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

Energy conservation: $\frac{\mathrm{D}}{\mathrm{D}t}\left(\varrho e + \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}\right) = -\nabla \cdot \boldsymbol{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) + \varrho \boldsymbol{g} \cdot \boldsymbol{u} + h.$

Here: e is the *intrinsic energy* per unit mass, $\boldsymbol{q}$ is the *heat flux vector*, and h is the *power of heat source* per unit volume. Moreover, notice that the term $\frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}$ is the *kinetic energy*, $\nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u})$ is the *energy change due to internal stresses*, and $\varrho \boldsymbol{g} \cdot \boldsymbol{u}$ is the change of *potential energy* of gravity forces.

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{u}) = 0.$

Momentum conservation: $\varrho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma} + \varrho\boldsymbol{g},$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

Energy conservation: $\frac{\mathrm{D}}{\mathrm{D}t}\left(\varrho e + \frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u}\right) = -\nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{u}) + \varrho\boldsymbol{g}\cdot\boldsymbol{u} + h.$

Equations of state and constitutive relations:

- **Thermal equation of state**: $\varrho = \varrho(p, T)$.
- **Constitutive law for fluid**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}) - p\boldsymbol{I}$.
- **Thermodynamic relation** between state variables: $e = e(p, T)$.
  For a calorically perfect fluid: $e = c_V T$, where c_V is the *specific heat at constant volume*. This equation is sometimes called the *caloric equation of state*.

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho\nabla\cdot\boldsymbol{u} = \frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{u}) = 0.$

Momentum conservation: $\varrho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma} + \varrho\boldsymbol{g},$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\top}$).

Energy conservation: $\frac{\mathrm{D}}{\mathrm{D}t}\left(\varrho e + \frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u}\right) = -\nabla\cdot\boldsymbol{q} + \nabla\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{u}) + \varrho\boldsymbol{g}\cdot\boldsymbol{u} + h.$

Equations of state and constitutive relations:

- **Thermal equation of state**: $\varrho = \varrho(p,T)$.
- **Constitutive law for fluid**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u},p) = \boldsymbol{\tau}(\boldsymbol{u}) - p\boldsymbol{I}$.
- **Thermodynamic relation** between state variables: $e = e(p,T)$.
- **Heat conduction law**: $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{u},T)$.
  Fourier's law of conduction with convection: $\boldsymbol{q} = -k\nabla T + \varrho c\boldsymbol{u}T$, where k is the *thermal conductivity* and c is the *thermal capacity* (the *specific heat*).

Mass conservation: $\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \nabla \cdot \boldsymbol{u} = \frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{u}) = 0.$

Momentum conservation: $\varrho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} + \varrho \boldsymbol{g}, \quad$ (here: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{T}}$).

Energy conservation: $\frac{\mathrm{D}}{\mathrm{D}t}\left(\varrho e + \frac{1}{2}\varrho \boldsymbol{u} \cdot \boldsymbol{u}\right) = -\nabla \cdot \boldsymbol{q} + \nabla \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{u}) + \varrho \boldsymbol{g} \cdot \boldsymbol{u} + h.$

Equations of state and constitutive relations:

- **Thermal equation of state**: $\varrho = \varrho(p, T)$.
- **Constitutive law for fluid**: $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{\tau}(\boldsymbol{u}) - p\boldsymbol{I}$.
- **Thermodynamic relation** between state variables: $e = e(p, T)$.
- **Heat conduction law**: $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{u}, T)$.

- There are **5** conservation equations for **14** unknowns: ϱ, $\boldsymbol{u}$, $\boldsymbol{\sigma}$, e, $\boldsymbol{q}$.
- Constitutive and state relations provide another **11** equations and introduce **2** additional state variables: p, T.
- That gives the total number of **16** equations for **16** unknown field variables: ϱ, $\boldsymbol{u}$, $\boldsymbol{\sigma}$ (or $\boldsymbol{\tau}$), e, $\boldsymbol{q}$, p, T.

Reynolds equations (turbulence)

- **The majority of natural and engineering flows are turbulent, hence the necessity of modeling turbulence.**

Buoyant plume of smoke rising from a stick of incense
Photo credit: https://www.flickr.com/photos/jlhopgood/

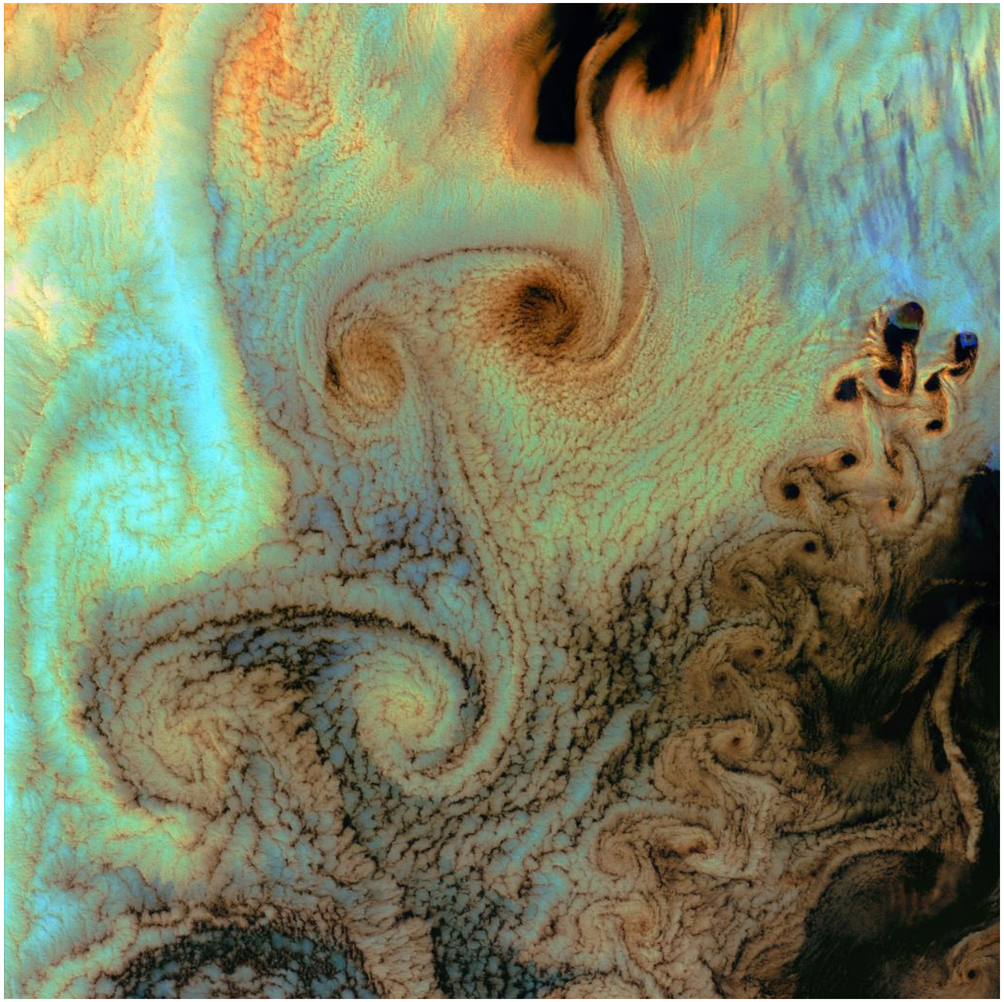
Buoyant plume of smoke rising from a stick of incense
Photo credit: M. Rosic

Tugboat riding on the turbulent wake of a ship
Photo credit: https://www.flickr.com/photos/oneeighteen/

Turbulent waters
Photo credit: https://www.flickr.com/photos/thepaegan

Spring vortex in turbulent waters
Photo credit: https://www.flickr.com/photos/kenii/

Wake turbulence behind individual wind turbines
Photo credit: NREL's wind energy research group.

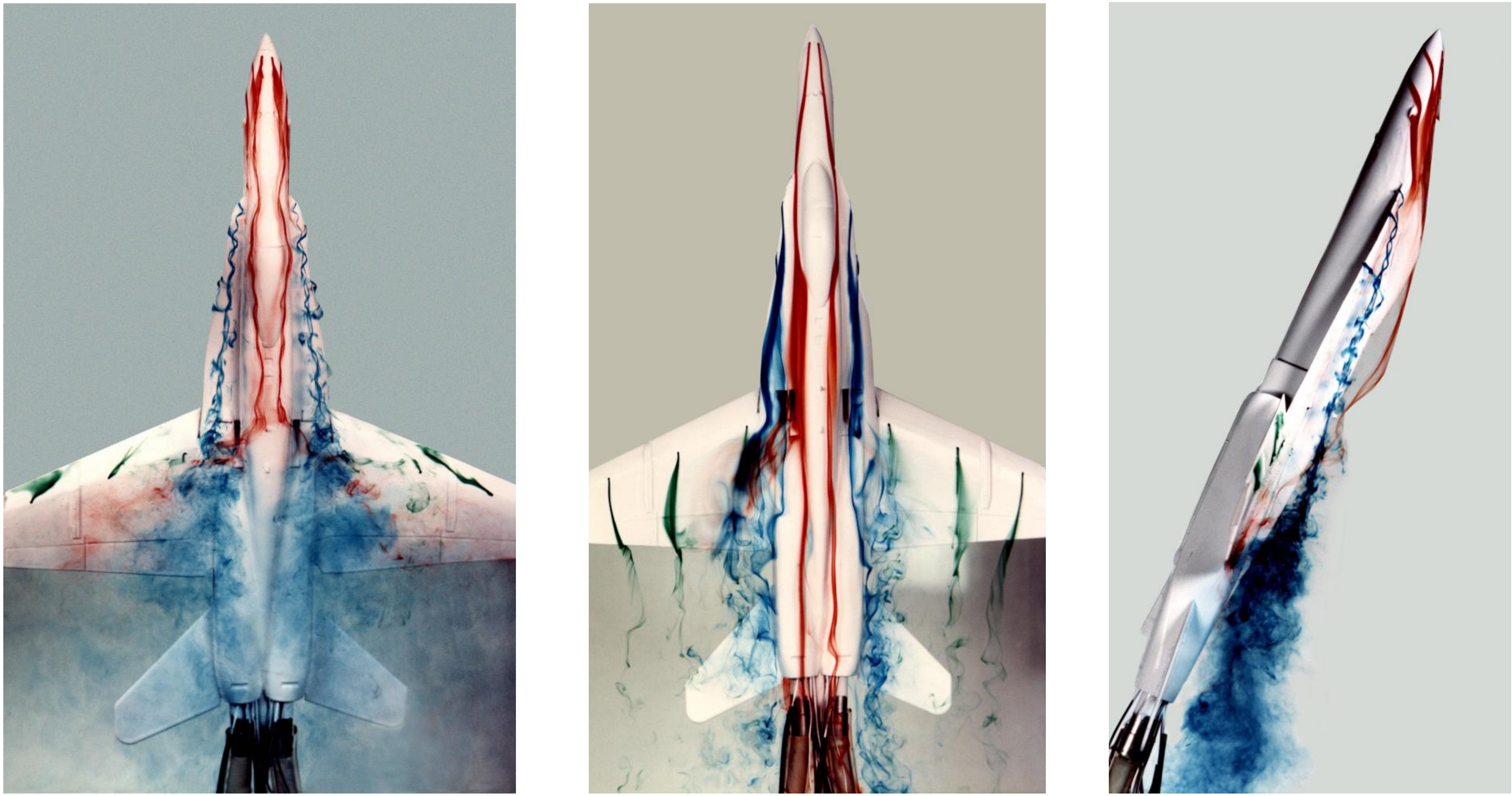


Von Karman vortices created when prevailing winds sweeping east across the northern Pacific Ocean encounters Alaska's Aleutian Islands

Photo credit: USGS EROS Data Center Satellite Systems Branch.

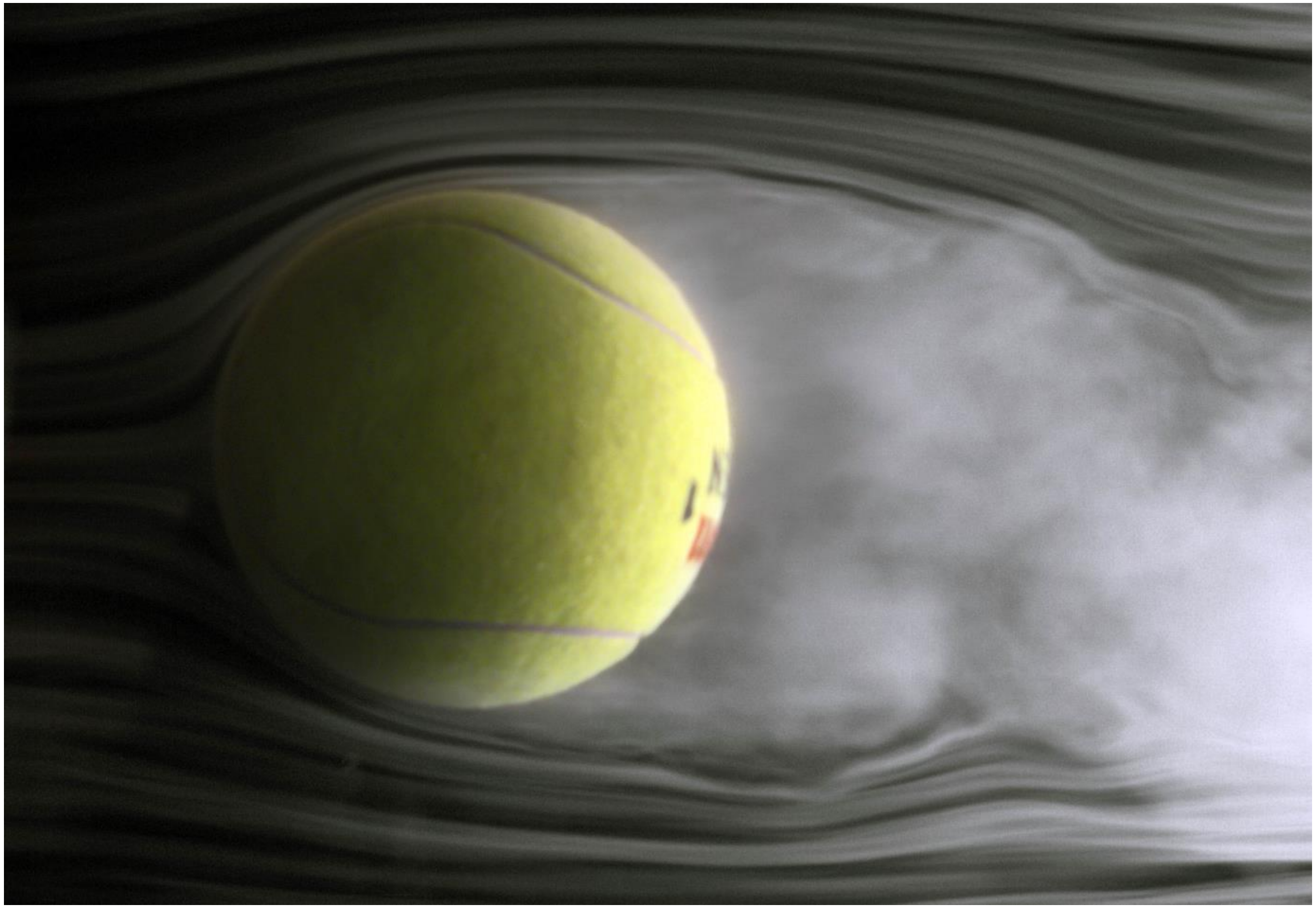
Von Karman Vortex Streets in the northern Pacific Photographed from the International Space Station
Photo credit: NASA

Trailing vortices
Photo credit: Steve Morris. AirTeamImages.

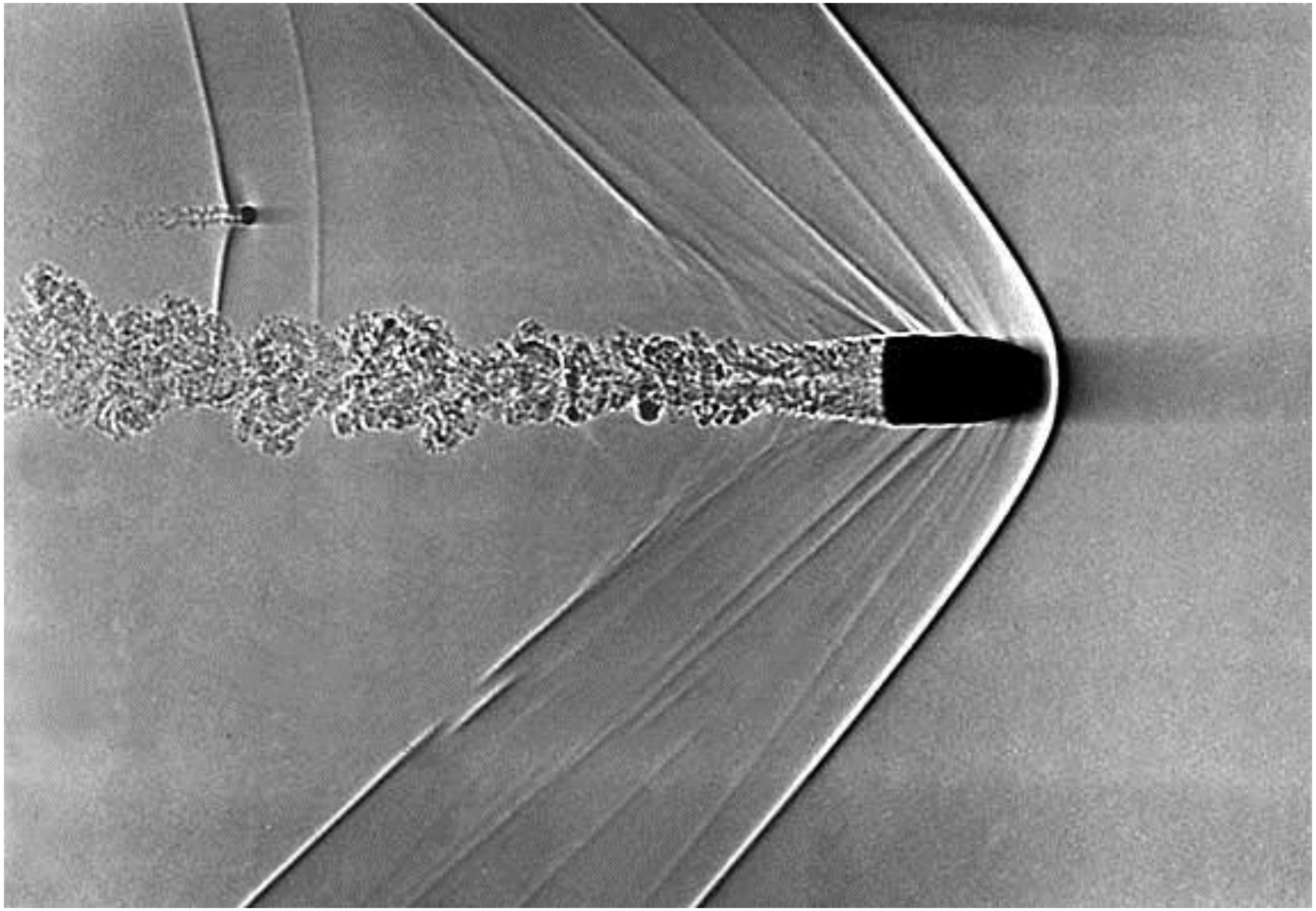


Vortices on a 1/48-scale model of an F/A-18 aircraft inside a Water Tunnel

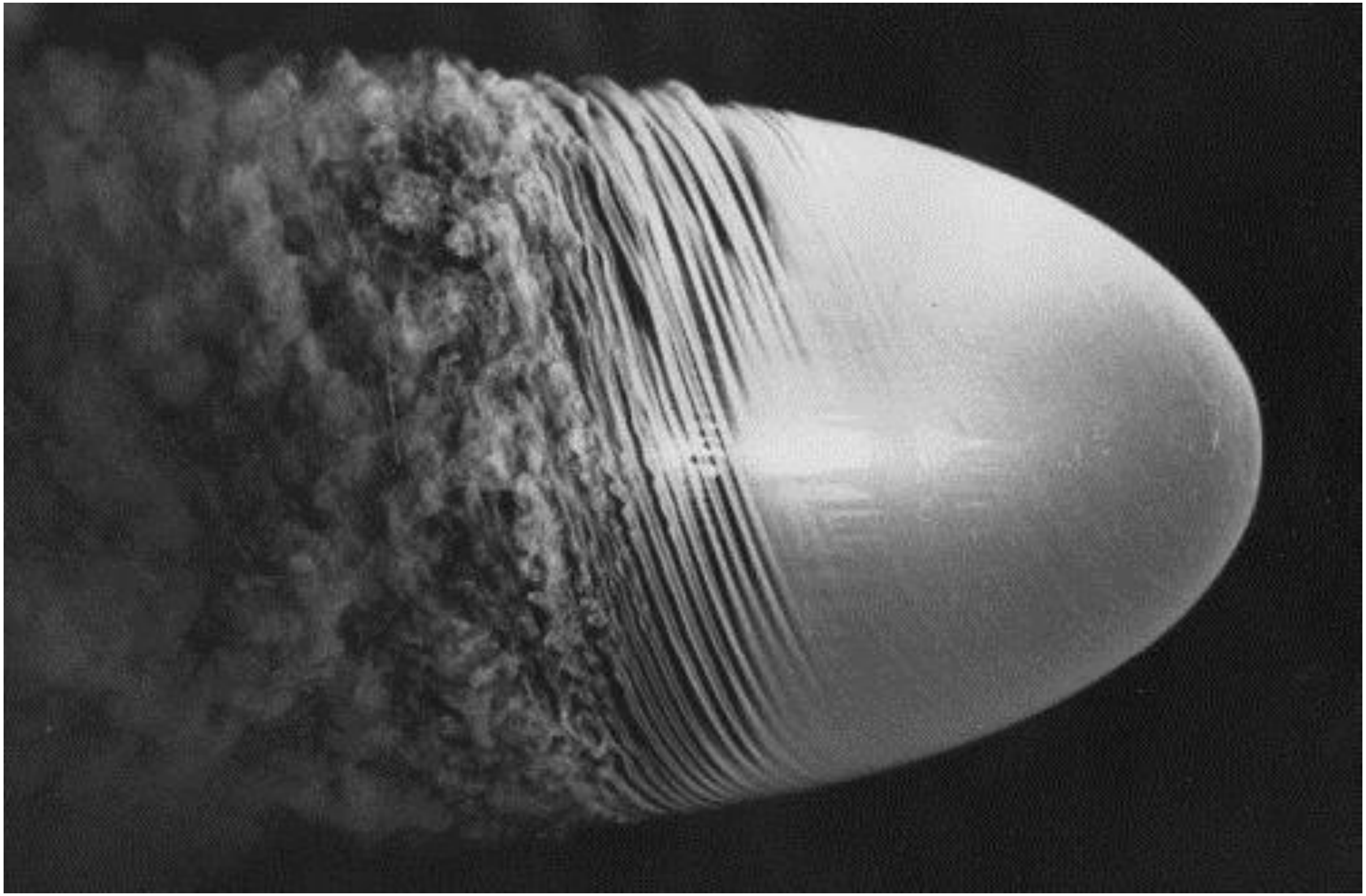
Photo credit: NASA Dryden Flow Visualization Facility. http://www.nasa.gov/centers/armstrong/multimedia/imagegallery/FVF



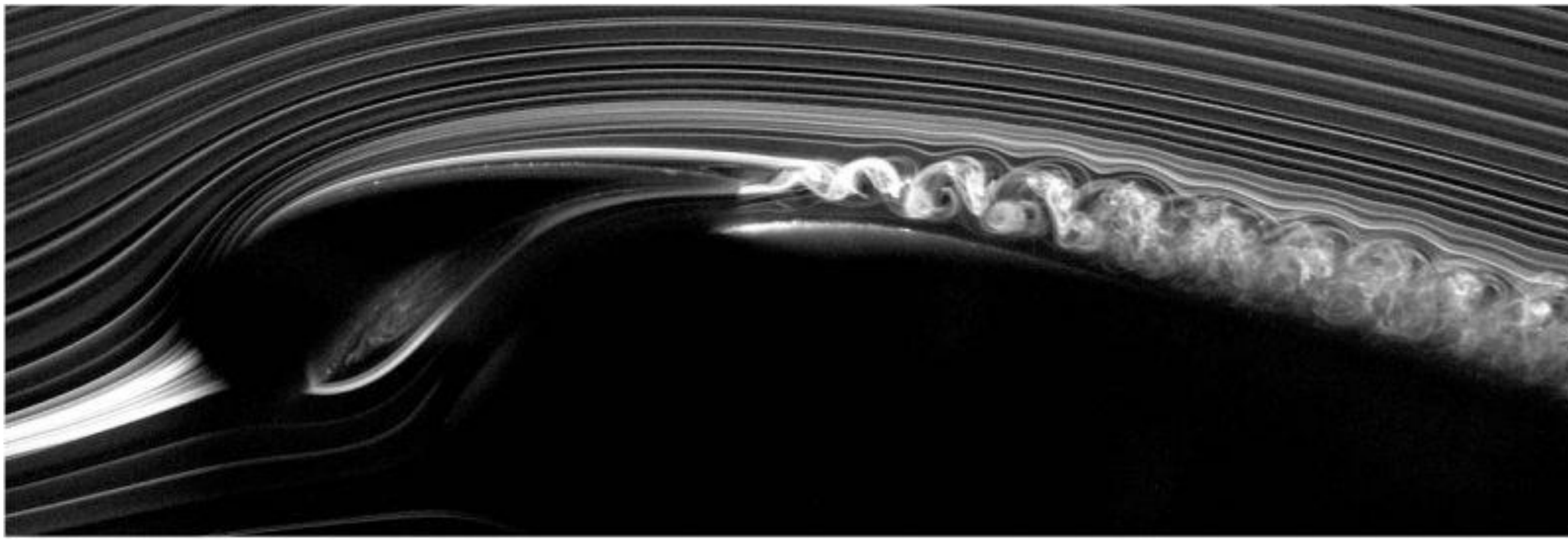
Wind Tunnel Test of New Tennis Ball
Photo credit: NASA http://tennisclub.gsfc.nasa.gov/tennis.windtunnelballs.html



Bullet at Mach 1.5
Photo credit: Andrew Davidhazy. Rochester Institute of Technology.



Flow visualization over a spinning spheroid
Photo credit: Y. Kohama.



Flow around an airfoil with a leading-edge slat
Photo credit: S. Makiya et al.

- Modeling turbulence is not an easy task.

- There is no universal turbulence model, hence you need to know the capabilities and limitations of the turbulence models.

- Turbulence can be wall bounded or free shear. Depending of what you want to simulate, you will need to choose an appropriate turbulence model and simulation techniques.

- Simulating turbulent flows in any CFD solver requires selecting a turbulence model, providing initial conditions and boundary conditions for the closure equations of the turbulent model, and selecting a near-wall modeling.

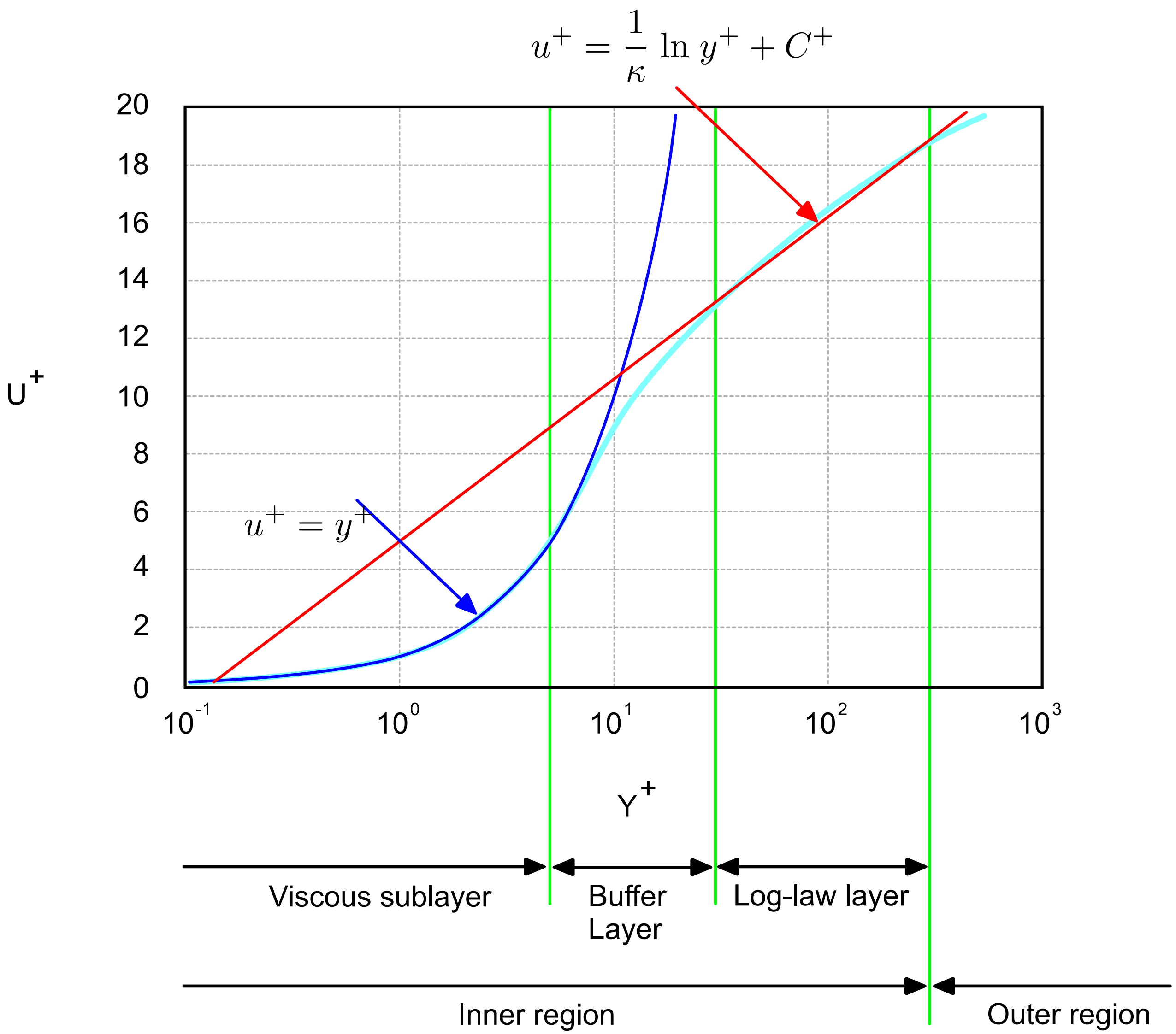
- You also need to choose the discretization scheme of the turbulent equations.

“Essentially, all models are wrong,
but some are useful”

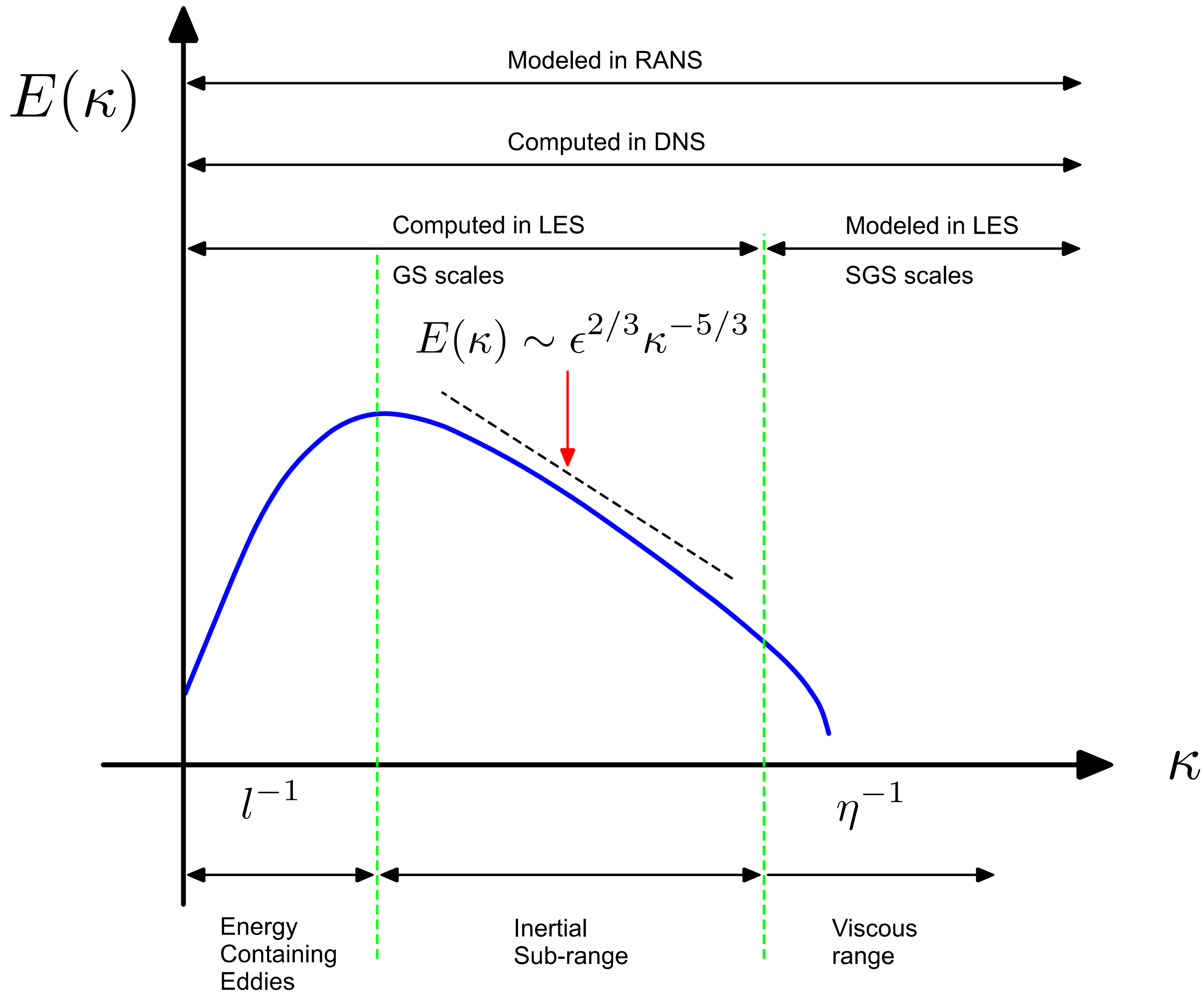
G.E.P Box

- Equations cannot be derived from fundamental principles.
- All turbulence models contain some sort of empiricism.
- Some calibration to observed physical solutions is contained in the turbulence models.
- Also, some intelligent guessing is used.
- A lot of uncertainty is involved!

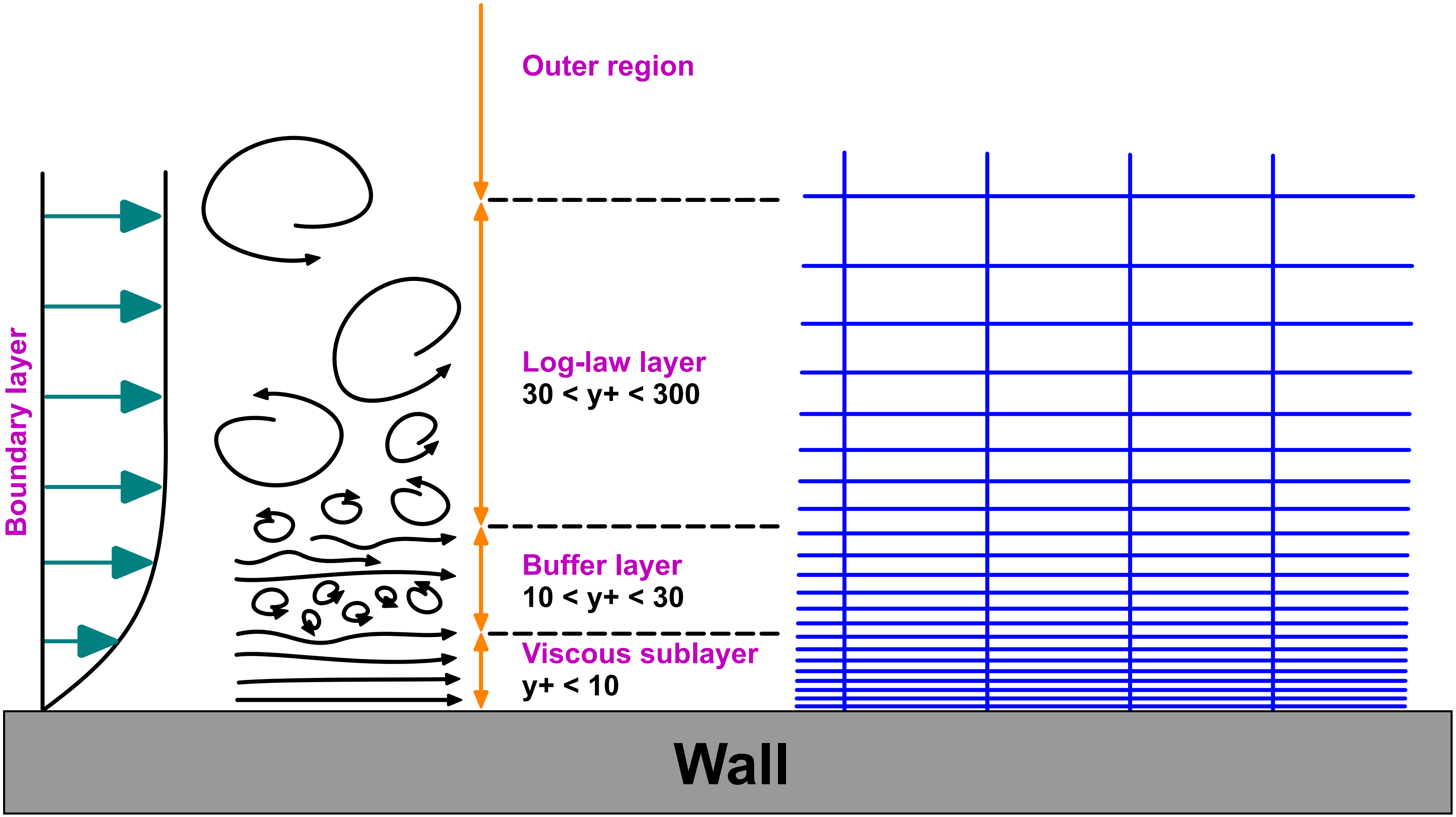
Turbulence near the wall - Law of the wall



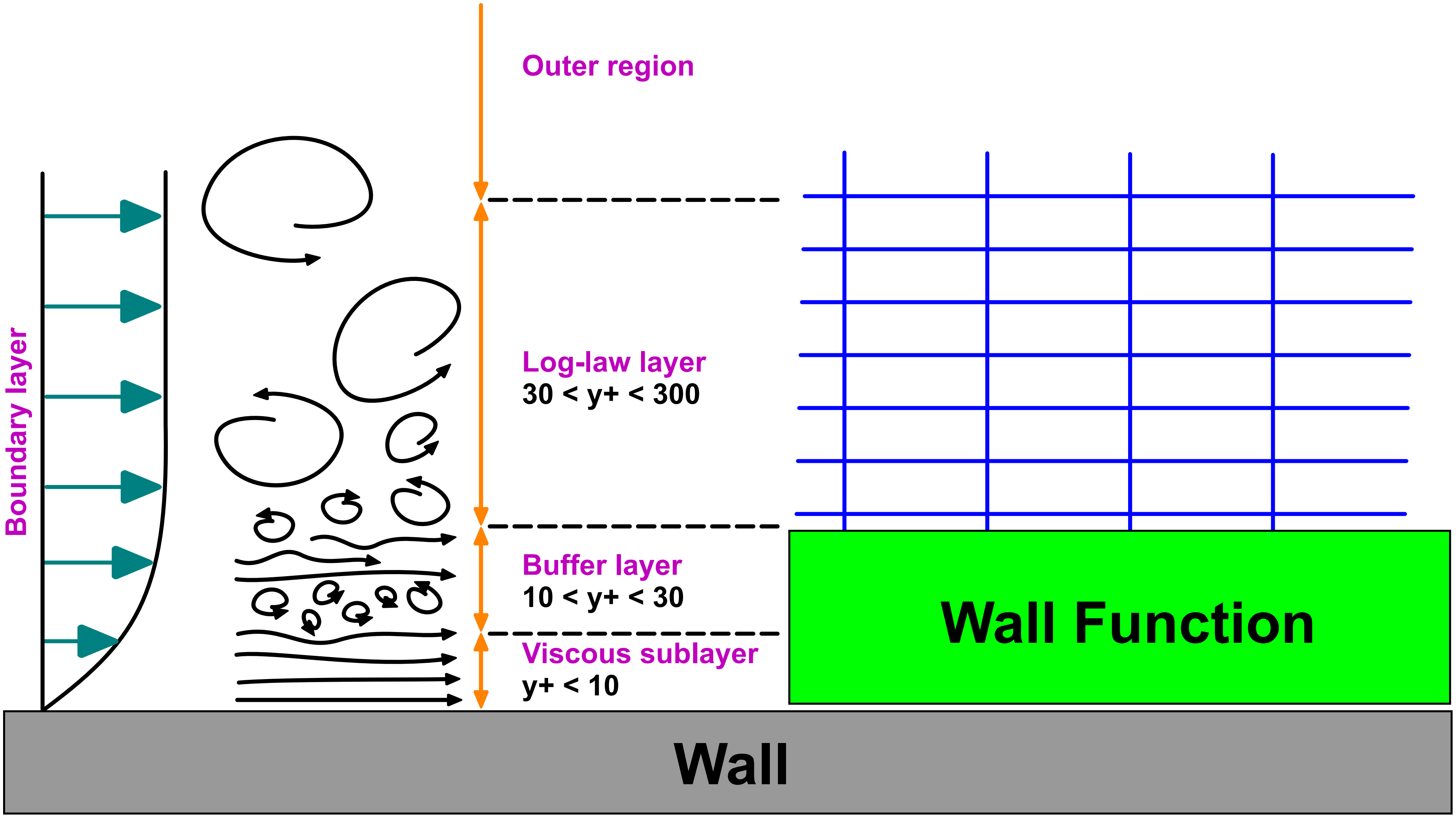
Energy spectrum for a turbulent flow – log-log scales



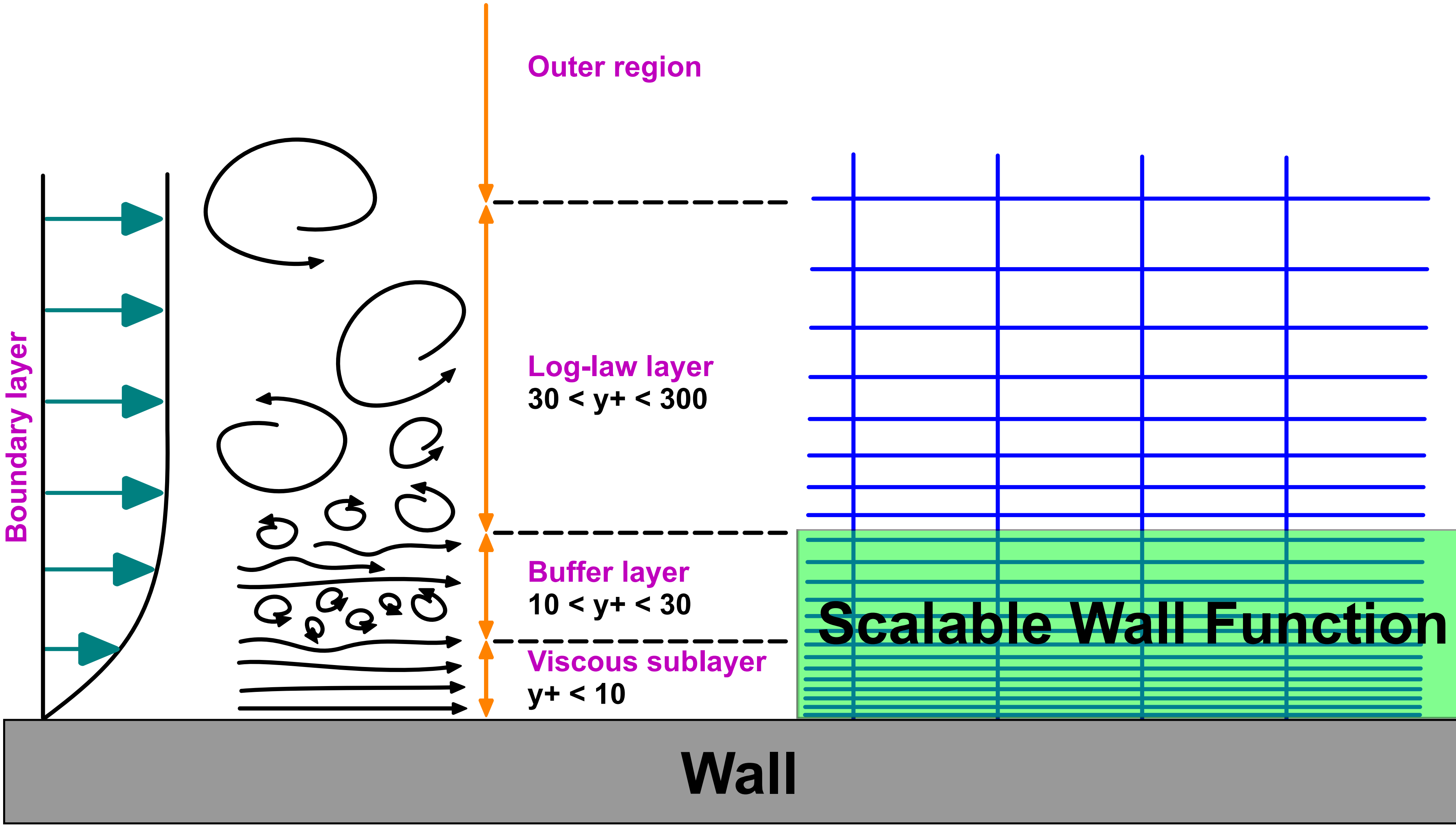
Turbulence near the wall



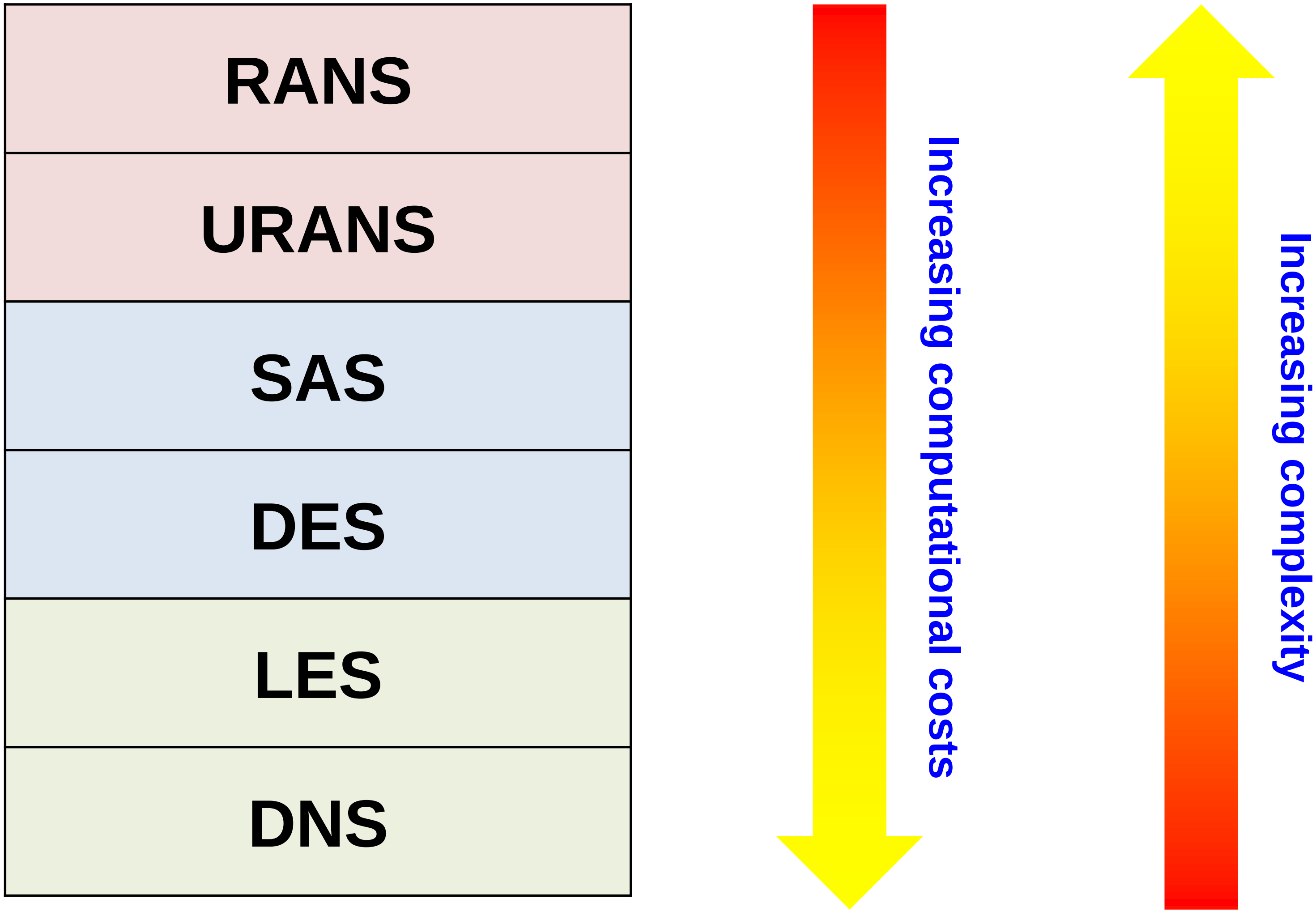
Turbulence near the wall



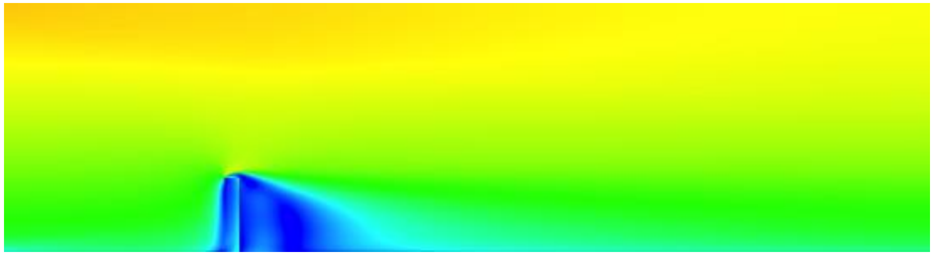
Turbulence near the wall



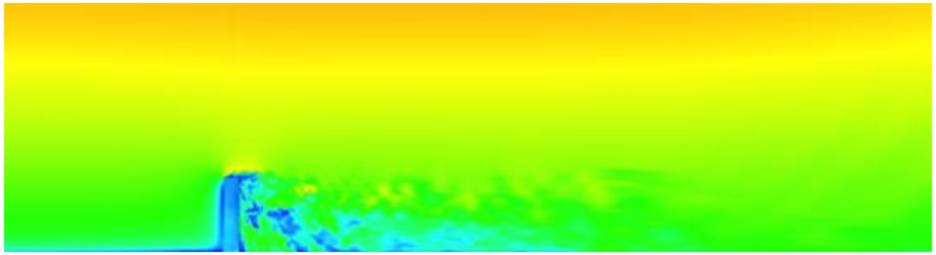
Overview of turbulence modeling approaches



Overview of turbulence modeling approaches



RANS



LES

RANS/URANS	DES/LES	DNS
• Solves the time-averaged NSE. • All turbulent spatial scales are modeled. • Many models are available. One equation models, two equation models, Reynolds stress models, transition models, and so on. • This is the most widely approach for industrial flows. • Unsteady RANS (URANS), use the same equations as the RANS but with the transient term retained. • It can be used in 2D and 3D cases.	• Solves the filtered unsteady NSE. • SGS scales are modeled, GS are resolved. • Resolves the temporal scales, hence requires small time-steps. • For most industrial applications, it is computational expensive. However, thanks to the current advances in parallel and scientific computing it is becoming affordable. • Many models are available. • It is intrinsically 3D and asymmetric.	• Solves the unsteady laminar NSE. • Solves all spatial and temporal scales; hence, requires extremely fine meshes and small time-steps. • No modeling is required. • It is extremely computational expensive. • Not practical for industrial flows. • It is intrinsically 3D and asymmetric.

Short description of some of the turbulence models available in commercial CFD codes

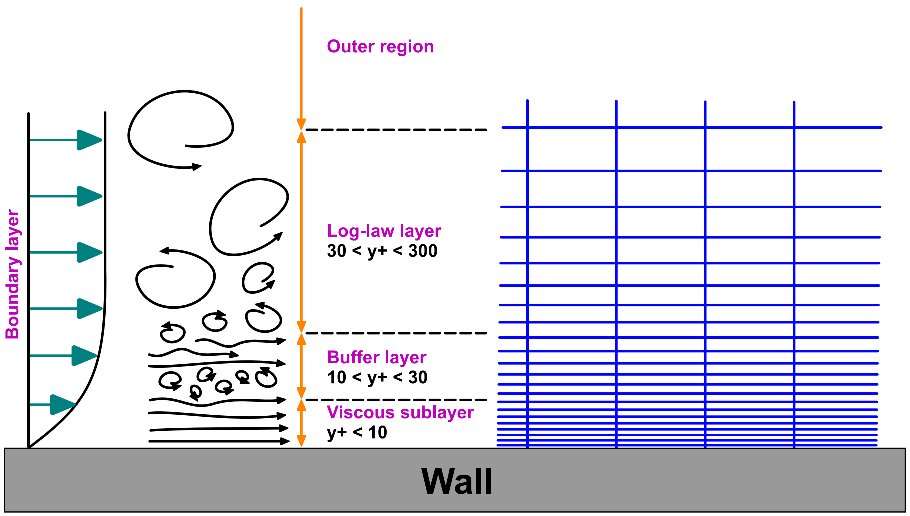
Model	Short description
Spalart-Allmaras	Suitable for external aerodynamics, tubomachinery and high speed flows. Good for mildly complex external/internal flows and boundary layer flows under pressure gradient. Performs poorly for free shear flows and flows with strong separation.
Standard k–epsilon	Robust. Widely used despite the known limitations of the model. Performs poorly for complex flows involving severe pressure gradient, separation, strong streamline curvature. Suitable for initial iterations, initial screening of alternative designs, and parametric studies.
Realizable k–epsilon	Suitable for complex shear flows involving rapid strain, moderate swirl, vortices, and locally transitional flows (e.g. boundary layer separation, massive separation, and vortex shedding behind bluff bodies, stall in wide-angle diffusers, room ventilation). It overcome the limitations of the standard k-epsilon model.
Standard k–omega	Superior performance for wall-bounded boundary layer, free shear, and low Reynolds number flows compared to models from the k-epsilon family. Suitable for complex boundary layer flows under adverse pressure gradient and separation (external aerodynamics and turbomachinery).
SST k–omega	Offers similar benefits as standard k–omega. Not overly sensitive to inlet boundary conditions like the standard k–omega. Provides more accurate prediction of flow separation than other RANS models.

Turbulence models available in commercial CFD software

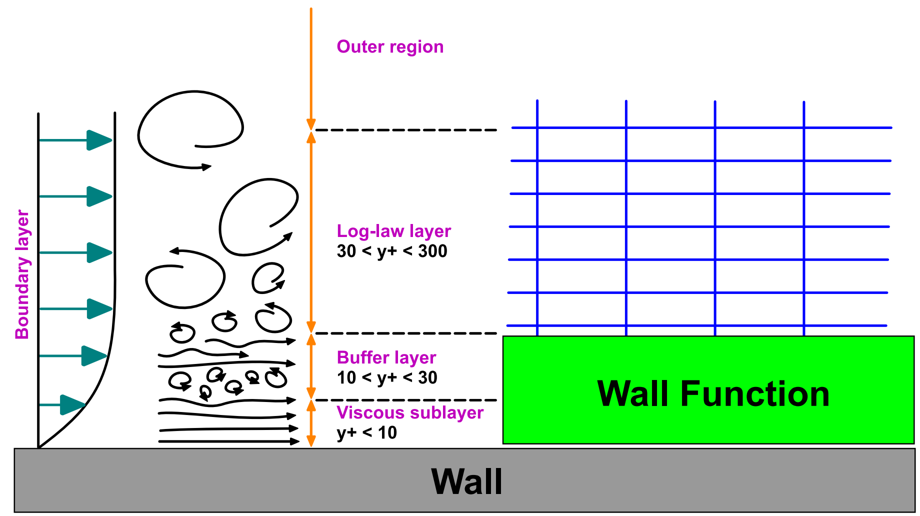
- If you have absolutely no idea of what model to use, I highly recommend you the $\kappa - \omega$ family models or the realizable $\kappa - \epsilon$
- Remember, when a turbulent flow enters a domain, turbulent boundary conditions and initial conditions must be specified.
- Also, if you are dealing with wall bounded turbulence you will need to choose the near-wall treatment. You can choose to solve the viscous sub-layer or use wall functions.
- You will need to give the appropriate boundary conditions to the near-wall treatment.

Near-wall treatment and wall functions

Near-wall treatment and wall functions



No wall-functions



Wall-functions

- Generally speaking, wall functions is the approach to use if you are more interested in the mixing in the outer region, rather than the forces on the wall.
- By the way, wall functions should never be used if $y^+ < 30$. What is y^+? We are going to talk about this later on.
- If accurate prediction of forces or heat transfer on the wall are key to your simulation (aerodynamic drag, turbomachinery blade performance, heat transfer) you might not want to use wall functions.

Near-wall treatment and wall functions

- If the first node normal to the wall is in the viscous sub-layer region, you do not use wall functions. This approach is computationally expensive.
- Instead, if the first node normal to the wall is in the log-law layer, you need to use wall functions.
- When positioning the first node normal to the wall, try to avoid as much as possible the buffer layer, this is the transition region and things are not very clear there.

$\kappa - \omega$ Turbulence model overview

$\kappa - \omega$ Turbulence model overview

- It is called $\kappa - \omega$ because it solves two additional equations for modeling the turbulence, namely, the turbulent kinetic energy κ and the specific kinetic energy ω.

$$\rho \frac{\partial \kappa}{\partial t} + \rho \nabla \cdot (\bar{\mathbf{u}} \kappa) = \tau^R \mathbf{:} \nabla \bar{\mathbf{u}} - \beta^* \rho \kappa \omega + \nabla \cdot \left[\left(\mu + \sigma^* \mu_T \right) \nabla \kappa \right]$$

$$\rho \frac{\partial \omega}{\partial t} + \rho \nabla \cdot (\bar{\mathbf{u}} \omega) = \alpha \frac{\omega}{\kappa} \tau^R \mathbf{:} \nabla \bar{\mathbf{u}} - \beta \rho \omega^2 + \nabla \cdot \left[\left(\mu + \sigma \mu_T \right) \nabla \omega \right]$$

$\kappa - \omega$ Turbulence model overview

- At the end of the day, we want to determine the turbulent eddy viscosity

$$\mu_T = \frac{\rho\kappa}{\omega}$$

- The turbulent eddy viscosity is used to compute the Reynolds stress tensor,

$$\tau^R = -\rho\left(\overline{\mathbf{u}'\mathbf{u}'}\right) = 2\mu_T \bar{\mathbf{D}}^R - \frac{2}{3}\rho\kappa\mathbf{I} = \mu_T\left[\nabla\overline{\mathbf{u}} + \left(\nabla\overline{\mathbf{u}}\right)^{\mathrm{T}}\right] - \frac{2}{3}\rho\kappa\mathbf{I},$$

- The Reynolds stress tensor is derived from the Boussinesq Approximation.

Incompressible RANS equations

- The following equations are the incompressible Reynolds-Averaged Navier-Stokes equations (RANS). These are the equations we want to solve.

$$\nabla \cdot (\bar{\mathbf{u}}) = 0,$$

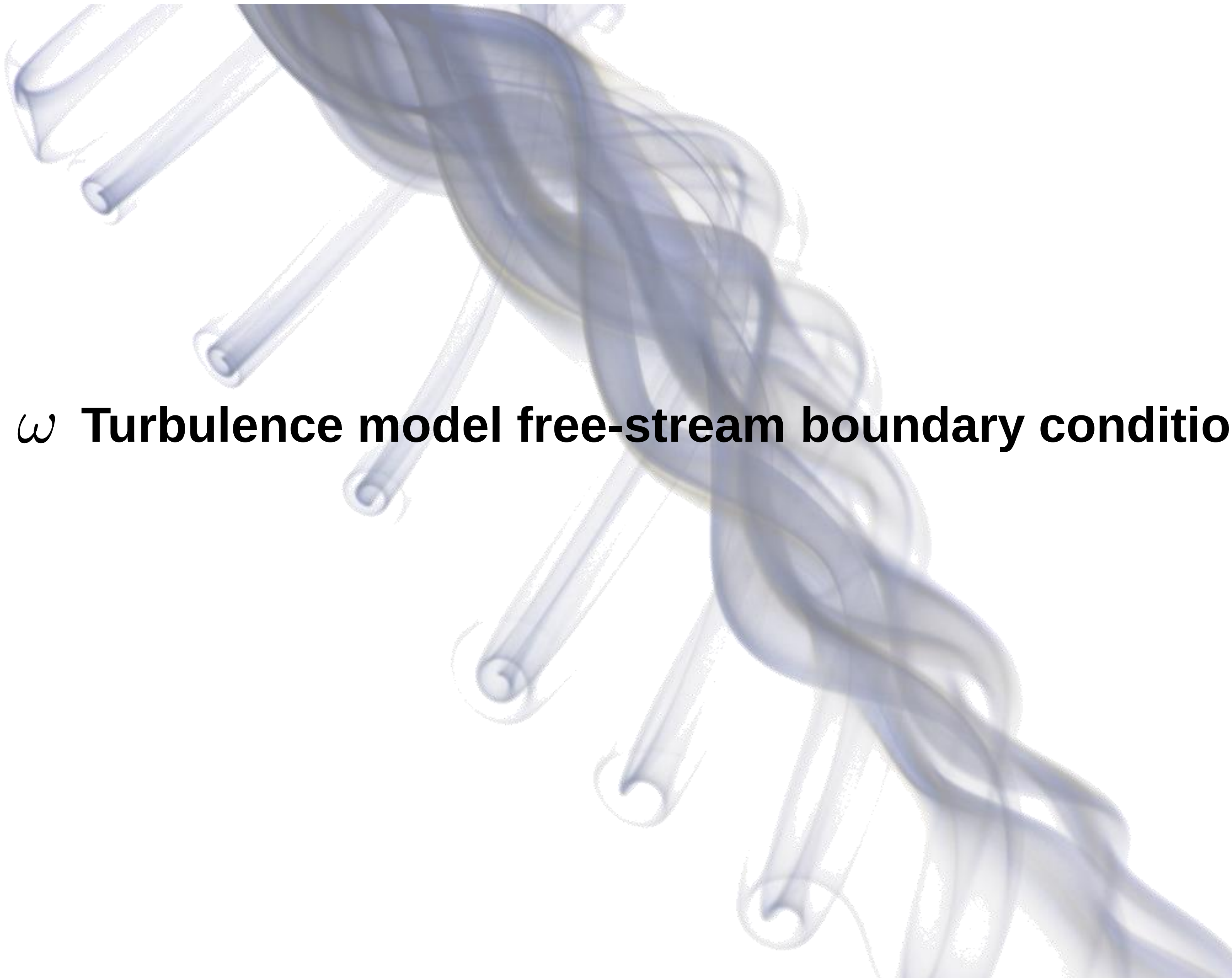
$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) = \frac{-\nabla \bar{p}}{\rho} + \nu \nabla^2 \bar{\mathbf{u}} - \frac{1}{\rho} \nabla \cdot \tau^R.$$

where τ^R is the Reynolds stress tensor and is given by,

$$\tau^R = -\rho \left(\overline{\mathbf{u}'\mathbf{u}'} \right) = 2\mu_T \bar{\mathbf{D}}^R - \frac{2}{3}\rho\kappa\mathbf{I} = \mu_T \left[\nabla \overline{\mathbf{u}} + \left(\nabla \overline{\mathbf{u}} \right)^{\mathrm{T}} \right] - \frac{2}{3}\rho\kappa\mathbf{I},$$

- To arrive to these equations, we used Reynolds averaging.

$\kappa - \omega$ Turbulence model free-stream boundary conditions



$\kappa - \omega$ Turbulence model free-stream boundary conditions

- The initial value for the turbulent kinetic energy κ can be found as follows

$$\kappa = \frac{3}{2}(UI)^2$$

- The initial value for the specific kinetic energy ω can be found as follows

$$\omega = \frac{\rho\kappa}{\mu}\frac{\mu_t}{\mu}^{-1}$$

- Where $\dfrac{\mu_t}{\mu}$ is the viscosity ratio and $I = \dfrac{u'}{\bar{u}}$ is the turbulence intensity.

$\kappa - \omega$ Turbulence model free-stream boundary conditions

- If you are totally lost, you can use these reference values. They work most of the times, but it is a good idea to have some experimental data or initial estimate.

	Low	Medium	High
I	1.0 %	5.0 %	10.0 %
μ_t/μ	1	10	100

- By the way, use these guidelines for external aerodynamics only.

$\kappa - \omega$ Turbulence model wall functions

$\kappa - \omega$ Turbulence model wall functions

- Follow these guidelines if you are struggling to find the boundary conditions for the near-wall treatment.
- I highly recommend you to read the source code and find the references used to implement the model.
- As for the free-stream boundary conditions, you need to give the boundary conditions for the near-wall treatment.
- When it comes to near-wall treatment, you have three options:
    - Use wall functions:

    $$30 \leq y^+ \leq 300$$

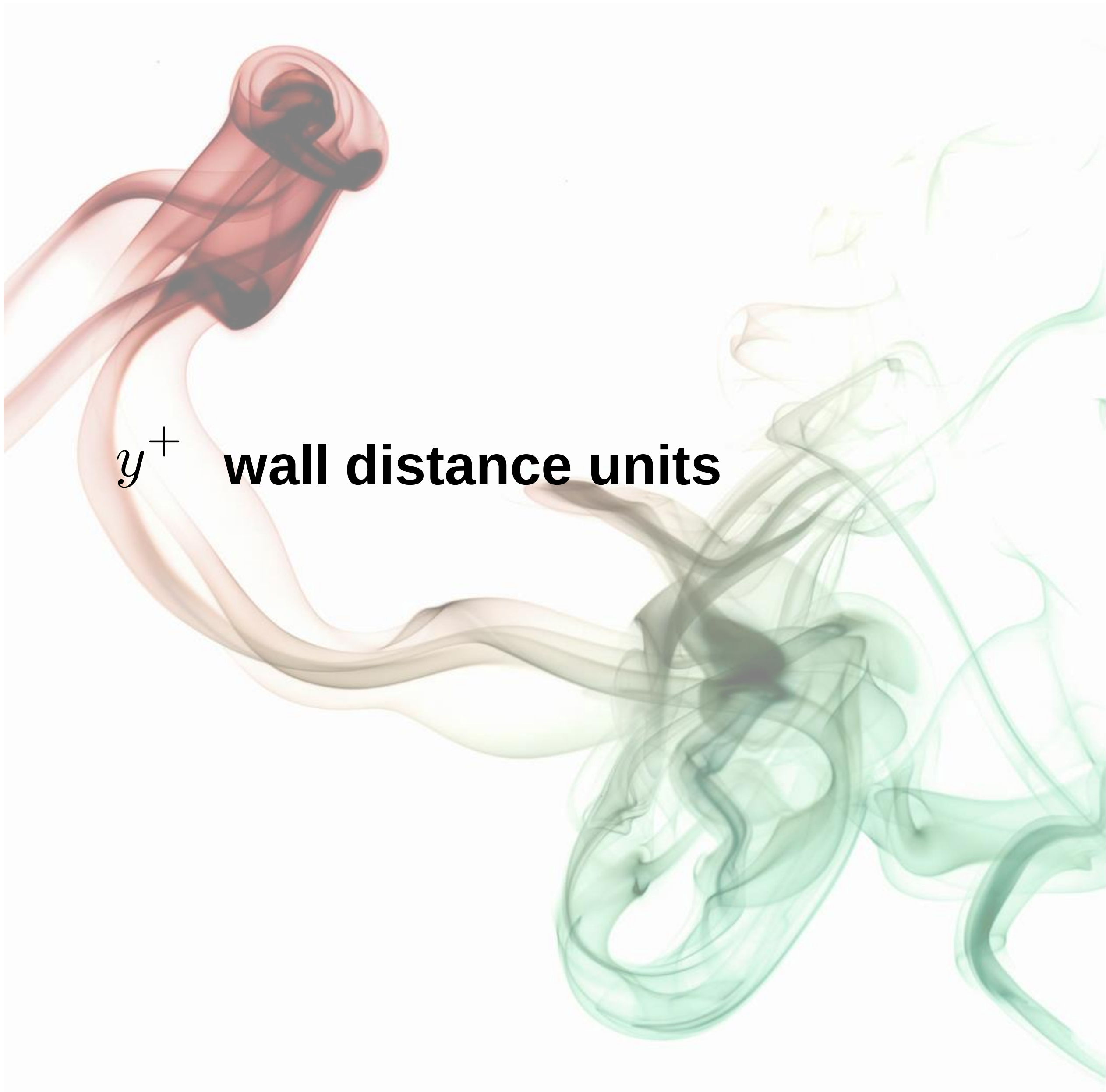
    - Use scalable wall functions, this only applies with the $\kappa - \omega$ family models:

    $$1 \leq y^+ \leq 300$$

    - Resolve the boundary layer (no wall functions):

    $$y^+ \leq 6$$

y^+ wall distance units



y^+ wall distance units

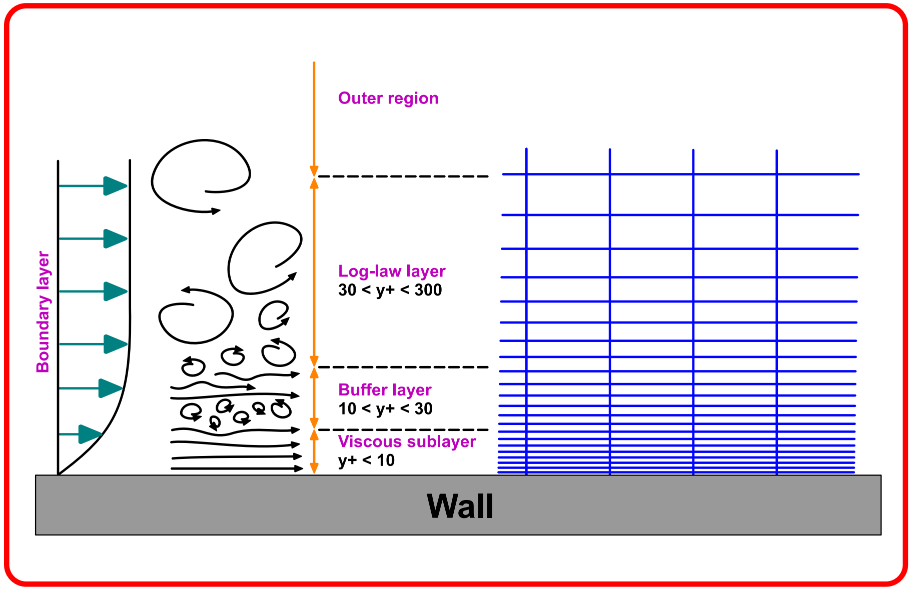
- To compute the wall distance units y^+, use the following equation

$$y^+ = \frac{\rho \times U_\tau \times y}{\mu}$$

- Where y is the distance to the first cell center normal to the wall, and U_τ is the friction velocity and is equal to

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

where τ_w is the wall shear stresses.



y^+ wall distance units

- We never know a priori the y^+ value.
- What we usually do is to run the simulation for a few time-steps or iterations, and then we get an estimate of the y^+ value.
- After determining where we are in the boundary layer (viscous sub-layer, buffer layer or log-law layer), we take the mesh as a good one or we modify it if is deemed necessary.
- It is an iterative process and it can be very time consuming.
- Have in mind that it is quite difficult to get uniform y^+ values at the walls. This does not mean that what you have done is wrong, use common sense.
- To get an initial estimate of the distance from the wall to the first cell center y, you can proceed as follows,

Estimating normal wall distance

- To estimate the distance from the wall to the first cell center y you can proceed as follows,

1. $$Re = \frac{\rho \times U \times L}{\mu}$$

2. $$C_f = 0.058 \times Re^{-0.2}$$

Skin friction coefficient of a flat plate

3. $$\tau_w = \frac{1}{2} \times C_f \times \rho \times U_\infty^2$$

4. $$U_\tau = \sqrt{\frac{\tau_w}{\rho}}$$

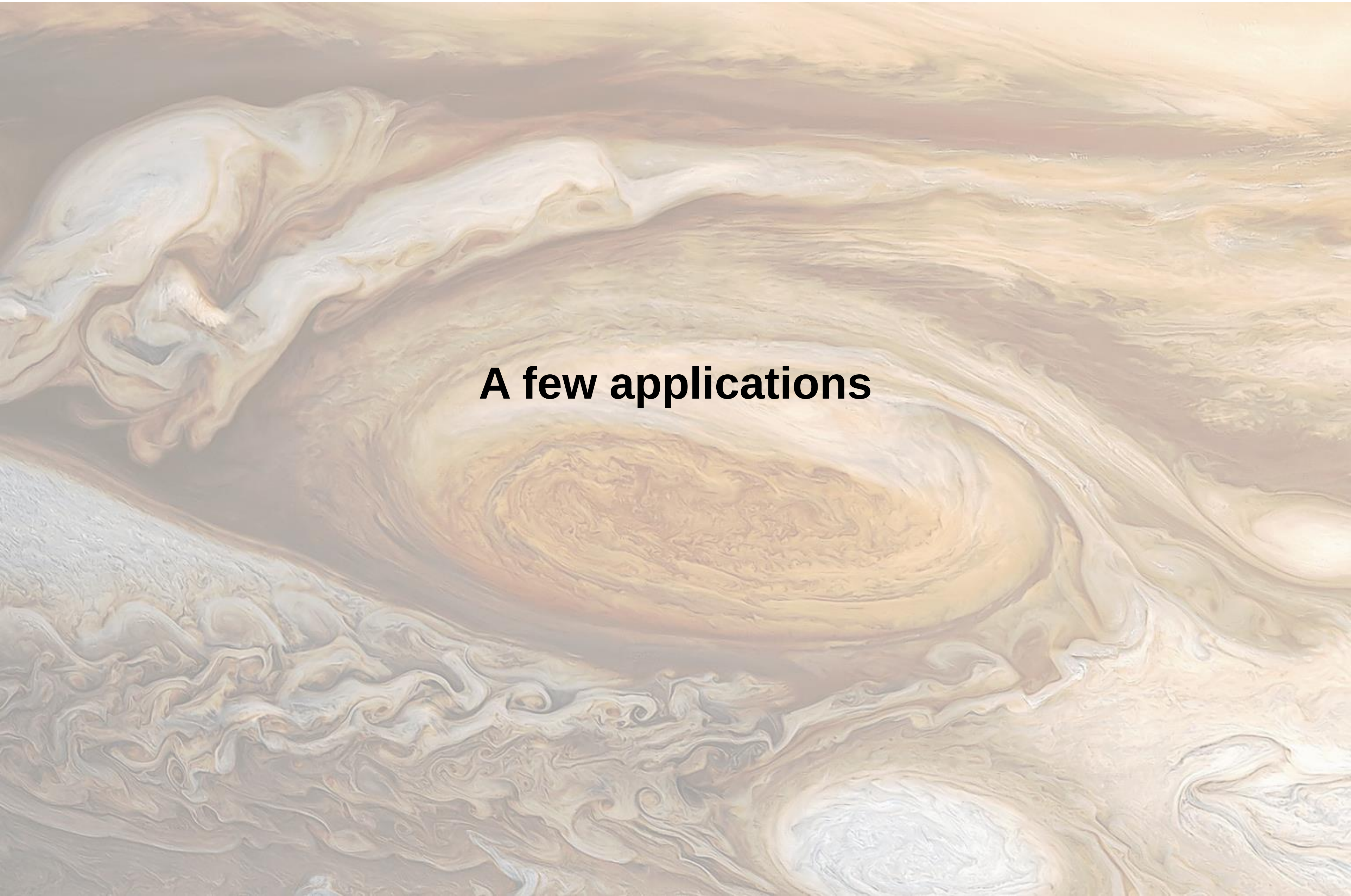
5. $$y^+ = \frac{\rho \times U_\tau \times y}{\mu} \quad \Rightarrow \quad y = \frac{\mu \times y^+}{\rho \times U_\tau}$$

Turbulence modeling guidelines and tips

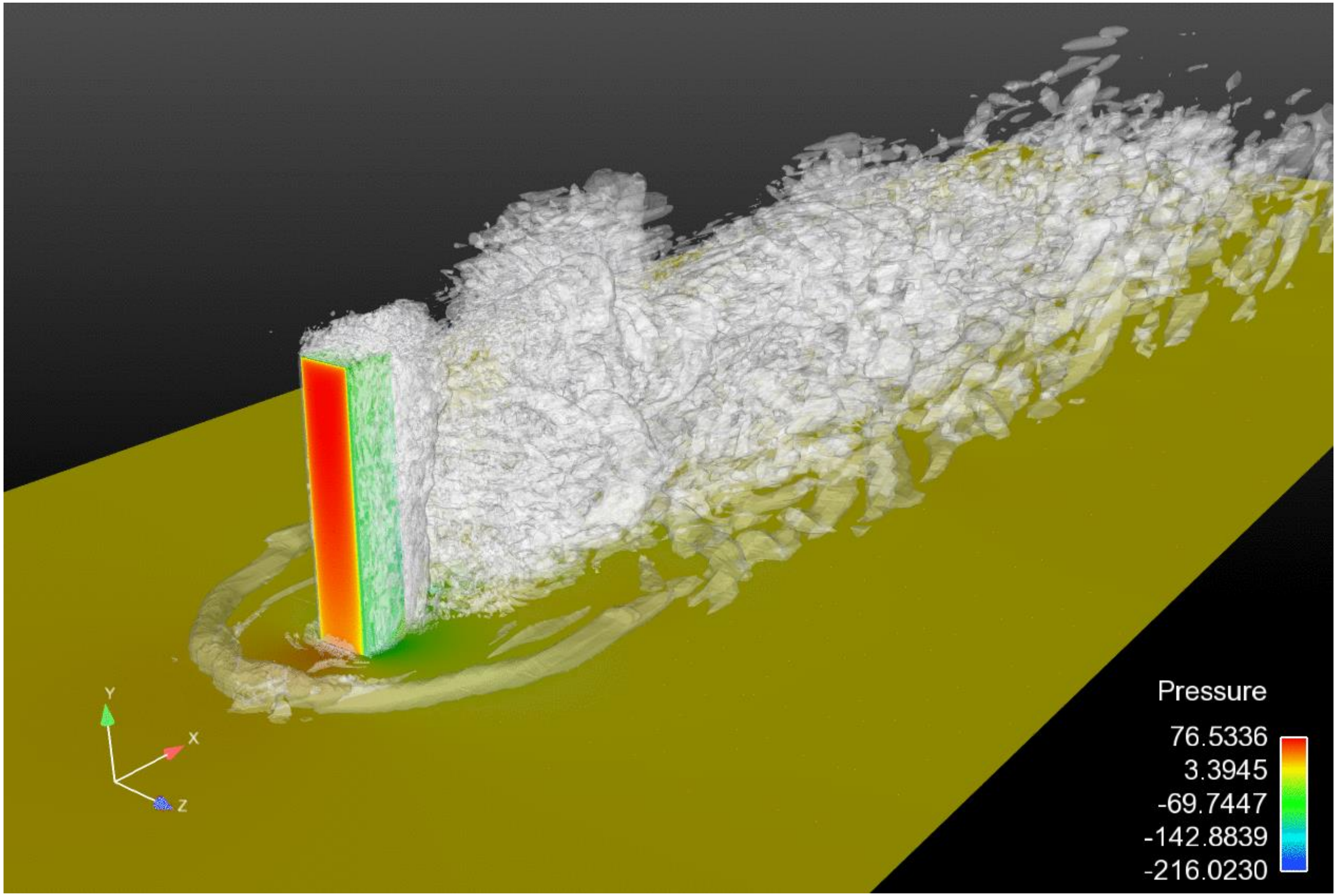
Turbulence modeling guidelines and tips

- Compute Reynolds number and determine whether the flow is turbulent.
- Estimate y before generating the mesh.
- Run the simulation for a few time steps and get a better prediction of y^+ and correct your initial prediction of y.
- The realizable $\kappa - \epsilon$ or $\mathrm{SST}\,\kappa - \omega$ models are good choices for general applications.
- If you are interesting in modeling the smallest eddies, DES or LES is the right choice.
- If you do not have any restriction in the near wall treatment method, use wall functions.
- Choose your near-wall modeling strategy ahead of time and check y^+ and y values to make sure the near-wall mesh is suitable.

A few applications

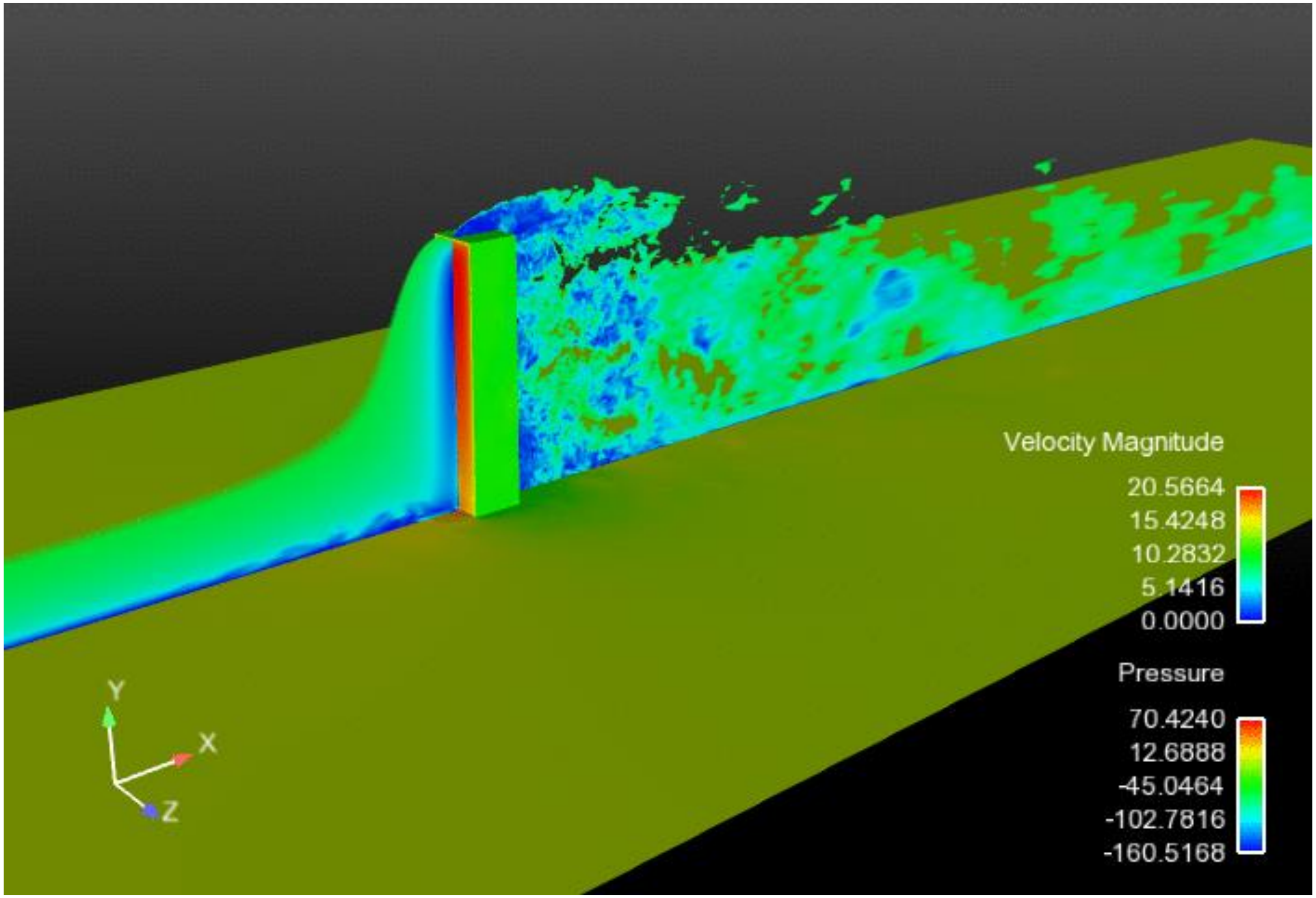


LES simulation



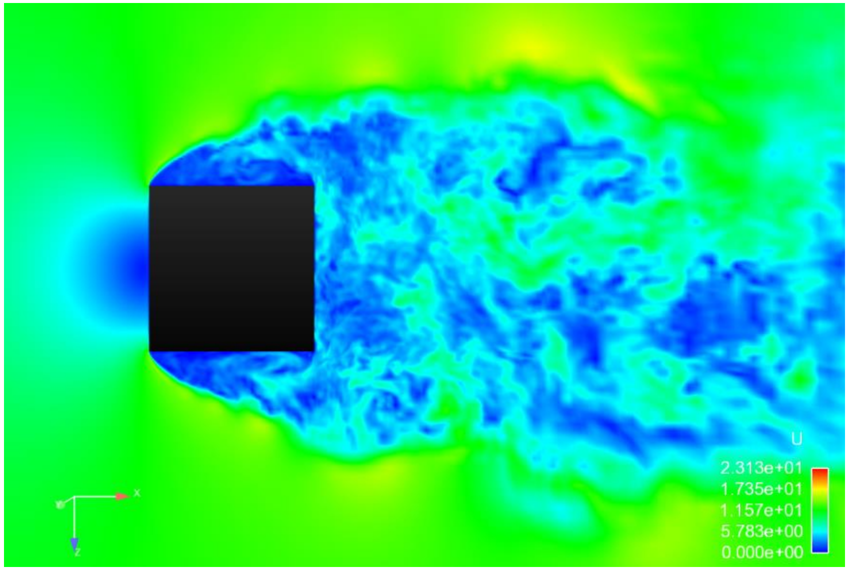
Iso-surfaces of Q criterion. Walls colored by instantaneous pressure.

LES simulation

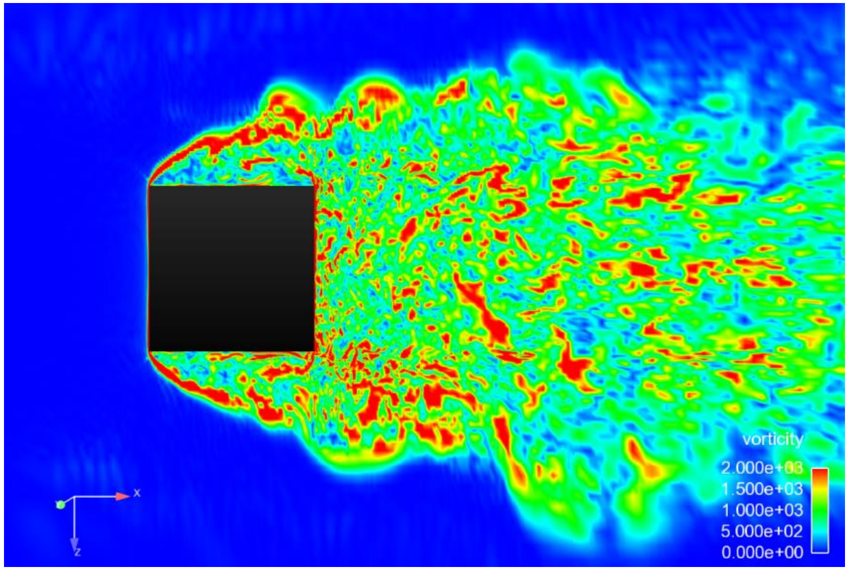


Cut plane with instantaneous velocity magnitude. Walls colored by instantaneous pressure.

LES simulation

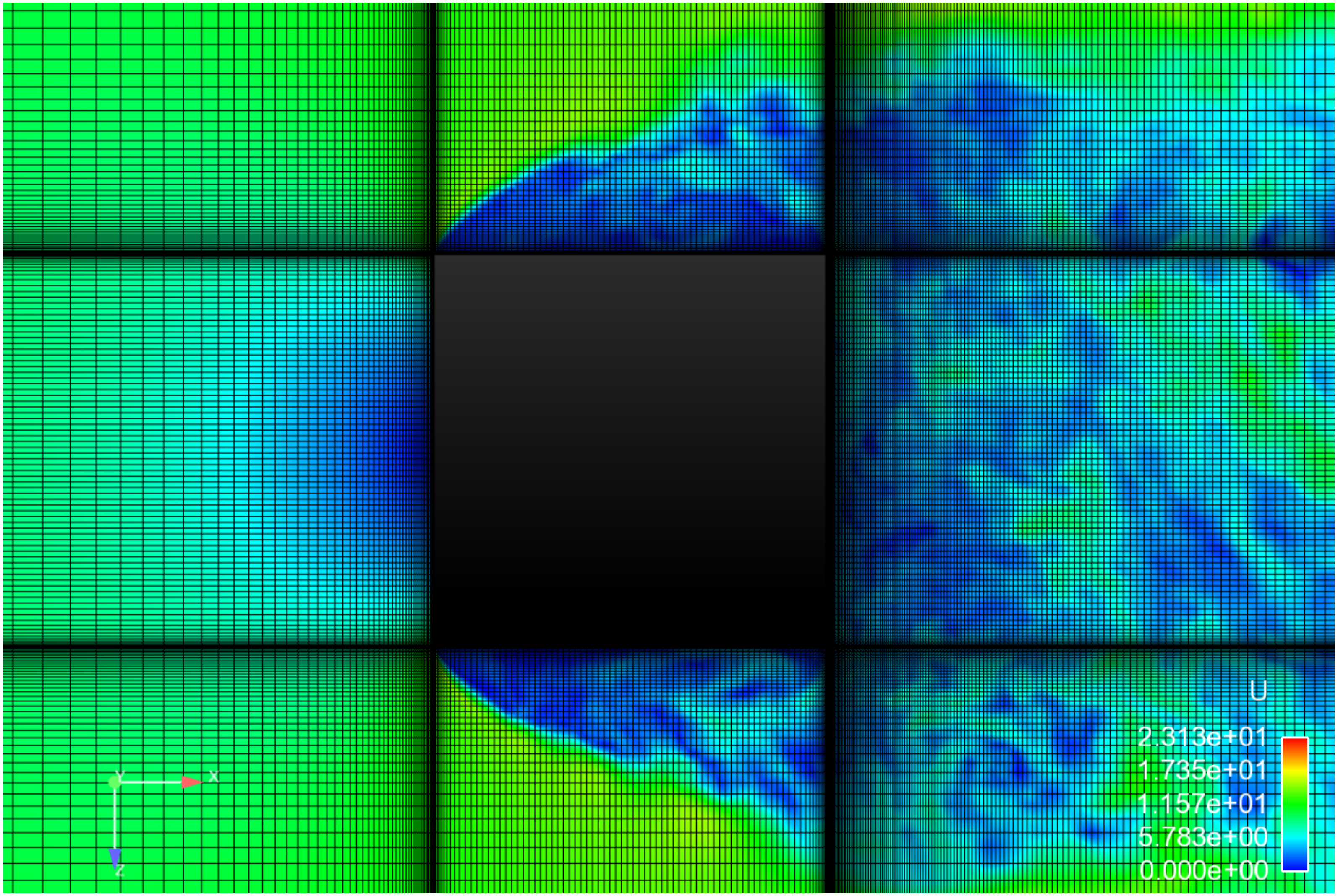


Instantaneous velocity magnitude.



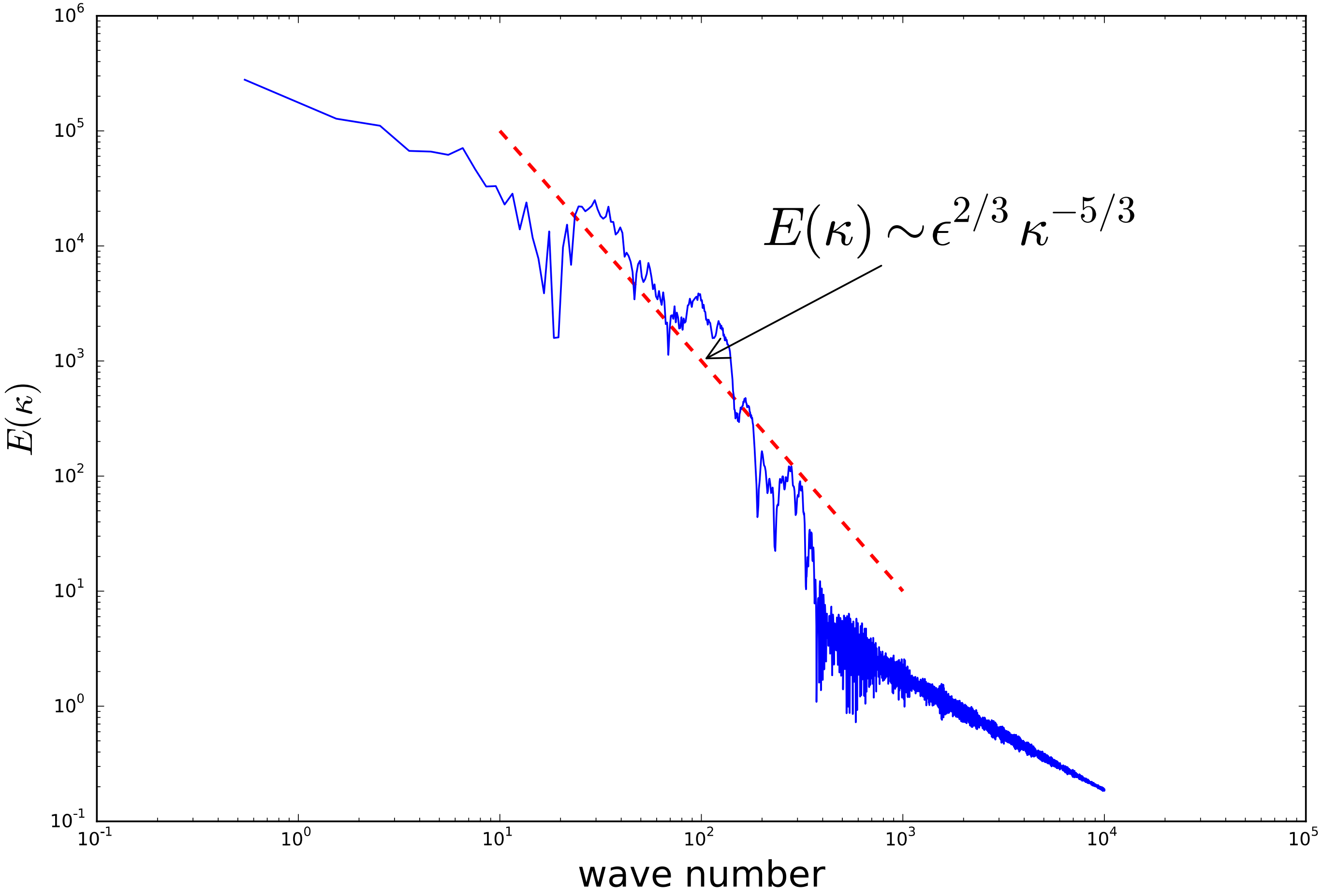
Instantaneous vorticity magnitude.

LES simulation



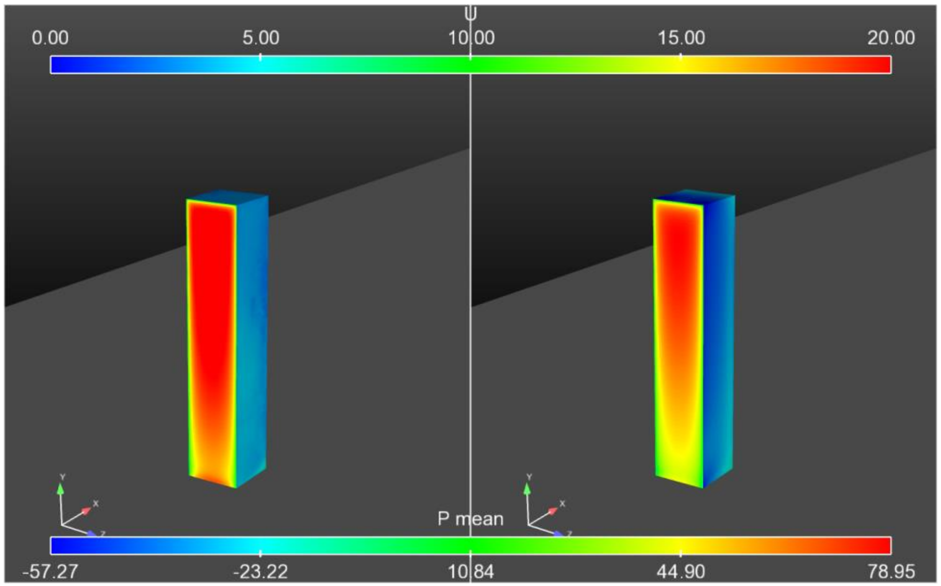
Mesh and instantaneous velocity magnitude.

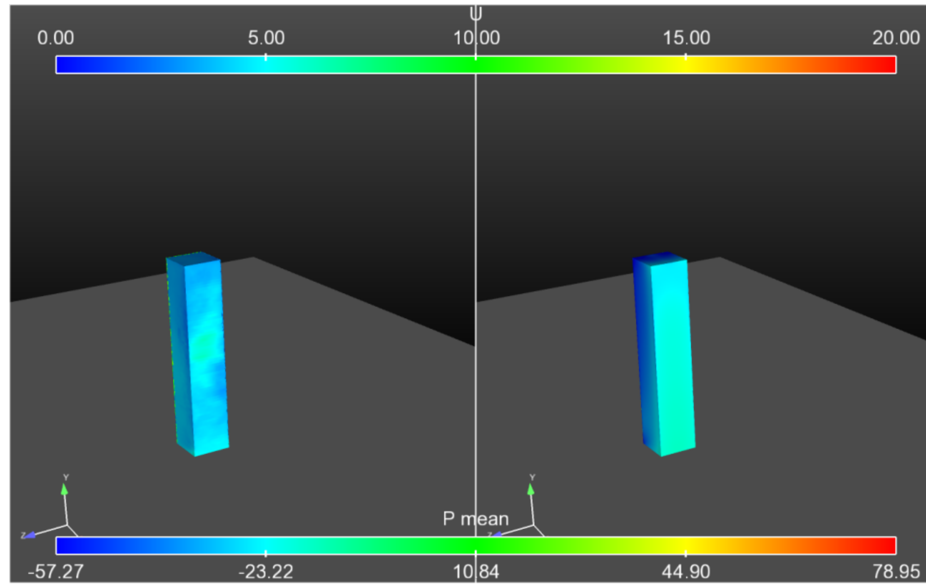
LES simulation



Energy spectrum and -5/3 slope.

LES simulation vs. RANS simulation

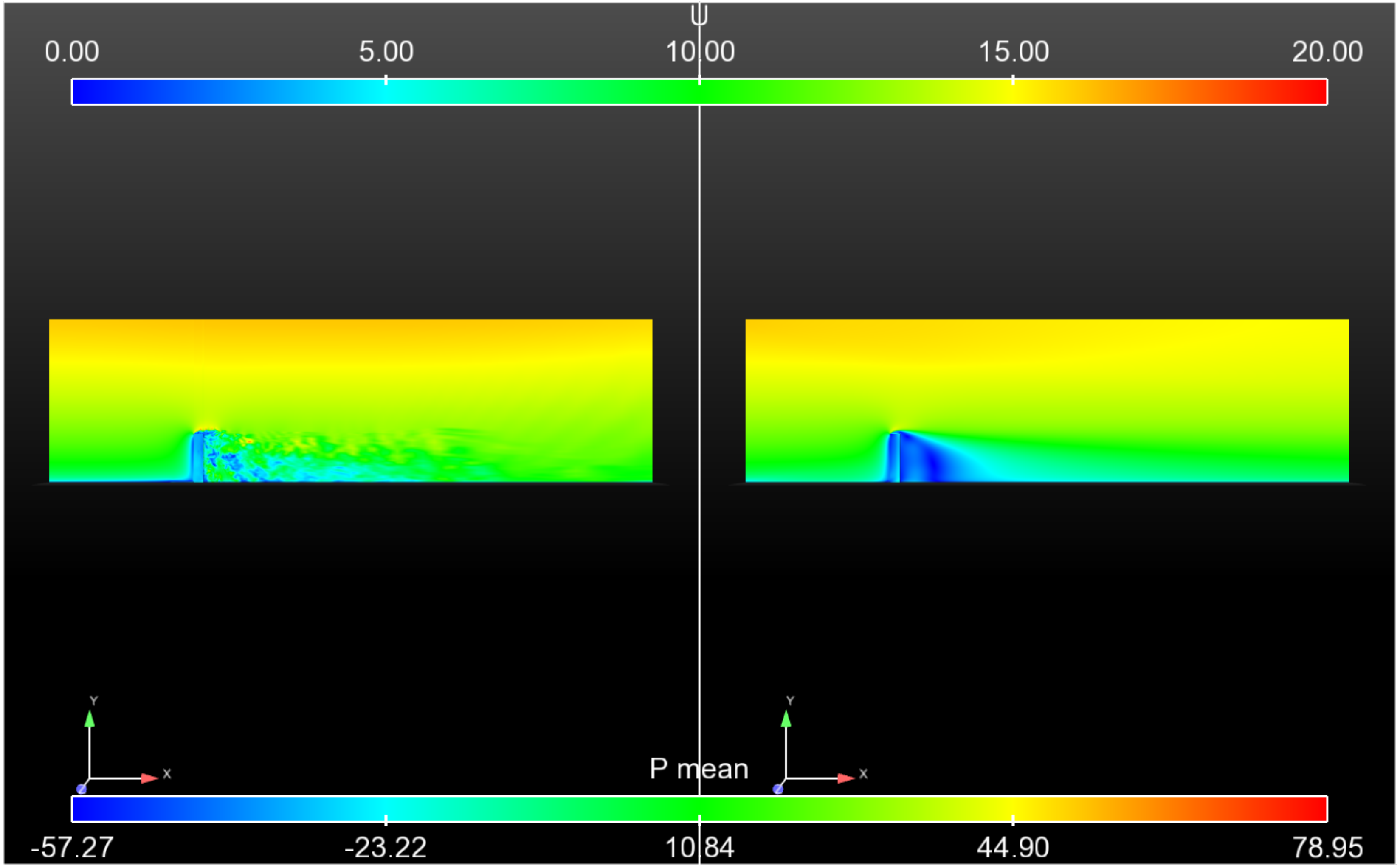




LES RANS LES RANS

- LES simulations are colored by instantaneous values.
- RANS simulations are colored by mean values.

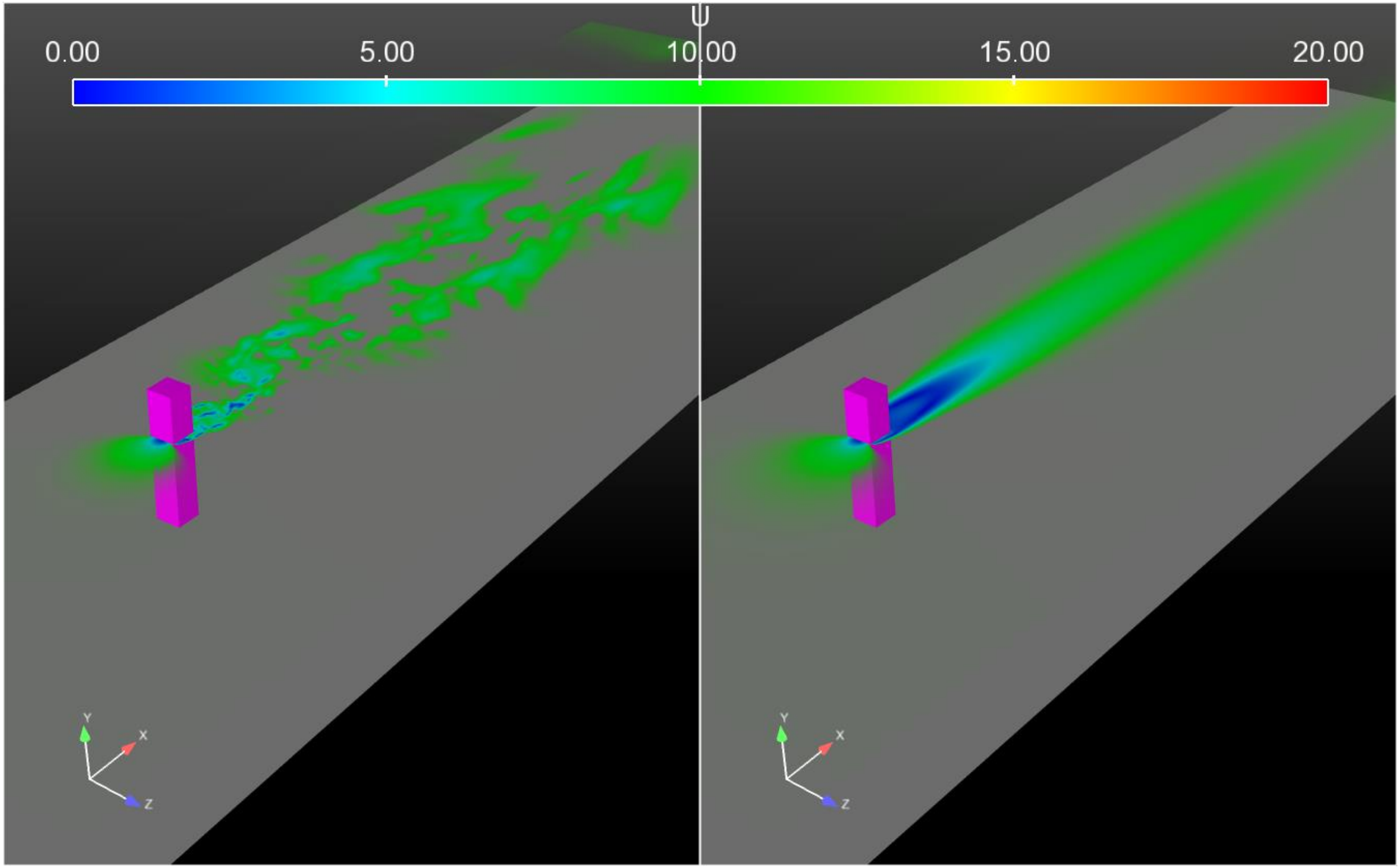
LES simulation vs. RANS simulation



LES (instantaneous velocity)

RANS

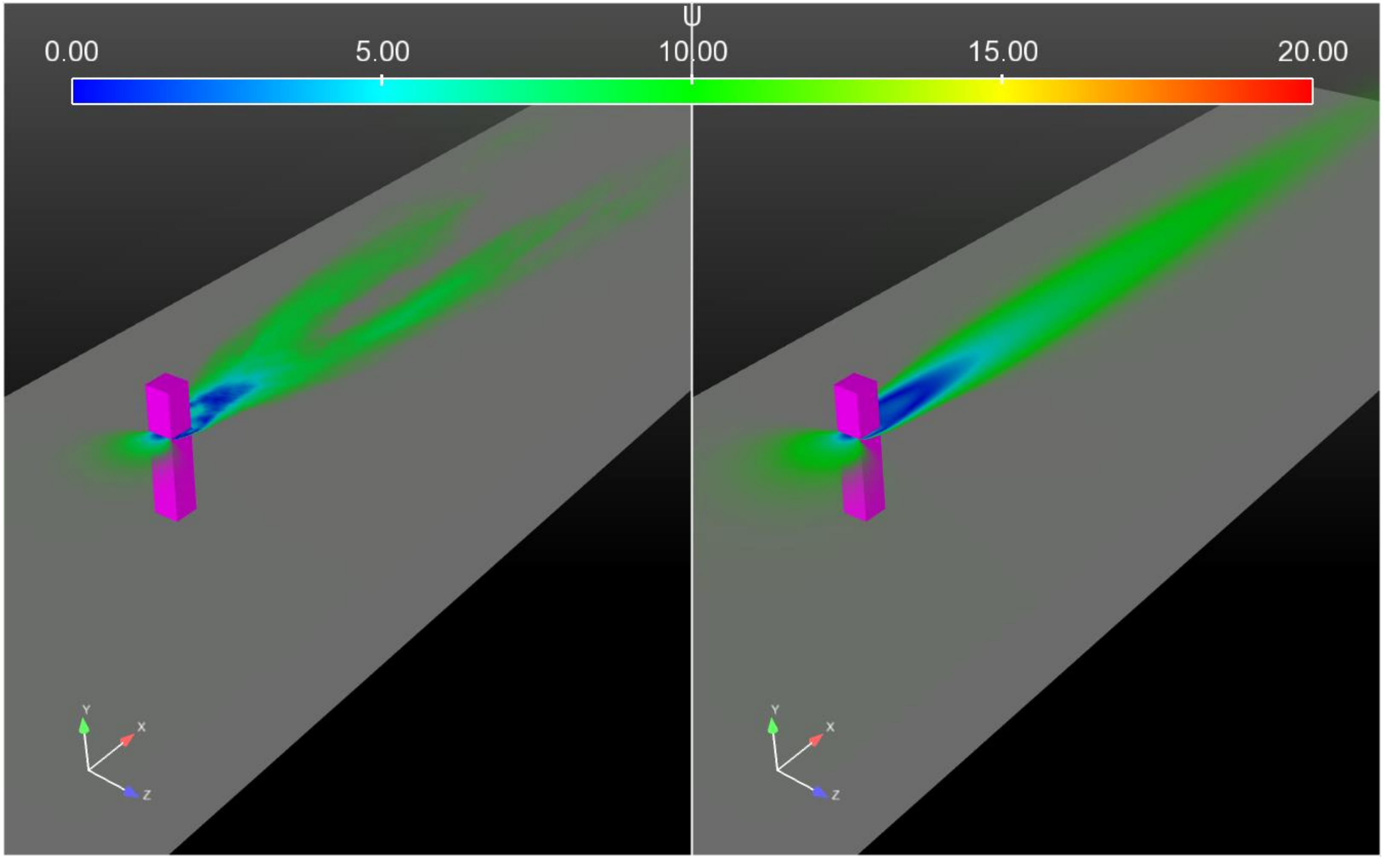
LES simulation vs. RANS simulation



LES (instantaneous velocity)

RANS

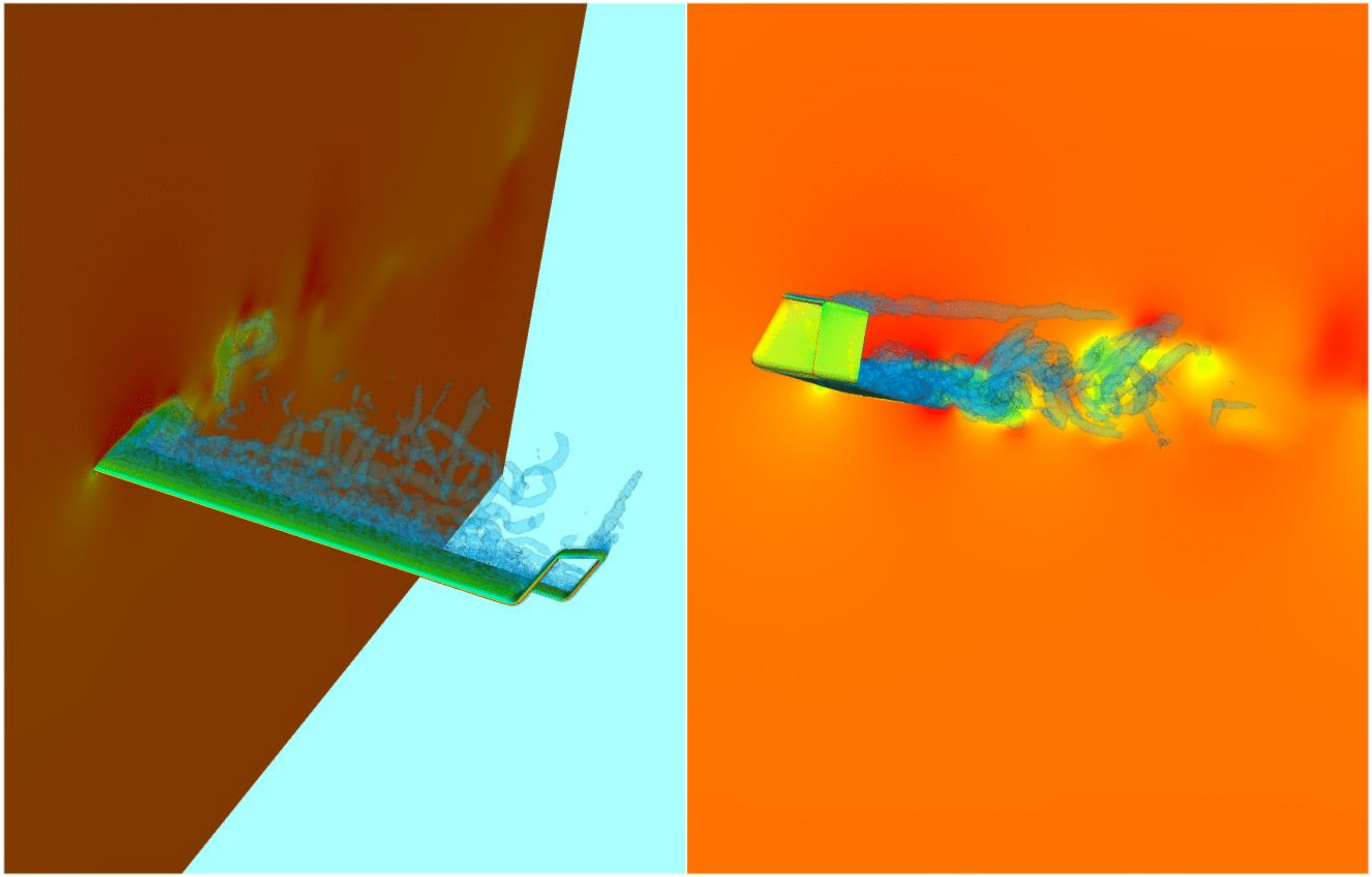
LES simulation vs. RANS simulation



LES (Mean velocity)

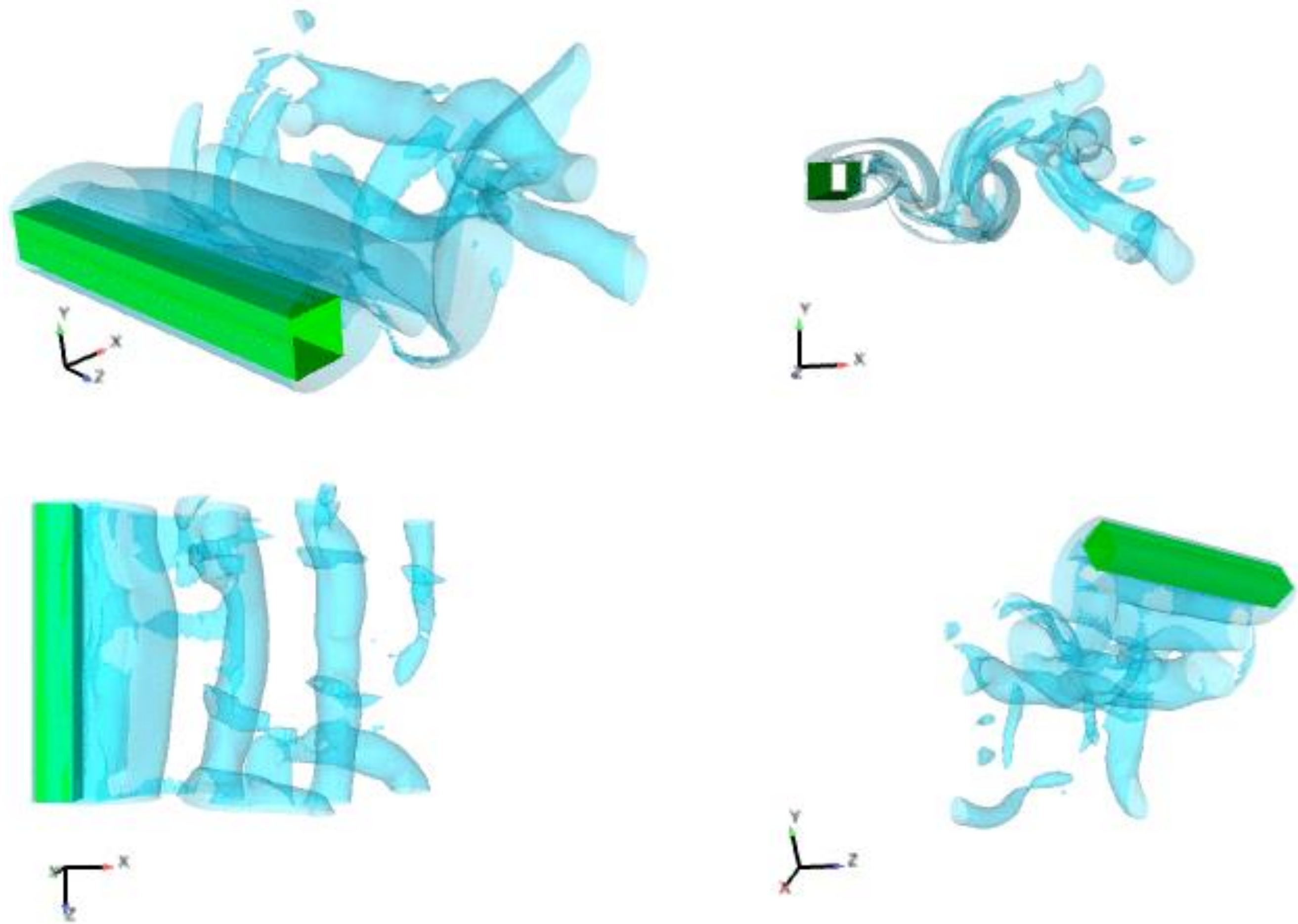
RANS

DES simulation



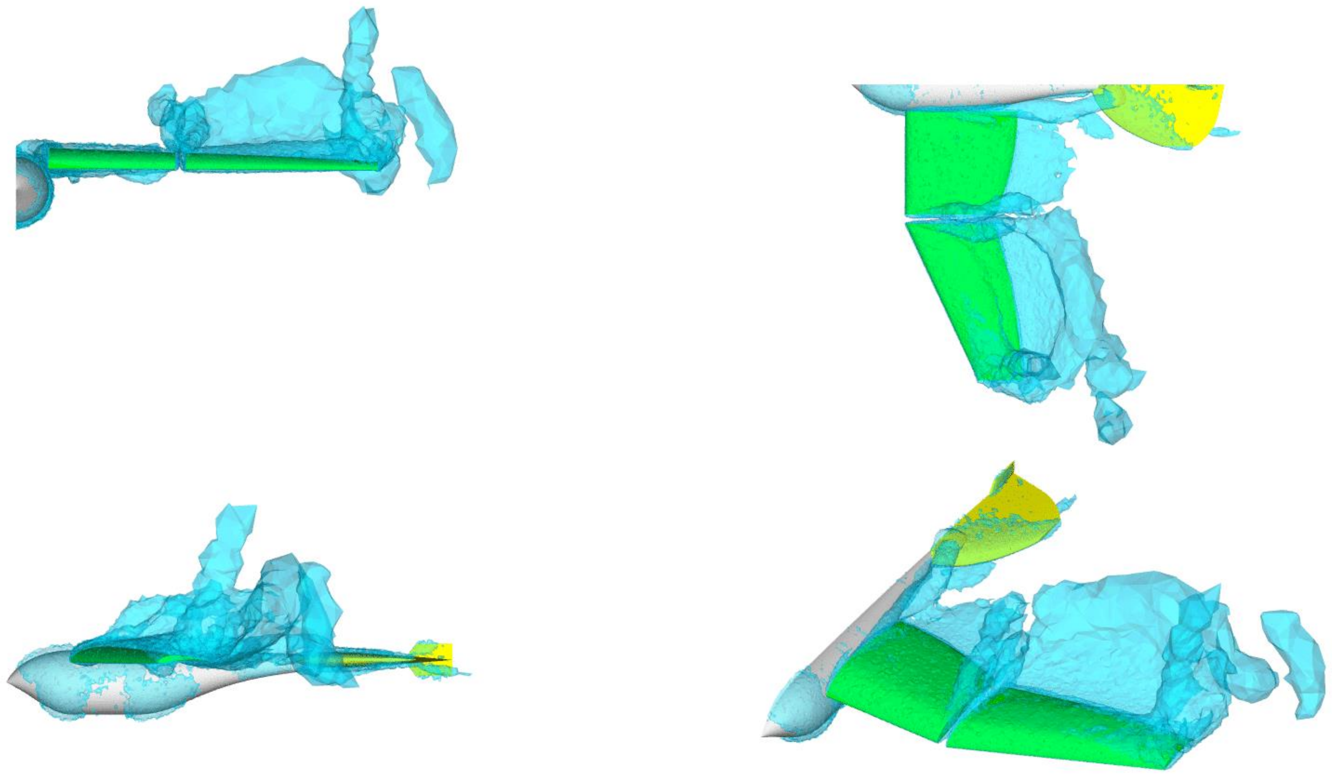
Iso-surfaces of Q criterion.

SAS simulation



Iso-surfaces of vorticity magnitude.

URANS simulation



Iso-surfaces of Q criterion.

Computational Fluid Dynamics (CFD)

Introduction

- Practice of engineering and science has been dramatically altered by the development of
  - Scientific computing
  - Mathematics of numerical analysis
  - The Internet

- Computational Fluid Dynamics is based upon the logic of applied mathematics
  - provides tools to unlock previously unsolved problems
  - is used in nearly all fields of science and engineering
    - Aerodynamics, acoustics, bio-systems, cosmology, geology, heat transfer, hydrodynamics, river hydraulics, etc…

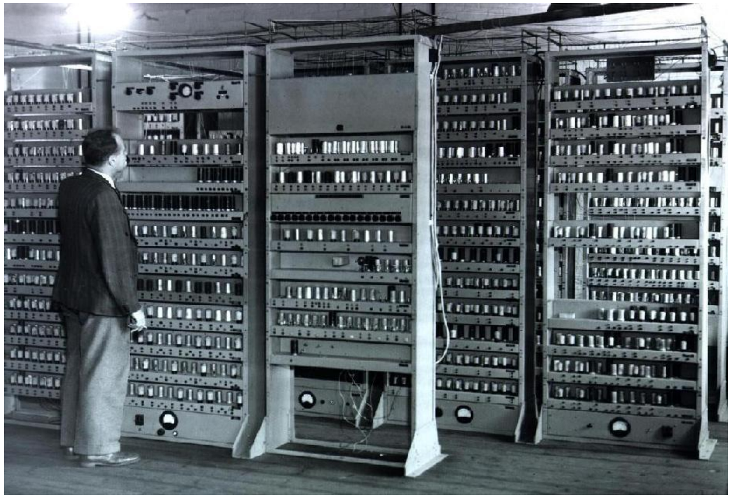
Introduction

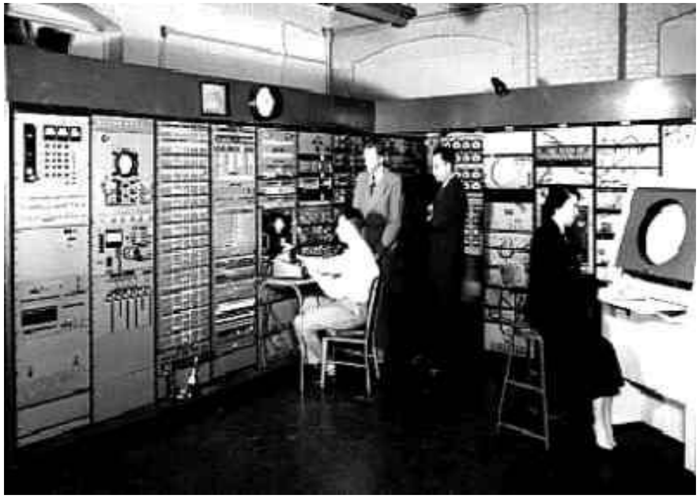
- We are in the midst of a new Scientific Revolution as significant as that of the 16th and 17th centuries when Galilean methods of systematic experiments and observation supplanted the logic-based methods of Aristotelian physics

- Modern tools, i.e., computational mechanics, are enabling scientists and engineers to return to logic-based methods for discovery and invention, research and development, and analysis and design

Introduction

History of computing



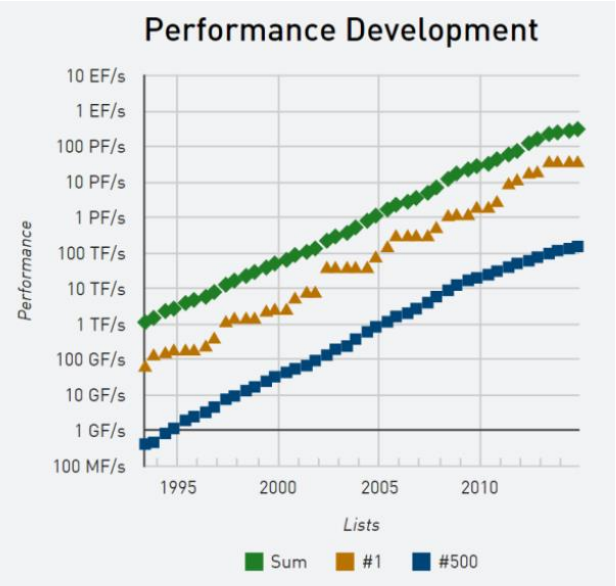


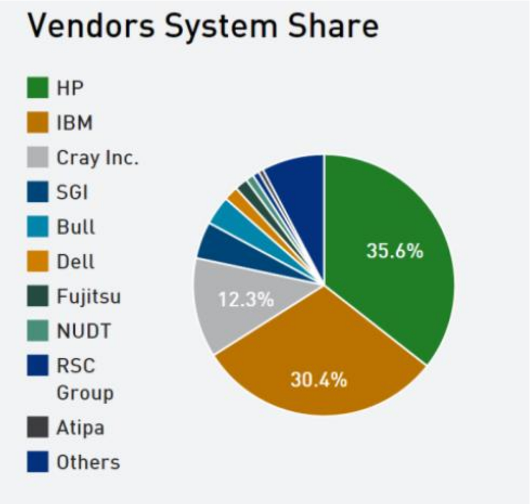
- Mastodons of computing, 1945-1960
  - Early computer engineers thought that only a few dozen computers required worldwide
  - Applications: cryptography (code breaking), ***fluid dynamics***, artillery firing tables, atomic weapons
  - ENIAC, or Electronic Numerical Integrator Analyzor and Computer, was developed by the Ballistics Research Laboratory in Maryland and was built at the University of Pennsylvania's Moore School of Electrical Engineering and completed in November 1945

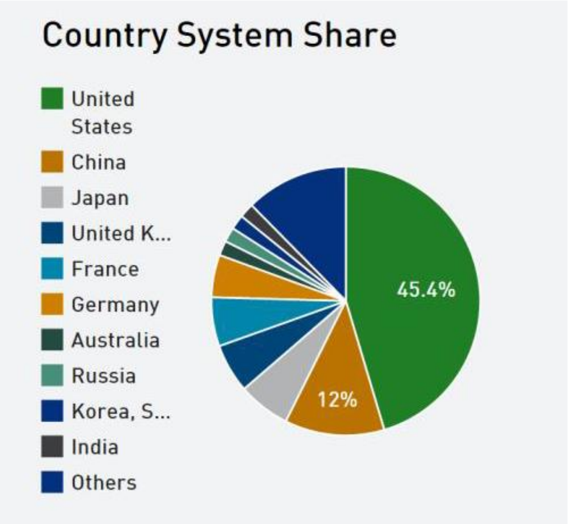
Introduction

High-performance computing

- Top 500 computers in the world compiled: www.top500.org
- Computers located at major centers connected to researchers via Internet







Outline

- CFD Process
  - Model Equations
  - Discretization
  - Grid Generation
  - Boundary Conditions
  - Solve
  - Post-Processing
  - Uncertainty Assessment

Model Equations

- Most commercial CFD codes solve the continuity, Navier-Stokes, and energy equations
  - Coupled, ***non-linear***, partial differential equations
  - For example, incompressible form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

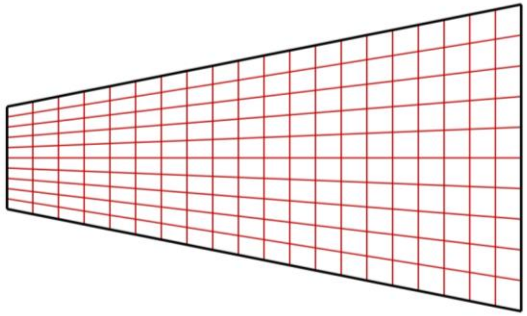
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial \rho v}{\partial y}+w\frac{\partial \rho v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

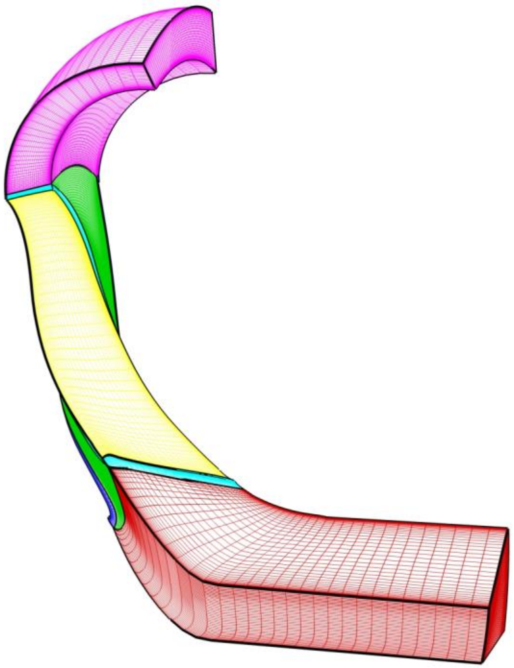
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

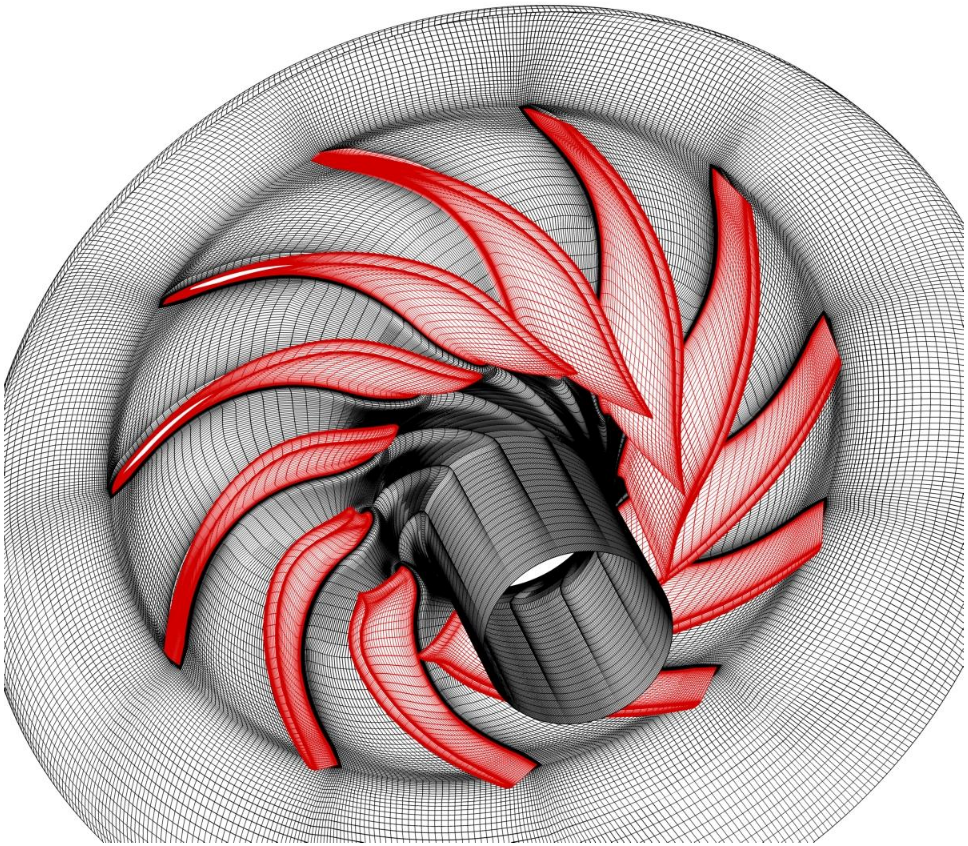
Discretization
Grid Generation

- Flow field must be treated as a discrete set of points (or volumes) where the governing equations are solved.
- Many types of grid generation: type is usually related to capability of flow solver.
  - Structured grids
  - Unstructured grids
  - Hybrid grids: some portions of flow field are structured (viscous regions) and others are unstructured
  - Overset (Chimera) grids

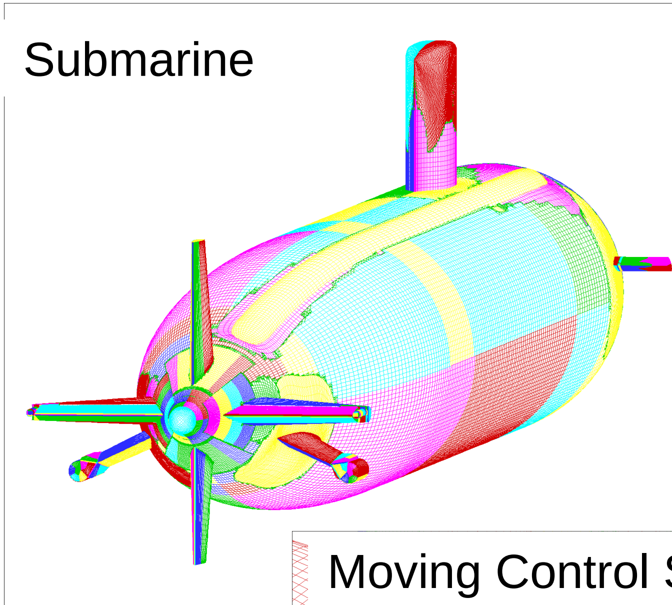
Structured Grids

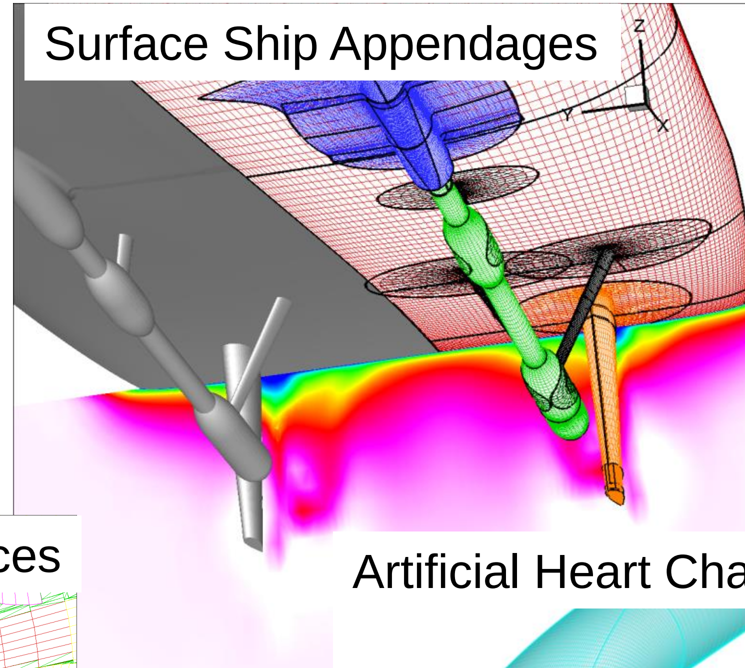


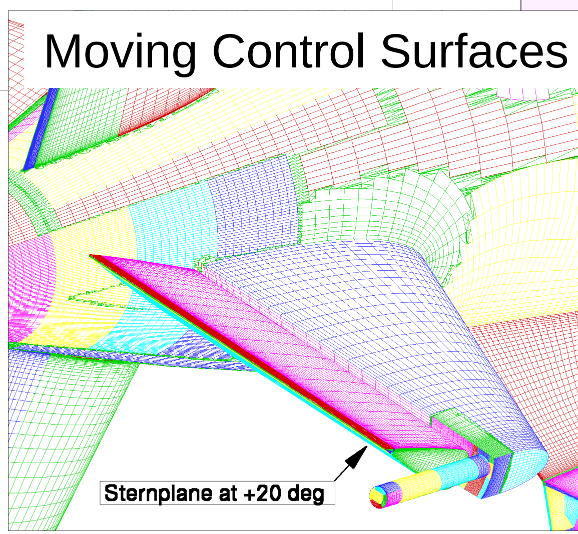


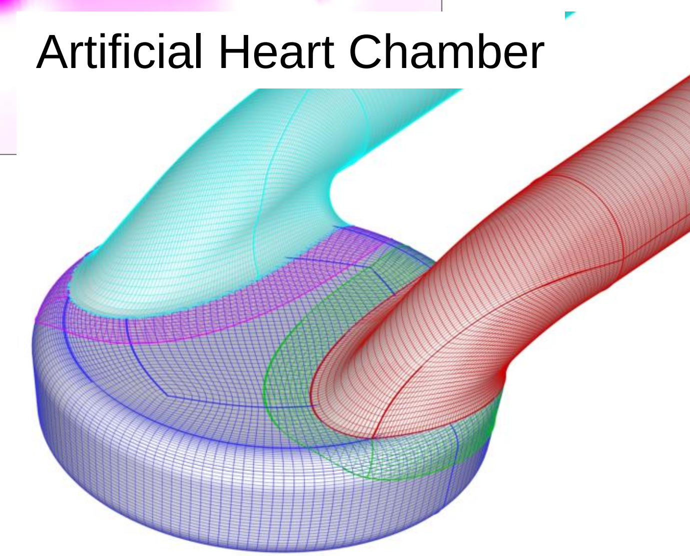


Structured Overset Grids

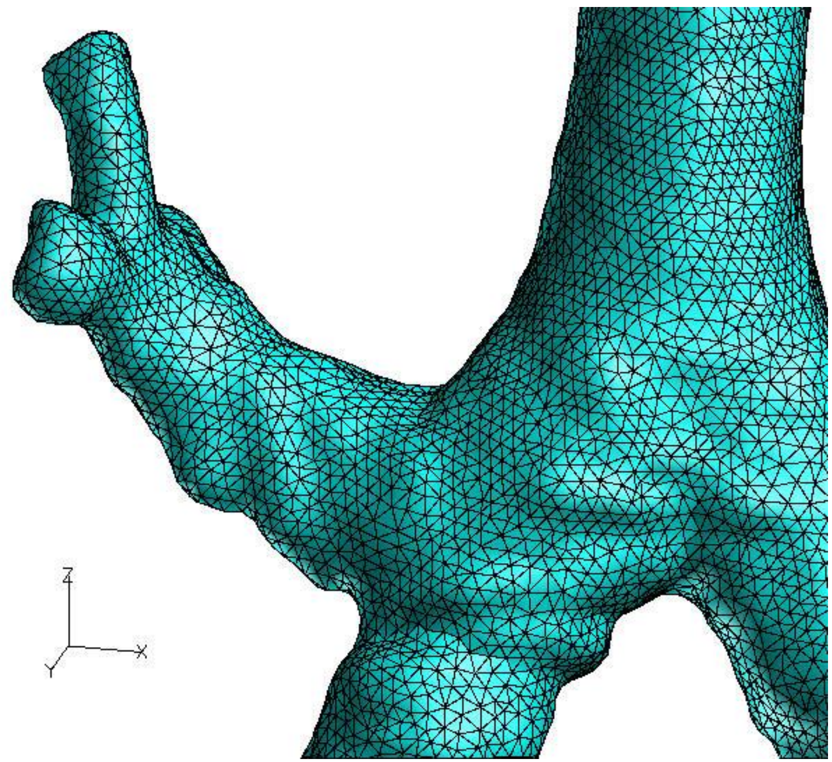




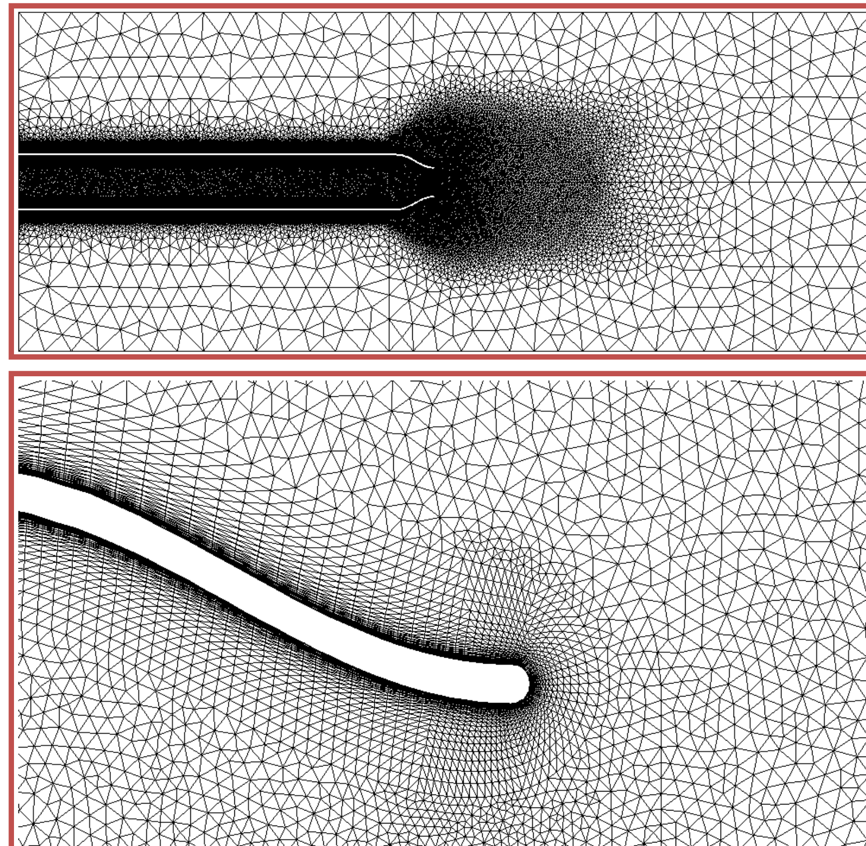




Unstructured Grids



Branches in Human Lung



Structured-Unstructured Nozzle Grid

Discretization
Algebraic equations

- To solve NSE, we must convert governing PDE's to algebraic equations
  - Finite difference methods (FDM)
    - Each term in NSE approximated using Taylor series, e.g.,

$$\frac{\partial U}{\partial x} = \frac{U_{i+1} - U_i}{\Delta x} + O(\Delta x)$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{U_{i+1} - 2U_i + U_{i-1}}{(\Delta x)^2} + O(\Delta x)^2$$

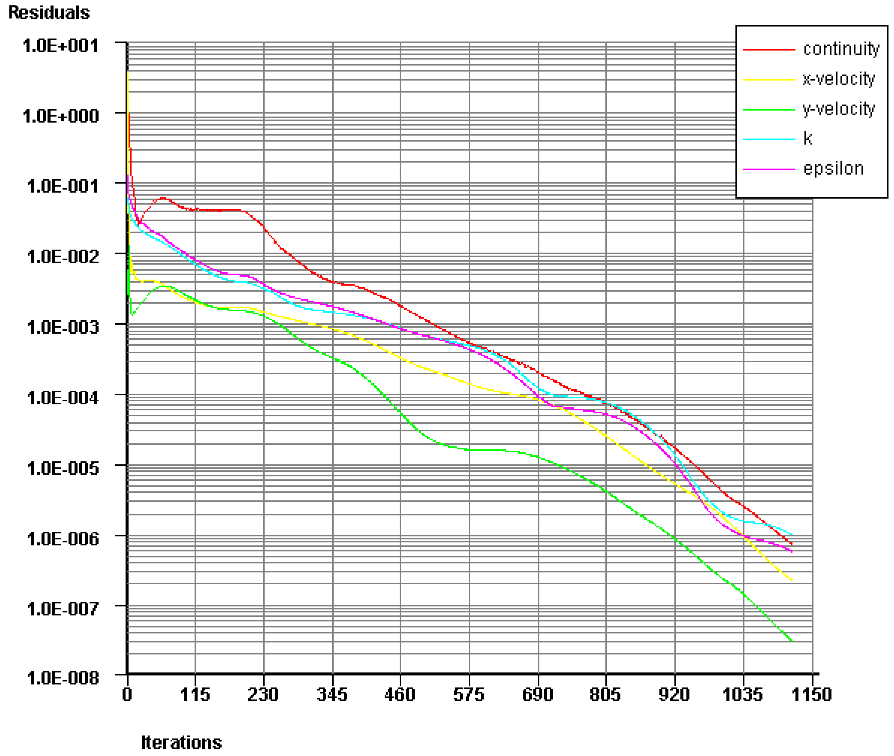
  - Finite volume methods (FVM)
    - Use CV form of NSE equations on each grid cell !
    - Most popular approach, especially for commercial codes
  - Finite element methods (FEM)
    - Solve PDE's by replacing continuous functions by piecewise approximations defined on polygons, which are referred to as elements.

Boundary Conditions

- Typical conditions
  - Wall
    - No-slip ($u = v = w = 0$)
    - Slip (tangential stress = 0, normal velocity = 0)
    - With specified suction or blowing
    - With specified temperature or heat flux
  - Inflow
  - Outflow
  - Interface Condition, e.g., Air-water free surface
  - Symmetry and Periodicity
- Usually set through the use of a graphical user interface (GUI) – click & set

Solve

- Run CFD code on computer
  - 2D and small 3D simulations can be run on desktop computers (e.g., FlowLab)
  - Unsteady 3D simulations still require large parallel computers
- Monitor Residuals
  - Defined two ways
    - Change in flow variables between iterations
    - Error in discrete algebraic equation



Uncertainty Assessment

- Process of estimating errors due to numerics and modeling
  - Numerical errors
    - Iterative non-convergence: monitor residuals
    - Spatial errors: grid studies and Richardson extrapolation
    - Temporal errors: time-step studies and Richardson extrapolation
  - Modeling errors (turbulence modeling, multi-phase physics, closure of viscous stress tensor for non-Newtonian fluids)
    - Only way to assess is through comparison with benchmark data which includes EFD uncertainty assessment.

Conclusions

- Capabilities of Current Technology
  - Complex real-world problems solved using Scientific Computing
  - Commercial software available for certain problems
  - Simulation-based design (i.e., logic-based) is being realized.
  - Ability to study problems that are either expensive, too small, too large, or too dangerous to study in laboratory
    - Very small: nano- and micro-fluidics
    - Very large: cosmology (study of the origin, current state, and future of our Universe)
    - Expensive: engineering prototypes (ships, aircraft)
    - Dangerous: explosions, response to weapons of mass destruction

Conclusions

- Limitations of Current Technology
  - For fluid mechanics, many problems not adequately described by Navier-Stokes equations or are beyond current generation computers.
    - Turbulence
    - Multi-phase physics: solid-gas (pollution, soot), liquid-gas (bubbles, cavitation); solid-liquid (sediment transport)
    - Combustion and chemical reactions
    - Non-Newtonian fluids (blood; polymers)
  - Similar modeling challenges in other branches of engineering and the sciences

Conclusions

- Because of limitations, need for experimental research is great
- However, focus has changed
  - From
    - Research based solely upon experimental observations
    - Build and test (although this is still done)
  - To
    - High-fidelity measurements in support of validation and building new computational models.
- Currently, the best approach to solving engineering problems often uses simulation and experimentation

Thank you for your attention