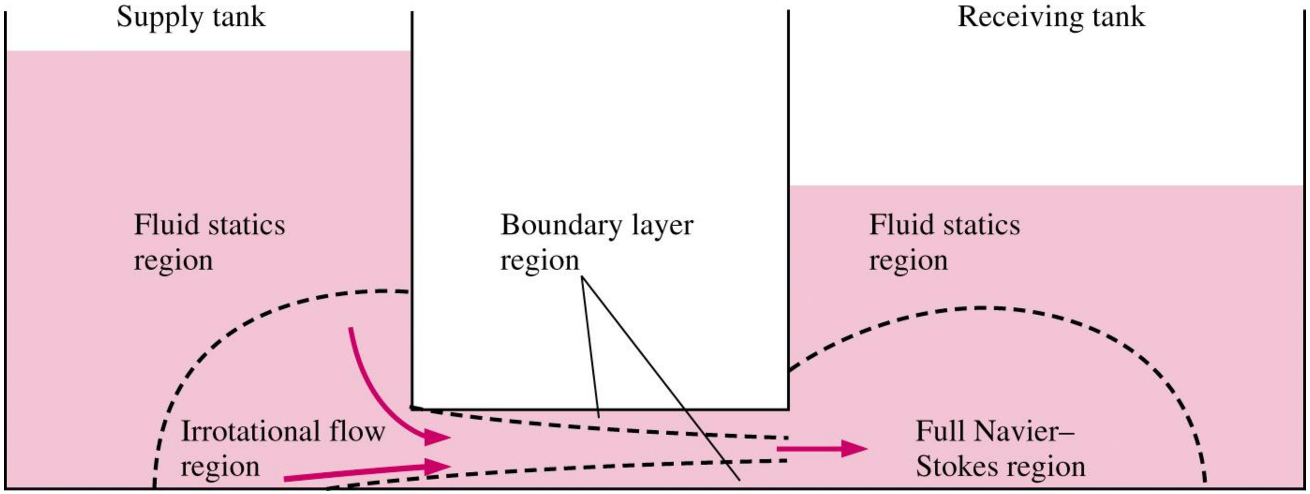
Chapter 10: Approximate Solutions of the Navier-Stokes Equation

Objectives

1. Appreciate why approximations are necessary, and know when and where to use.
2. Understand effects of lack of inertial terms in the creeping flow approximation.
3. Understand superposition as a method for solving potential flow.
4. Predict boundary layer thickness and other boundary layer properties.

Introduction

- In Chap. 9, we derived the NSE and developed several exact solutions.
- In this Chapter, we will study several methods for simplifying the NSE, which permit use of mathematical analysis and solution
  - These approximations often hold for certain regions of the flow field.



Nondimensionalization of the NSE

- Purpose: Order-of-magnitude analysis of the terms in the NSE, which is necessary for simplification and approximate solutions.
- We begin with the incompressible NSE

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

- Each term is *dimensional, and ea*ch variable or property (ρ, *V*, *t*, μ, etc.) is also dimensional.
- What are the primary dimensions of each term in the NSE equation?

$$\boxed{\text{Answer}: \left\{ \frac{m}{L^2 t^2} \right\}}$$

Nondimensionalization of the NSE

- To nondimensionalize, we choose ***scaling parameters*** as follows

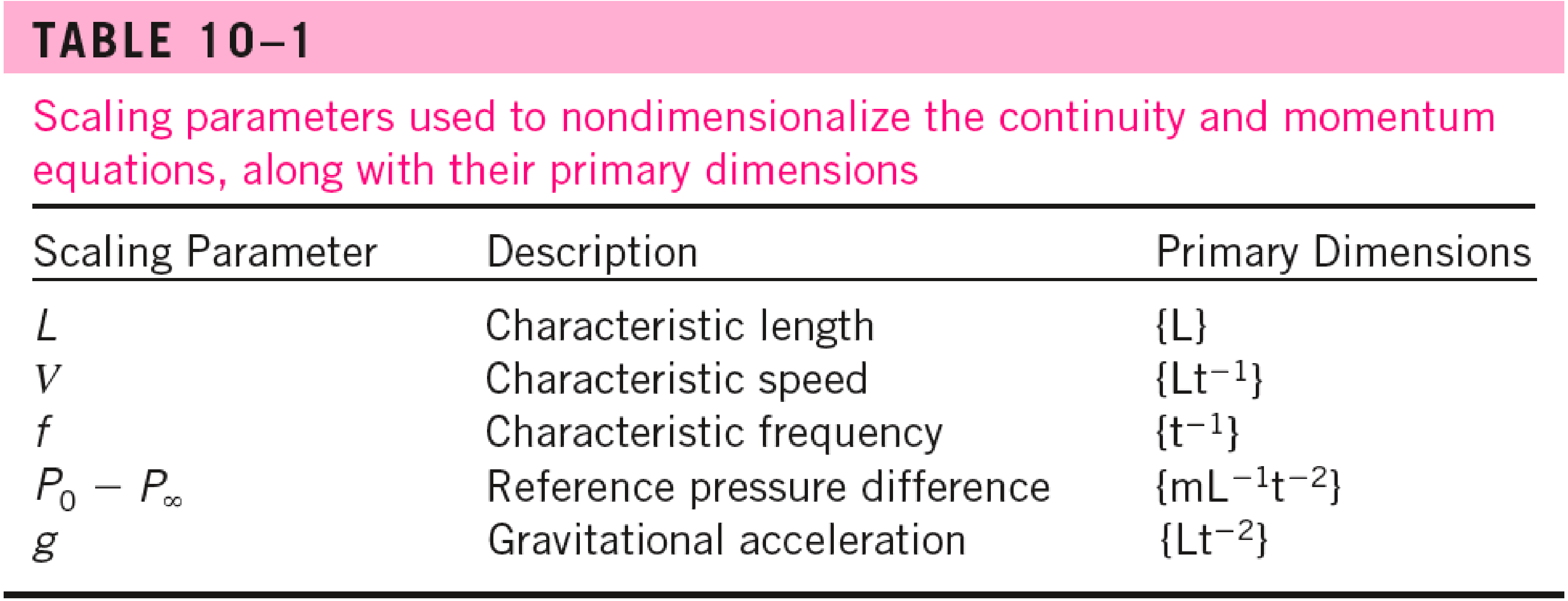


TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	$\{L\}$
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

Nondimensionalization of the NSE

- Next, we define ***nondimensional variables***, using the scaling parameters in Table 10-1

$$t^\star = ft \qquad \vec{x}^\star = \frac{\vec{x}}{L} \qquad \vec{V}^\star = \frac{\vec{V}}{V}$$

$$P^\star = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^\star = \frac{\vec{g}}{g} \qquad \nabla^\star = L\nabla$$

- To plug the nondimensional variables into the NSE, we need to first rearrange the equations in terms of the dimensional variables

$$t = \frac{1}{f}t^\star \qquad \vec{x} = L\vec{x}^\star \qquad \vec{V} = V\vec{V}^\star \qquad \nabla = \frac{1}{L}\nabla^\star$$

$$P = P_\infty + (P_0 - P_\infty)P^\star \qquad \vec{g} = g\vec{g}^\star$$

Nondimensionalization of the NSE

- Now we substitute into the NSE to obtain

$$\rho V f \frac{\partial \vec{V}^\star}{\partial t^\star} + \frac{\rho V^2}{L}\left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\frac{P_0 - P_\infty}{L}\nabla^\star P^\star + \rho g \vec{g}^\star + \frac{\mu V}{L^2}\nabla^{\star 2}\vec{V}^\star$$

- Every additive term has primary dimensions $\{m^{1}L^{-2}t^{-2}\}$. To nondimensionalize, we multiply every term by $L/(\rho V^{2})$, which has primary dimensions $\{m^{-1}L^{2}t^{2}\}$, so that the dimensions cancel. After rearrangement,

$$\left[\frac{fL}{V}\right]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\left[\frac{P_0 - P_\infty}{\rho V^2}\right]\nabla^\star P^\star + \left[\frac{gL}{V^2}\right]\vec{g}^\star + \left[\frac{\mu}{\rho V L}\right]\nabla^{\star 2}\vec{V}^\star$$

Nondimensionalization of the NSE

Terms in [ ] are nondimensional parameters

$$\underbrace{\boxed{\left[\frac{fL}{V}\right]}}_{\text{Strouhal number}}\frac{\partial \vec{V}^\star}{\partial t^\star}+\left(\vec{V}^\star\cdot\nabla^\star\right)\vec{V}^\star=-\underbrace{\boxed{\left[\frac{P_0-P_\infty}{\rho V^2}\right]}}_{\text{Euler number}}\nabla^\star P^\star+\underbrace{\boxed{\left[\frac{gL}{V^2}\right]}}_{\text{Inverse of Froude number squared}}\vec{g}^\star+\underbrace{\boxed{\left[\frac{\mu}{\rho V L}\right]}}_{\text{Inverse of Reynolds number}}\nabla^{\star 2}\vec{V}^\star$$

$$\boxed{[St]\frac{\partial \vec{V}^\star}{\partial t^\star}+\left(\vec{V}^\star\cdot\nabla^\star\right)\vec{V}^\star=-[Eu]\,\nabla^\star P^\star+\left[\frac{1}{Fr^2}\right]\vec{g}^\star+\left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star}$$

Navier-Stokes equation in nondimensional form

Nondimensionalization of the NSE

- Nondimensionalization vs. Normalization
  - NSE are now *nondimensional*, but not necessarily *normalized.* What is the difference?
  - ***Nondimensionalization*** concerns only the *dimensions* of the equation - we can use *any* value of scaling parameters L, V, etc.
  - ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters *L,V,* etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables are of order of magnitude unity,*** i.e., their minimum and maximum values are close to 1.0.

$$t^\star \sim 1 \qquad \vec{x}^\star \sim 1 \qquad \vec{V}^\star \sim 1 \qquad P^\star \sim 1 \qquad \vec{g}^\star \sim 1 \qquad \nabla^\star \sim 1$$

If we have properly normalized the NSE, we can compare the relative importance of the terms in the equation by comparing the relative magnitudes of the nondimensional parameters St, Eu, Fr, and Re.

Creeping Flow

- Also known as "Stokes Flow" or "Low Reynolds number flow"
- Occurs when Re << 1
    - ρ, *V*, or *L* are very small, e.g., micro-organisms, MEMS, nano-tech, particles, bubbles
    - μ is very large, e.g., honey, lava

Creeping Flow

- To simplify NSE, assume St ~ 1, Fr ~ 1

$$\boxed{[Eu]\,\nabla^\star P^\star = \left[\frac{1}{Re}\right] \nabla^{\star 2} \vec{V}^\star}$$

Pressure forces

Viscous forces

- Since $P^\star \sim 1, \quad \nabla^\star \sim 1$

$$Eu = \frac{P_0 - P_\infty}{\rho V^2} \sim \frac{1}{Re} = \frac{\mu}{\rho V L} \Longrightarrow \boxed{P_0 - P_\infty \sim \frac{\mu V}{L}}$$

Creeping Flow

- This is important $\boxed{P_0 - P_\infty \sim \frac{\mu V}{L}}$
  - Very different from inertia dominated flows where

$$P_0 - P_\infty \sim \rho V^2$$

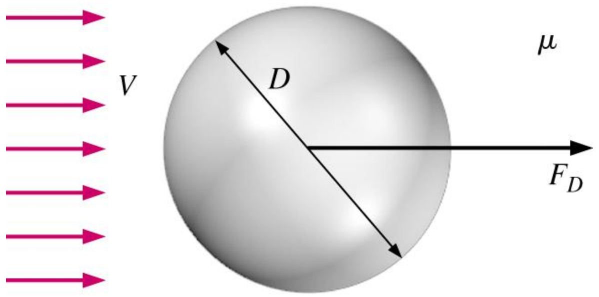
  - Density has completely dropped out of NSE. To demonstrate this, convert back to dimensional form.

$$\nabla P = \mu \nabla^2 \vec{V}$$

  - This is now a **LINEAR EQUATION** which can be solved for simple geometries.

Creeping Flow

- Solution of Stokes flow is beyond the scope of this course.
- Analytical solution for flow over a sphere gives a drag coefficient which is a linear function of velocity V and viscosity μ.



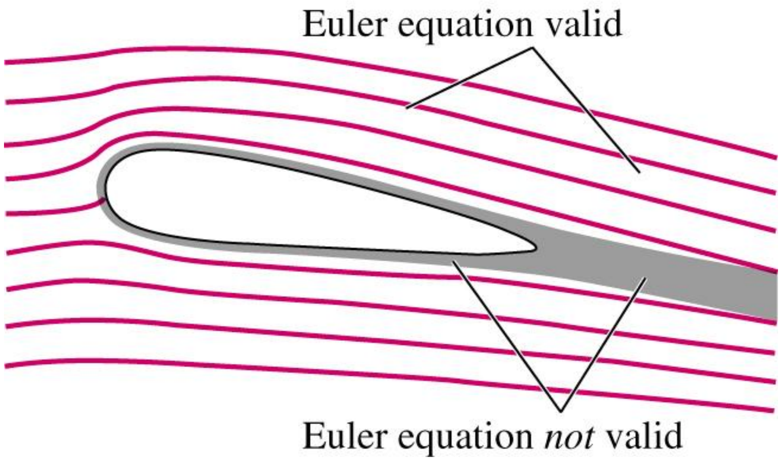
$$F_D = 3\pi\mu V D$$

Inviscid Regions of Flow

- Definition: Regions where net viscous forces are negligible compared to pressure and/or inertia forces

~0 if Re large

$$\underbrace{[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star}_{\text{Euler Equation}} + \cancel{\left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star}$$



Inviscid Regions of Flow

- Euler equation often used in aerodynamics
- Elimination of viscous term changes PDE from mixed elliptic-hyperbolic to hyperbolic. This affects the type of analytical and computational tools used to solve the equations.
- Must “relax” wall boundary condition from **no-slip** to **slip**

For example for the case of a fixed wall:

No-slip BC

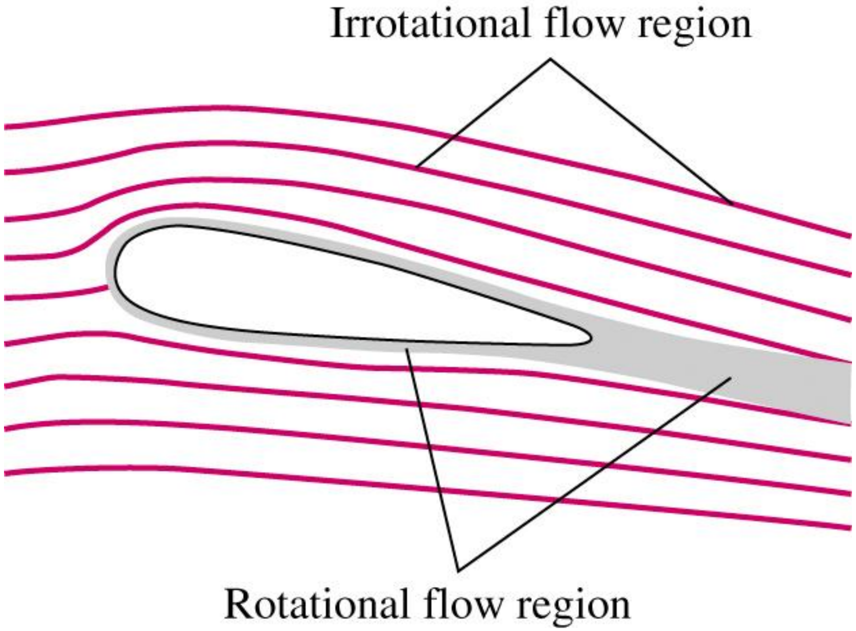
$u = v = w = 0$

Slip BC

$\tau_w = 0,\ v_n = 0$

v_n = normal velocity

Irrotational Flow Approximation



- Irrotational approximation: vorticity is negligibly small

$$\vec{\zeta} = \nabla \times \vec{V} \cong 0$$

- In general, inviscid regions are also irrotational, but there are situations where inviscid flow are rotational, e.g., solid body rotation (Ex. 10-3)

Irrotational Flow Approximation

- What are the implications of irrotational approximation. Look at continuity and momentum equations.
- Continuity equation
  - Use the vector identity $\nabla \times \nabla\phi = 0$
  - Since the flow is irrotational $\nabla \times \vec{V} = 0$

$$\boxed{\vec{V} = \nabla\phi}$$

ϕ is a scalar potential function

Irrotational Flow Approximation

- Therefore, regions of irrotational flow are also called regions of potential flow.
- From the definition of the gradient operator ∇

Cartesian

$$U = \frac{\partial \phi}{\partial x}, \quad V = \frac{\partial \phi}{\partial y}, \quad W = \frac{\partial \phi}{\partial z}$$

Cylindrical

$$U_r = \frac{\partial \phi}{\partial r}, \quad U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad U_z = \frac{\partial \phi}{\partial z}$$

- Substituting into the continuity equation gives

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0}$$

Irrotational Flow Approximation

- This means we only need to solve **1 linear scalar equation** to determine all 3 components of velocity!

$$\nabla^2 \phi = 0$$

Laplace Equation

- Luckily, the Laplace equation appears in numerous fields of science, engineering, and mathematics. This means that there are well developed tools for solving this equation.

Irrotational Flow Approximation

- Momentum equation
  - If we can compute ϕ from the Laplace equation (which came from continuity) and velocity from the definition $\vec{V} = \nabla\phi$, why do we need the NSE? $\Rightarrow$ To compute Pressure.
  - To begin analysis, apply irrotational approximation to viscous term of the NSE

$$\mu\nabla^2\vec{V} = \mu\nabla^2(\nabla\phi) = \mu\nabla\underbrace{(\nabla^2\phi)}_{= 0} = 0$$

Irrotational Flow Approximation

- Therefore, the NSE reduces to the Euler equation for irrotational flow

nondimensional

$$[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\,\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star$$

dimensional

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + \underbrace{\left(\vec{V} \cdot \nabla\right)\vec{V}}\right] = -\nabla P + \rho\vec{g}$$

- Instead of integrating to find P, use vector identity to derive Bernoulli equation

$$\left(\vec{V} \cdot \nabla\right)\vec{V} = \nabla\left(\frac{V^2}{2}\right) - \vec{V} \times \left(\nabla \times \vec{V}\right) = \boxed{\nabla\left(\frac{V^2}{2}\right) - \vec{V} \times \vec{\zeta}}$$

Irrotational Flow Approximation

- This allows the steady Euler equation to be written as

$$\boxed{= -g\vec{k} = -\nabla\left(gz\right)}$$

$$\nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{1}{\rho}\nabla P + \boxed{\vec{g}}$$

$$\boxed{\nabla\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta}}$$

- This form of Bernoulli equation is valid for inviscid and irrotational flow since we've shown that NSE reduces to the Euler equation.

Irrotational Flow Approximation

- However,

Inviscid

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C \quad \textbf{along a streamline}$$

Irrotational ($\vec{\zeta} = 0$)

$$\frac{P}{\rho} + \frac{V^2}{2} + gz = C \quad \textbf{everywhere}$$

Irrotational Flow Approximation

- Therefore, the process for irrotational flow
  1. Calculate ϕ from Laplace equation (from continuity)
  2. Calculate velocity from definition $\vec{V} = \nabla\phi$
  3. Calculate pressure from Bernoulli equation (derived from momentum equation)

$$P = P_\infty + \rho\left[\frac{V_\infty^2 - V^2}{2} + g\left(z_0 - z\right)\right]$$

Valid for 3D or 2D

Irrotational Flow Approximation

2D Flows

- For 2D flows, we can also use the streamfunction
- Recall the definition of streamfunction for planar (x-y) flows

$$U = \boxed{\frac{\partial \psi}{\partial y}}, \qquad V = \boxed{-\frac{\partial \psi}{\partial x}}$$

- Since vorticity is zero,

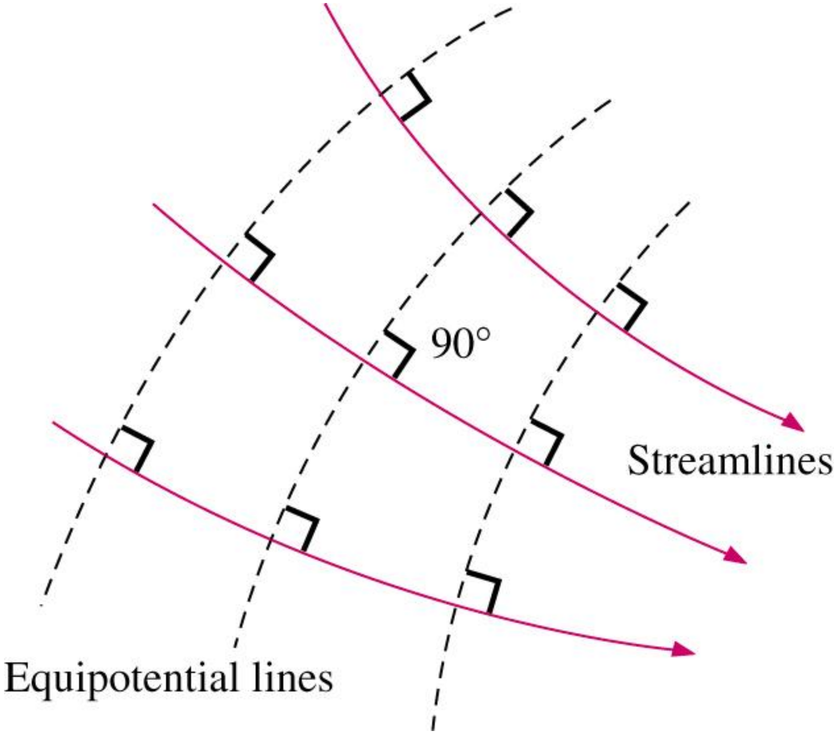
$$\zeta_z = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0$$

$$\boxed{\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2} = 0}$$

- This proves that the Laplace equation holds for the streamfunction and the velocity potential

Irrotational Flow Approximation

2D Flows



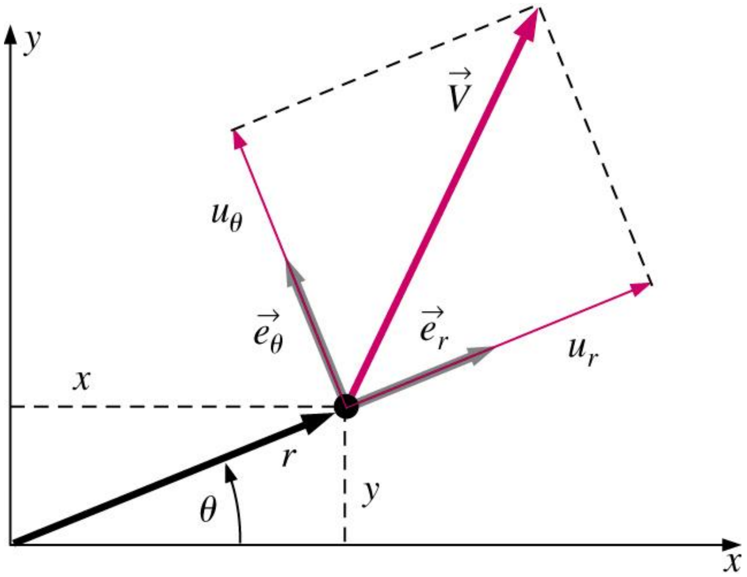
- Constant values of ψ: streamlines
- Constant values of ϕ: equipotential lines
- ψ and ϕ are mutually orthogonal
- ψ and ϕ are harmonic functions
- ψ is defined by continuity; $\nabla^2\psi$ results from irrotationality
- ϕ is defined by irrotationality; $\nabla^2\phi$ results from continuity

Flow solution can be achieved by solving either $\nabla^2\phi$ or $\nabla^2\psi$, however, BC are easier to formulate for ψ.

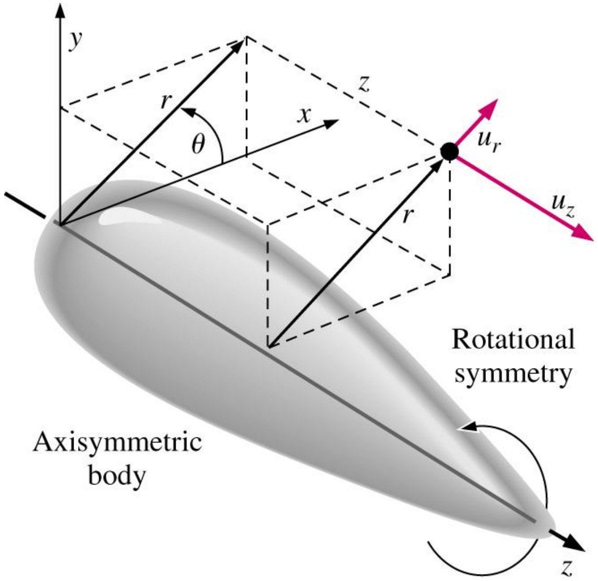
Irrotational Flow Approximation

2D Flows

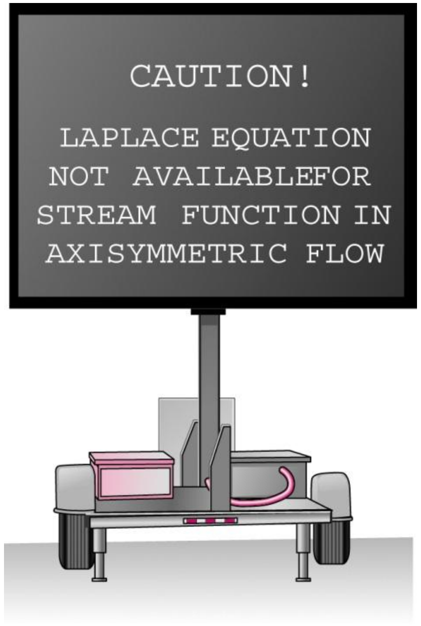
- Similar derivation can be performed for cylindrical coordinates (except for $\nabla^2\psi$ for axisymmetric flow)
  - Planar, cylindrical coordinates : flow is in (r,θ) plane
  - Axisymmetric, cylindrical coordinates : flow is in (r,z) plane



Planar



Axisymmetric



Irrotational Flow Approximation

2D Flows

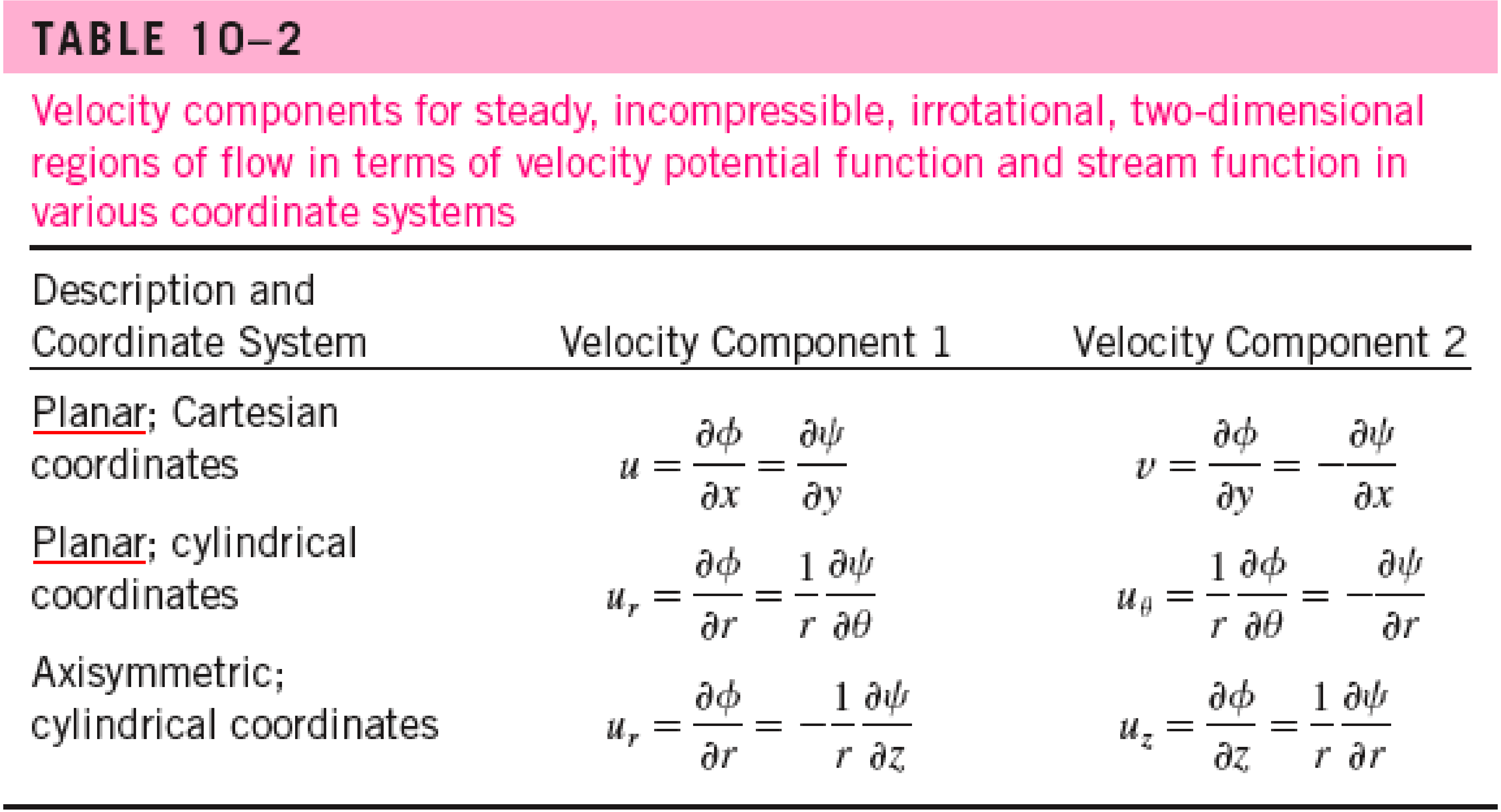


TABLE 10–2

Velocity components for steady, incompressible, irrotational, two-dimensional regions of flow in terms of velocity potential function and stream function in various coordinate systems

Description and Coordinate System	Velocity Component 1	Velocity Component 2
Planar; Cartesian coordinates	$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$	$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$
Planar; cylindrical coordinates	$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$	$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$
Axisymmetric; cylindrical coordinates	$u_r = \frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$u_z = \frac{\partial \phi}{\partial z} = \frac{1}{r}\frac{\partial \psi}{\partial r}$

Irrotational Flow Approximation

2D Flows

- Method of Superposition

1. Since $\nabla^2\phi=0$ is linear, a linear combination of two or more solutions is also a solution, e.g., if ϕ_1 and ϕ_2 are solutions, then $(A\phi_1)$, $(\phi_1+\phi_2)$, $(A\phi_1+B\phi_2)$ are also solutions
2. Also true for ψ in 2D flows ($\nabla^2\psi=0$)
3. <u>Velocity components</u> are also additive

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial\left(\phi_1 + \phi_2\right)}{\partial x} = \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial x}$$

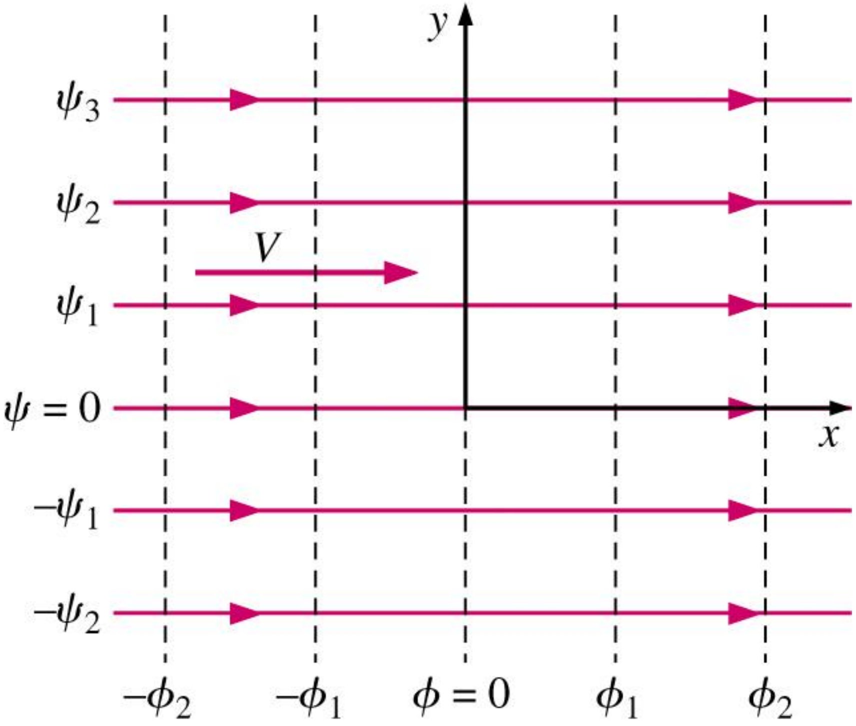
Irrotational Flow Approximation

2D Flows

- Given the principal of superposition, there are several elementary planar irrotational flows which can be combined to create more complex flows.
  - Uniform stream
  - Line vortex
  - Line source/sink
  - Doublet

Elementary Planar Irrotational Flows

Uniform Stream



- In Cartesian coordinates

$$\boxed{\phi = Vx, \quad \psi = Vy}$$

- Conversion to cylindrical coordinates can be achieved using the transformation

$$x = rcos\theta, \quad y = rsin\theta$$

$$\boxed{\phi = Vrcos\theta, \quad \psi = Vrsin\theta}$$

Elementary Planar Irrotational Flows

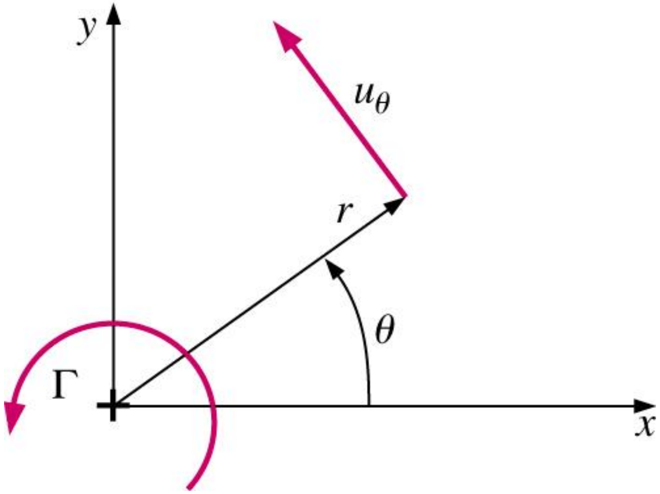
Line Vortex

- Vortex at the origin. First look at irrotationality condition which leads to the following velocity components

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

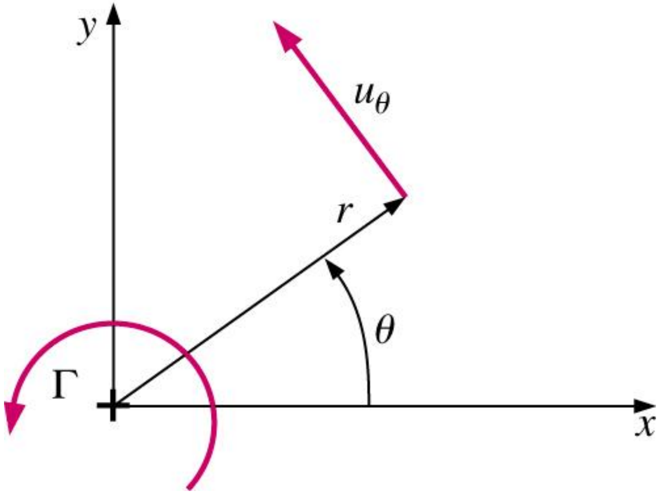
Γ is the circulation



Equations are for a vortex centered on the origin

Elementary Planar Irrotational Flows

Line Vortex



$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

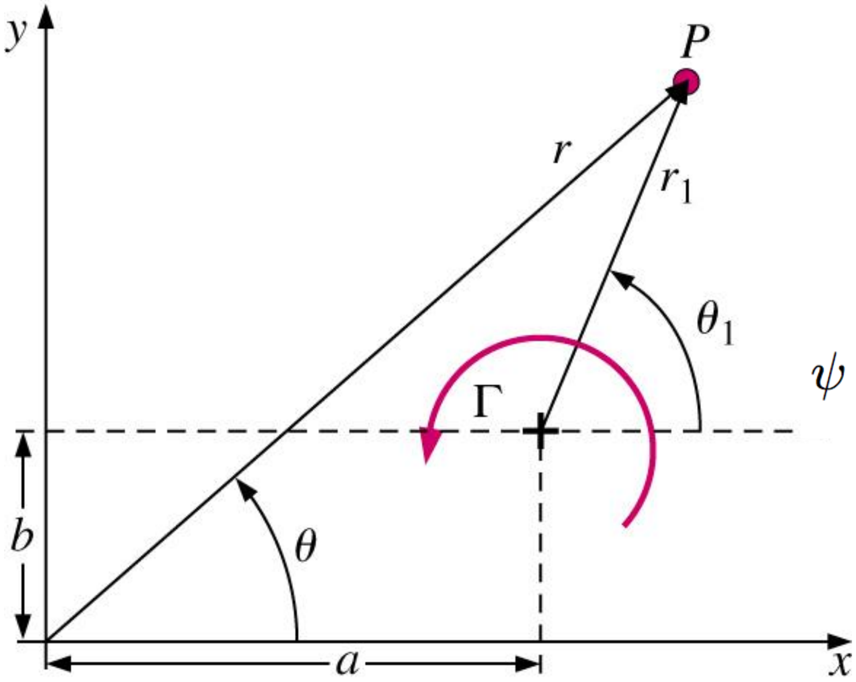
- Integrating:

$$\boxed{\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r}$$

Elementary Planar Irrotational Flows

Line Vortex

- If vortex is moved to $(x,y) = (a,b)$

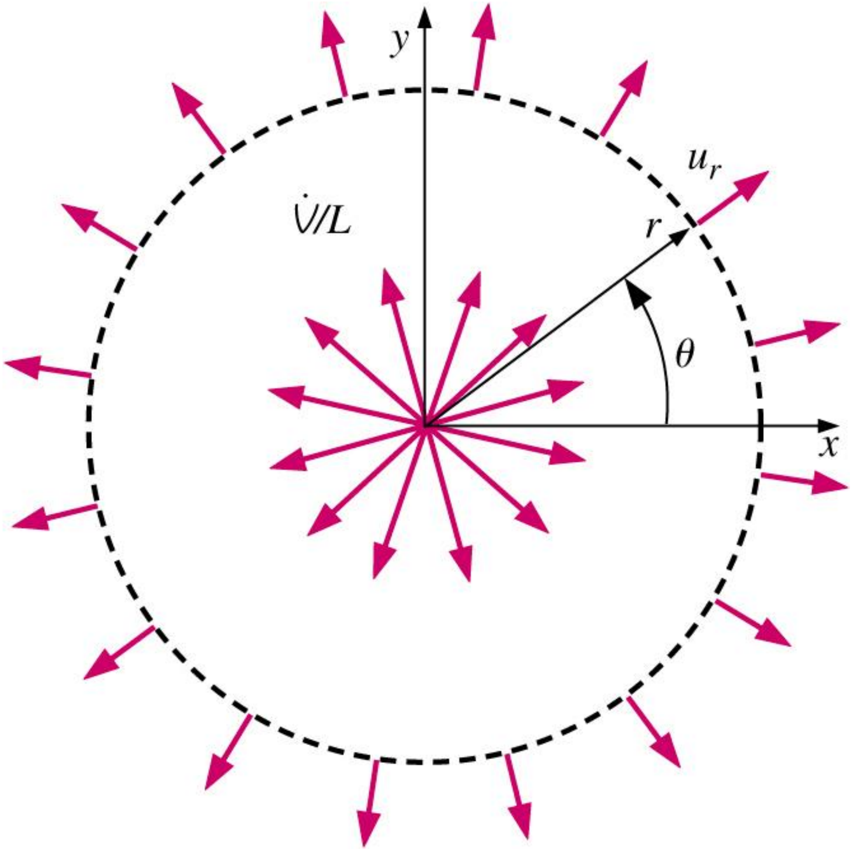


$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

Elementary Planar Irrotational Flows

Line Source/Sink

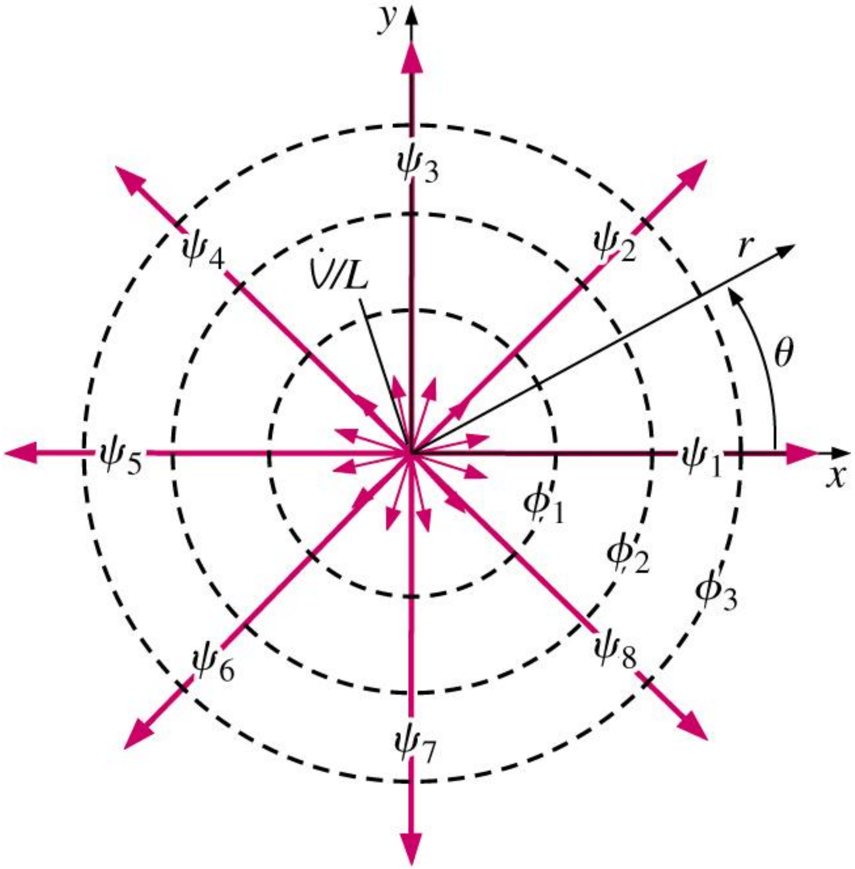


- Potential and stream-function are derived by observing that volume flow rate across any circle <u>in the x-y plane</u> is $\dot{V}/L$
- See also <u>continuity</u> equation
- This gives velocity components

$$U_r = \frac{\dot{V}/L}{2\pi r}, \quad U_\theta = 0$$

Elementary Planar Irrotational Flows

Line Source/Sink



- Using definition of (U_r, U_θ)

$$U_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$$

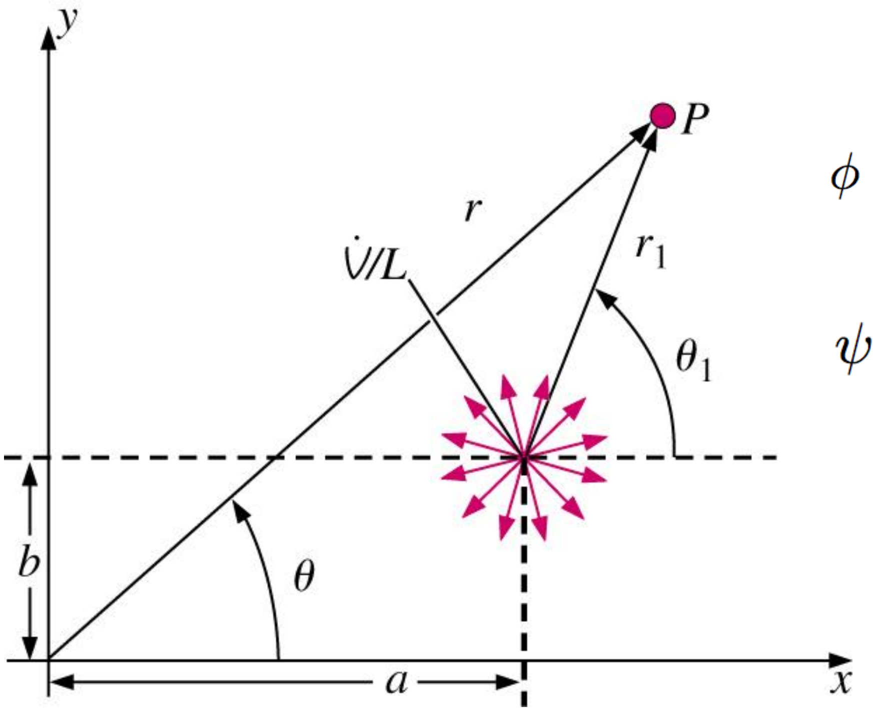
- These can be integrated to give ϕ and ψ

$$\boxed{\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r \qquad \psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta}$$

Equations are for a source/sink at the origin. Result is different in 3D.

Elementary Planar Irrotational Flows

Line Source/Sink



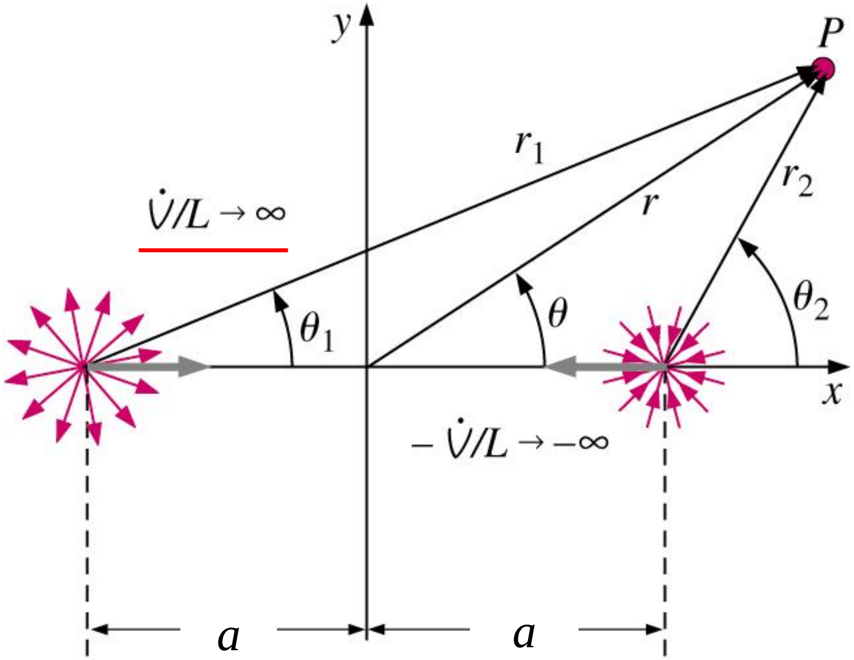
- If source/sink is moved to $(x,y) = (a,b)$

$$\phi = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln r_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\ln\sqrt{(x-a)^2+(y-b)^2}$$

$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 = \frac{\dot{\mathcal{V}}/L}{2\pi}\tan^{-1}\left(\frac{y-b}{x-a}\right)$$

Elementary Planar Irrotational Flows

Doublet



- A doublet is a combination of a line sink and source of equal magnitude
- Source

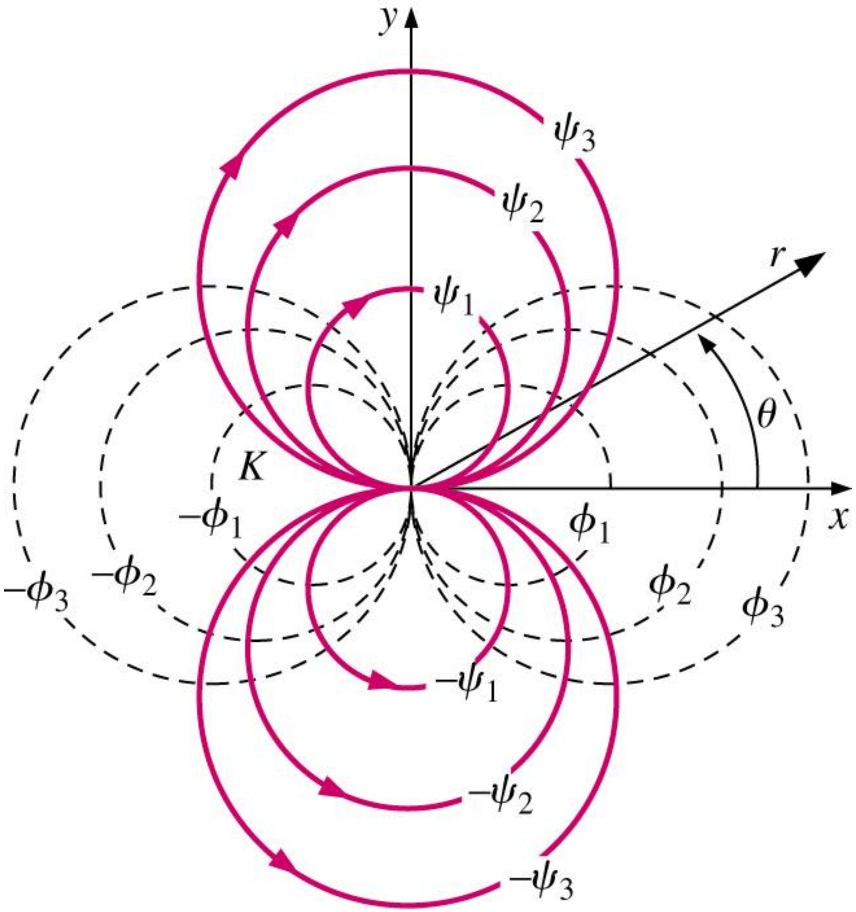
$$\psi = \frac{\dot{\mathcal{V}}/L}{2\pi}\theta_1 \quad \theta_1 = \tan^{-1}\left(\frac{y}{x+a}\right)$$

- Sink

$$\psi = -\frac{\dot{\mathcal{V}}/L}{2\pi}\theta_2 \quad \theta_2 = \tan^{-1}\left(\frac{y}{x-a}\right)$$

Elementary Planar Irrotational Flows

Doublet

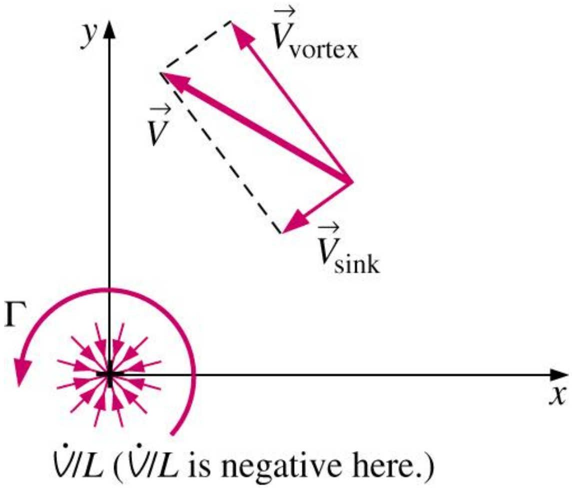


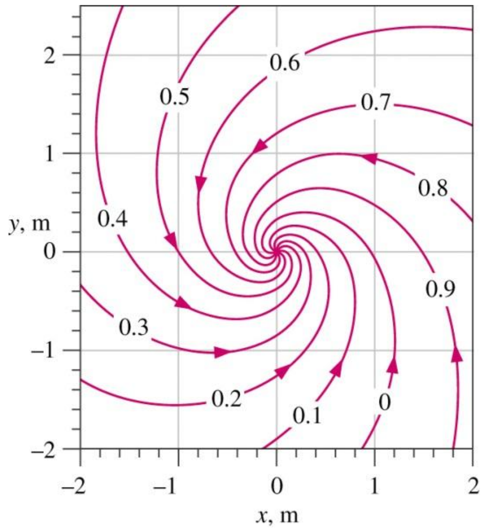
- Adding ψ_1 and ψ_2 together, performing some algebra, and taking $a \to 0$ gives

$$\boxed{\begin{aligned}\psi &= -K\frac{\sin\theta}{r}\\ \phi &= K\frac{\cos\theta}{r}\end{aligned}}$$

K is the doublet strength

Examples of Irrotational Flows Formed by Superposition





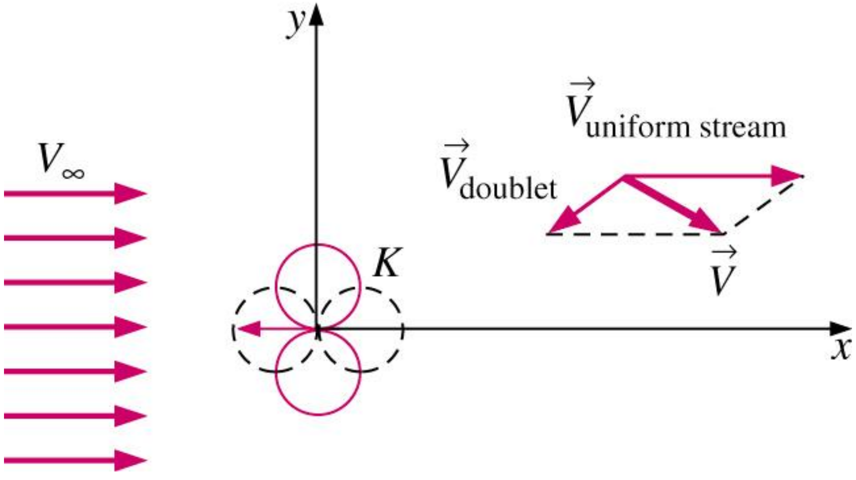
- Superposition of sink and vortex : bathtub vortex

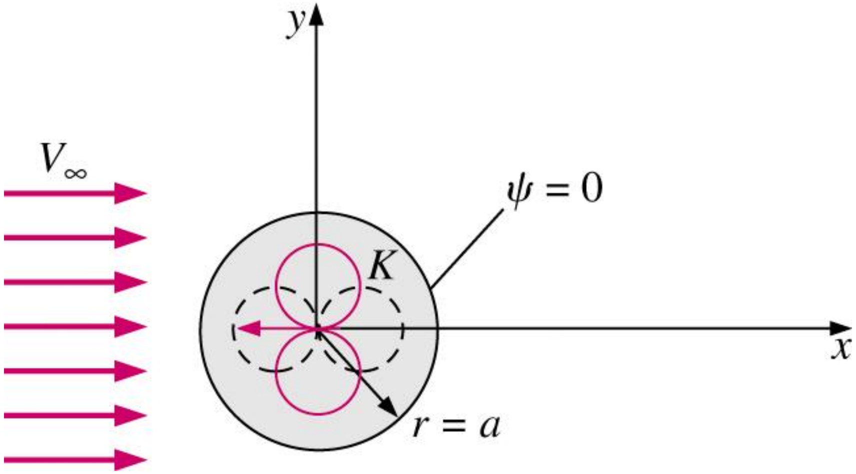
$$\psi = \underbrace{\frac{\dot{\mathcal{V}}/L}{2\pi}}_{\text{Sink}}\theta - \underbrace{\frac{\Gamma}{2\pi}\ln r}_{\text{Vortex}}$$

$$U_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\dot{\mathcal{V}}/L}{2\pi r}$$

$$U_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}$$

Examples of Irrotational Flows Formed by Superposition





- Flow over a circular cylinder: Free stream + doublet

$$\phi = V r \cos\theta + K \frac{\cos\theta}{r}$$

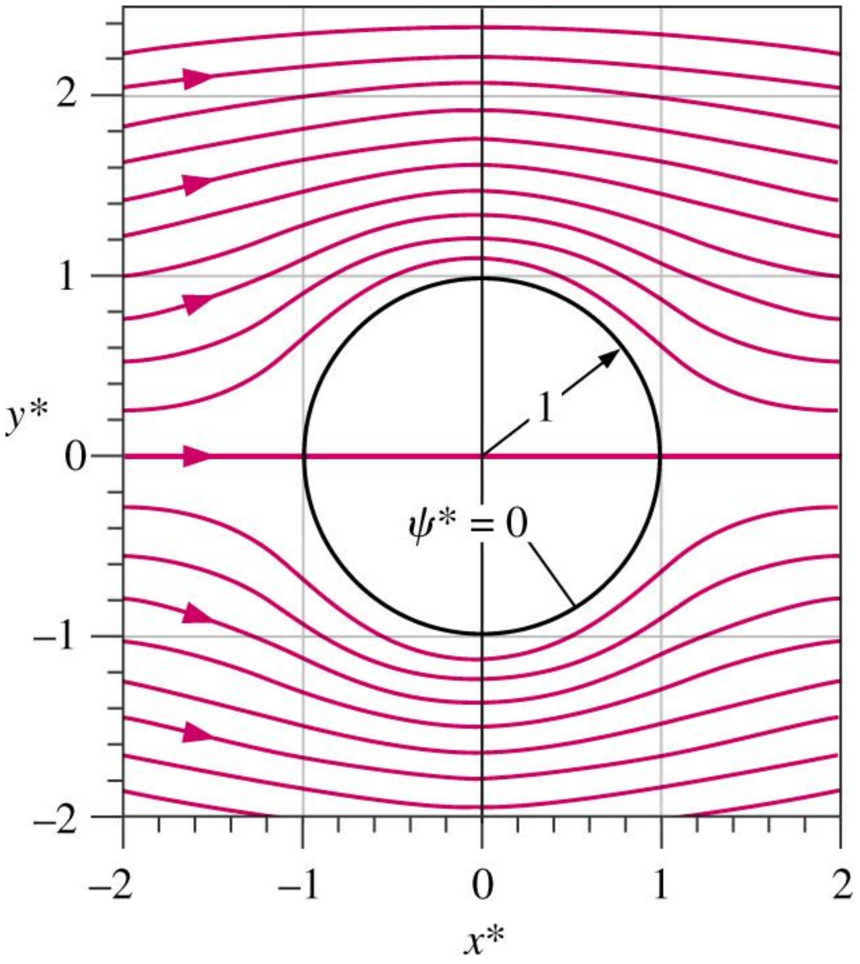
$$\psi = V r \sin\theta - K \frac{\sin\theta}{r}$$

- Assume body is $\psi = 0$

$$(r = a) \Rightarrow K = Va^2$$

$$\psi = V \sin\theta \left(r - a^2/r\right)$$

Examples of Irrotational Flows Formed by Superposition



- Velocity field can be found by differentiating streamfunction

$$U_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V\cos\theta\left(1 - a^2/r^2\right)$$

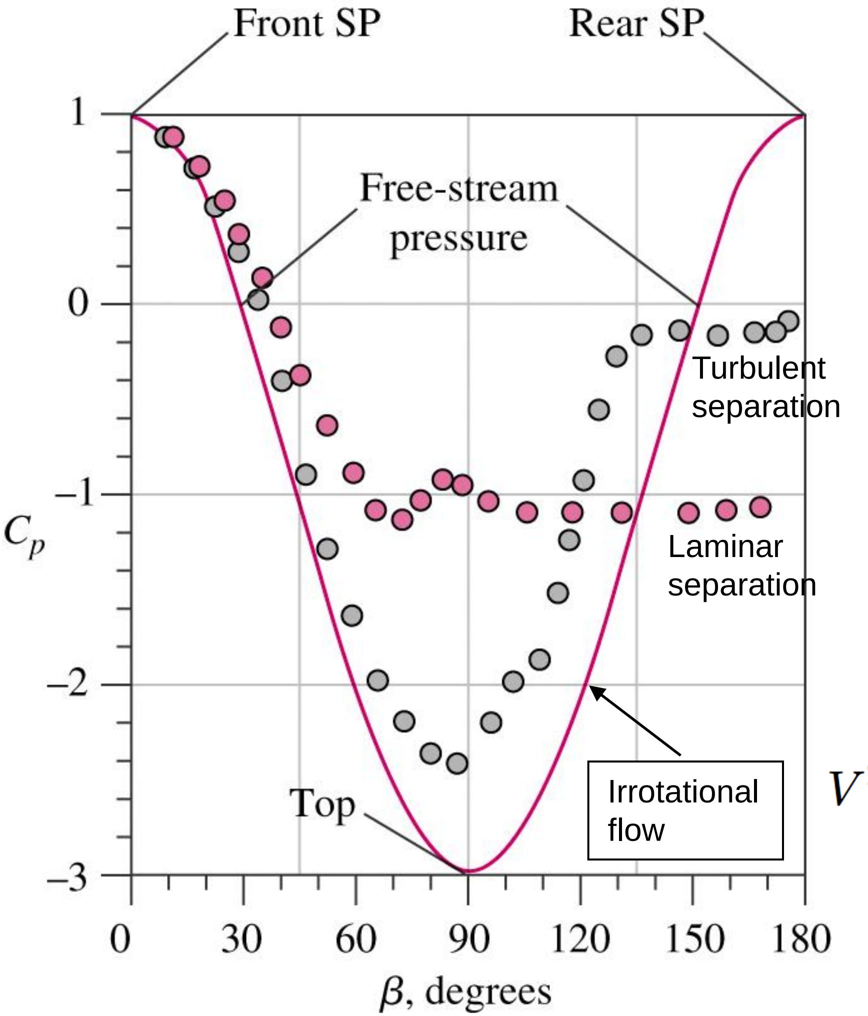
$$U_\theta = -\frac{\partial \psi}{\partial r} = -V\sin\theta\left(1 + a^2/r^2\right)$$

- On the cylinder surface ($r = a$)

$$U_r = 0, \quad U_\theta = -2V\sin\theta$$

Normal velocity (U_r) is zero, Tangential velocity (U_θ) is non-zero ⇒slip condition.

Examples of Irrotational Flows Formed by Superposition



- Compute pressure using Bernoulli equation and velocity on cylinder surface

$$\frac{P}{\rho}+\frac{V^2}{2}=\frac{P_\infty}{\rho}+\frac{V_\infty^2}{2}$$

$$C_P=\frac{P-P_\infty}{1/2\,\rho V^2}=1-\frac{V^2}{V_\infty^2}$$

$$V^2=U_r^2+U_\theta^2=0^2+(2V_\infty\sin\theta)^2=4V_\infty^2\sin^2\theta$$

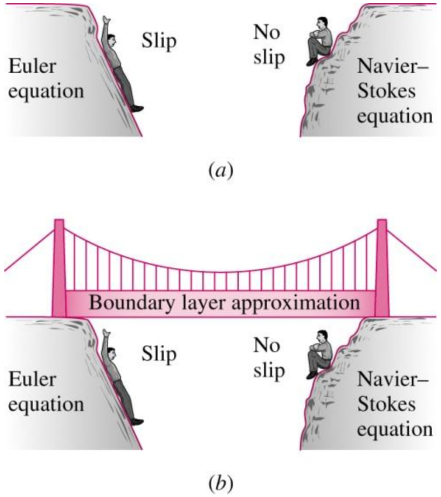
$$\boxed{C_P=1-4\sin^2\theta=1-4\sin^2\beta}$$

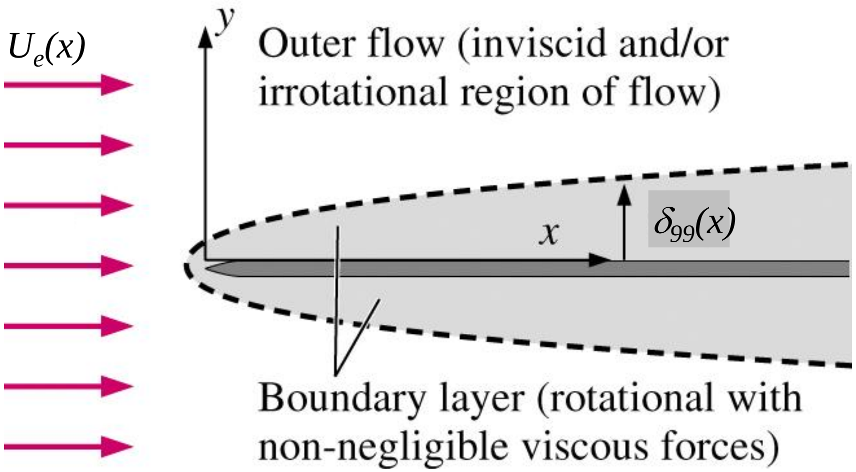
$$\beta=\pi-\theta$$

Examples of Irrotational Flows Formed by Superposition

- Integration of surface pressure (which is symmetric in x), reveals that the DRAG is ZERO. This is known as *D'Alembert's Paradox*
  - For the irrotational flow approximation, the drag force on ***any*** non-lifting body of ***any*** shape immersed in a uniform stream is **ZERO**
  - Why?
    - Viscous effects have been neglected. Viscosity and the no-slip condition are responsible for
      - Flow separation (which contributes to pressure drag)
      - Wall-shear stress (which contributes to friction drag)

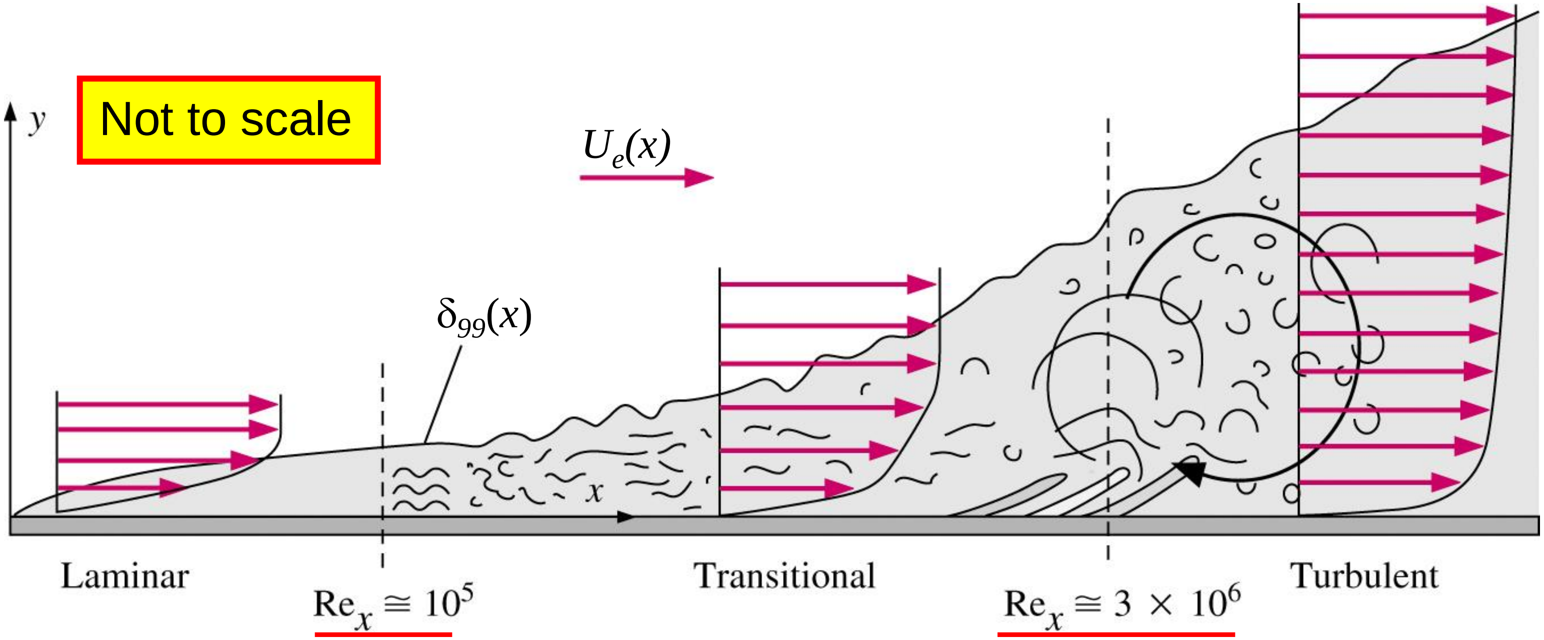
Boundary Layer (BL) Approximation



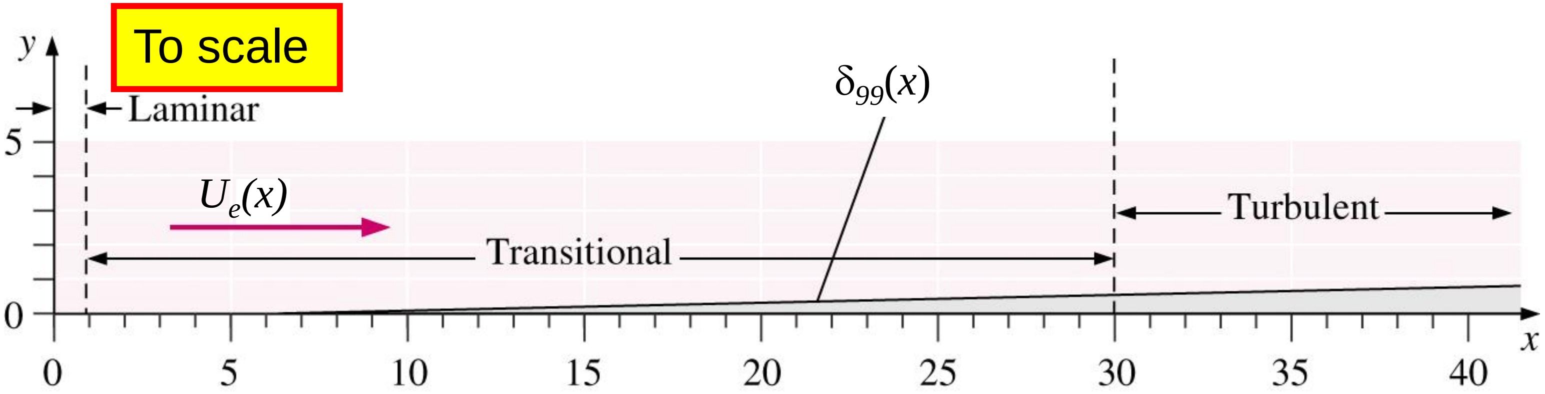


- BL approximation bridges the gap between the Euler and NS equations, and between the slip and no-slip BC at the wall.
- Prandtl (1904) introduced the BL approximation

Boundary Layer (BL) Approximation

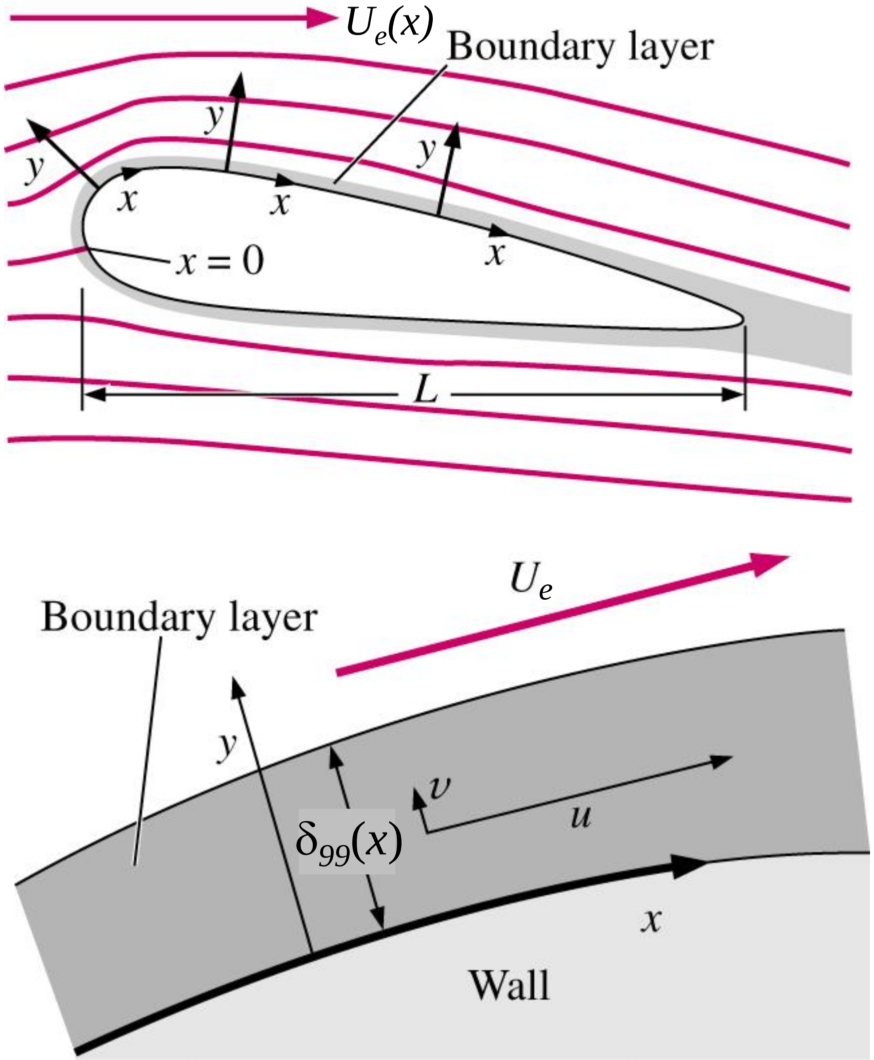


Not to scale



To scale

Boundary Layer (BL) Approximation



- BL Equations: we restrict attention to steady, 2D, laminar flow (although method is fully applicable to unsteady, 3D, turbulent flow)
- BL coordinate system
  - x : tangential direction
  - y : normal direction

Boundary Layer (BL) Approximation

- To derive the equations, start with the steady nondimensional NS equations

$$\left(\vec{V}^\star\cdot\nabla^\star\right)\vec{V}^\star = -\left[Eu\right]\nabla^\star P^\star + \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star$$

- Recall definitions $Eu = \frac{P_0 - P_\infty}{\rho\, U_e^2}$, $\quad Re = \frac{\rho\, U_e\, L}{\mu}$
- Since $\rho U_e^2 \sim P - P_\infty \rightarrow Eu \sim 1$
- $Re >> 1$, should we neglect viscous terms? No (!), because we would end up with the Euler equation along with deficiencies already discussed.
- Can we neglect **some** of the viscous terms?

Boundary Layer (BL) Approximation

- To answer this question, we need to do a normalization
  - Use L as length scale in streamwise direction and for derivatives of velocity and pressure with respect to x.
  - Use $\delta(x)$ (a quantity proportional to the boundary layer thickness δ_{99}) for distances and derivatives in y.
  - Use local outer (or edge) velocity U_e.

Boundary Layer (BL) Approximation

- Orders of Magnitude (OM)

$$U \sim U_e \quad P - P_\infty \sim \rho U_e^2, \quad \frac{\partial}{\partial x} \sim \frac{1}{L}, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

- What about *V*? Use continuity

$$\left.\underbrace{\frac{\partial U}{\partial x}}_{\sim U_e/L} + \underbrace{\frac{\partial V}{\partial y}}_{\sim V/\delta} = 0 \right\} \; V \sim \frac{U_e \delta}{L}$$

- Since $\delta/L \ll 1 \rightarrow V \ll U_e$

Boundary Layer (BL) Approximation

- Now, define new nondimensional variables

$$x^\star = \frac{x}{L}, \quad y^\star = \frac{y}{\delta}, \quad U^\star = \frac{U}{U_e}, \quad V^\star = \frac{VL}{U_e\delta}, \quad P^\star = \frac{P - P_\infty}{\rho U_e^2}$$

- All are order unity, therefore normalized
- Apply to x- and y-components of NSE
- ...

Boundary Layer (BL) Approximation

- Incompressible Laminar Boundary Layer Equations

Continuity

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

x-momentum

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = U_e\frac{\partial U_e}{\partial x} + \nu\frac{\partial^2 U}{\partial y^2}$$

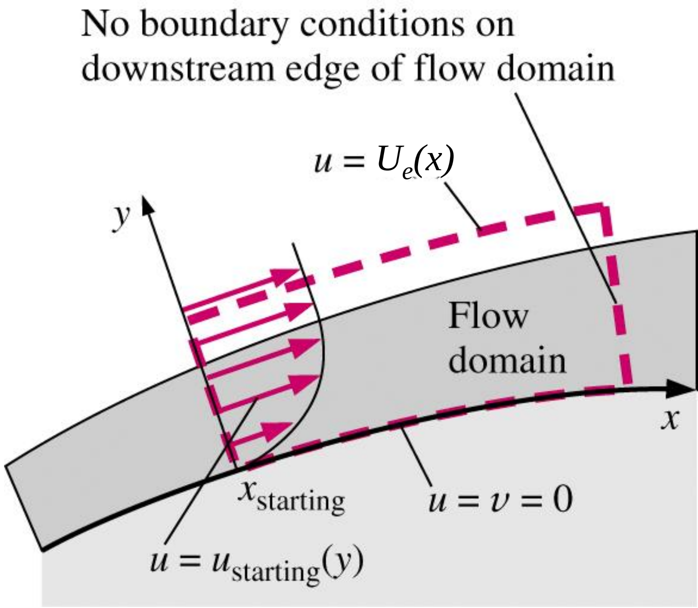
y-momentum

$$\frac{\partial P}{\partial y} = 0$$

(from now on use small letters to denote dependent variables)

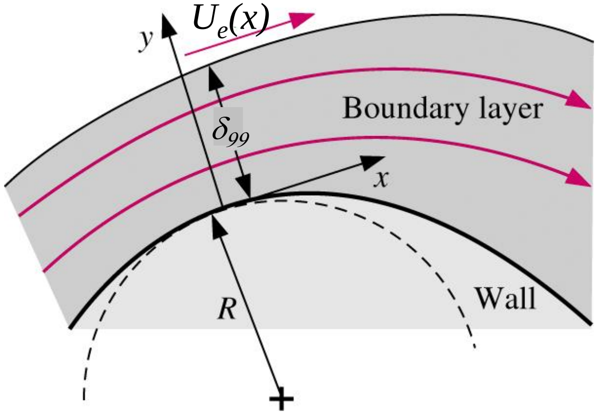
Boundary Layer Procedure

1. Solve for outer flow, ignoring the BL. Use potential flow (irrotational approximation) or Euler equation
2. Assume $\delta/L << 1$ (thin BL)
3. Solve BLE
   - $y = 0 \Rightarrow$ no-slip, $u=0, v=0$
   - $y = \delta_{99} \Rightarrow u = U_e(x)$
   - $x = x_0 \Rightarrow u = u_{starting}(x_0,y)$
4. Calculate δ, θ, δ^*, τ_w, Drag
5. Verify $\delta/L << 1$
6. If δ/L is not $<< 1$, use δ^* as body, go to step 1 and repeat



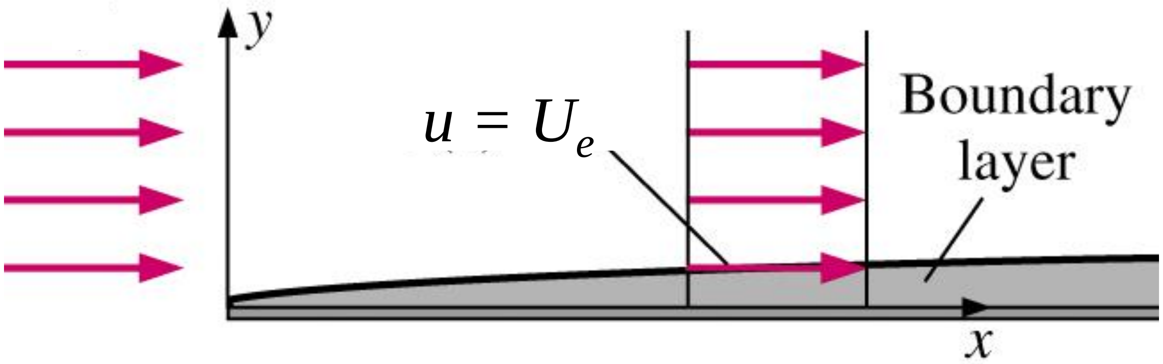
Boundary Layer Procedure

- Possible Limitations
  1. Re is not large enough ⇒ BL may be too thick for thin BL assumption.
  2. $\partial p/\partial y \neq 0$ due to wall curvature $\delta_{99} \sim R$
  3. Re too large ⇒ transitional flow starts at $Re \sim 10^5$. BL approximation still valid, but new terms required.
  4. Flow separation

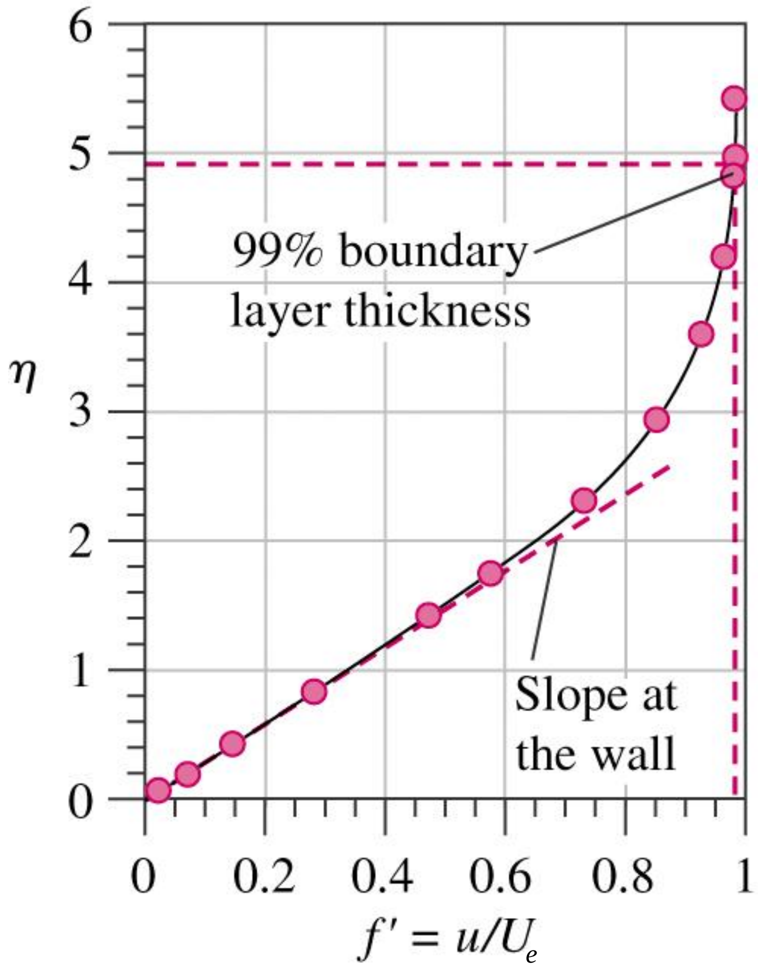


Boundary Layer Procedure

- Before defining and δ^* and θ, are there analytical solutions to the BL equations?
  - *Unfortunately, NO*
- *Blasius Similarity Solution* boundary layer on a flat plate, constant edge velocity, zero external pressure gradient (U_e = const.)



Blasius Similarity Solution



- Blasius introduced similarity variables

$$f' = \frac{u}{U_e} \qquad \eta = y\sqrt{\frac{U_e}{\nu x}}$$

- This reduces the BLE to

$$2f''' + ff'' = 0$$

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1$$

- This ODE can be solved using Runge-Kutta technique
- Result is a BL profile which holds at every station along the flat plate

Blasius Similarity Solution

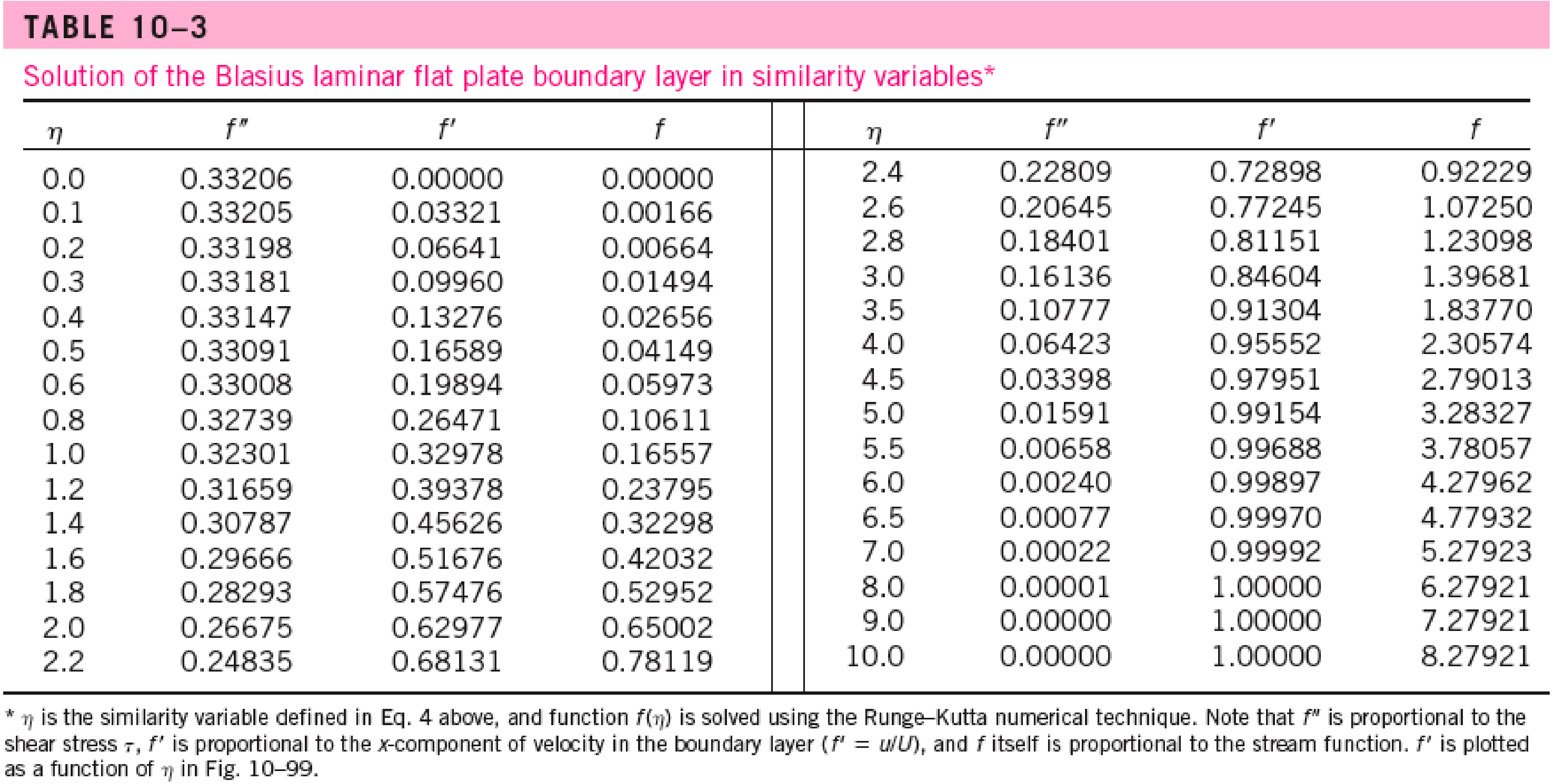


TABLE 10–3

Solution of the Blasius laminar flat plate boundary layer in similarity variables*

η	f''	f'	f	η	f''	f'	f
0.0	0.33206	0.00000	0.00000	2.4	0.22809	0.72898	0.92229
0.1	0.33205	0.03321	0.00166	2.6	0.20645	0.77245	1.07250
0.2	0.33198	0.06641	0.00664	2.8	0.18401	0.81151	1.23098
0.3	0.33181	0.09960	0.01494	3.0	0.16136	0.84604	1.39681
0.4	0.33147	0.13276	0.02656	3.5	0.10777	0.91304	1.83770
0.5	0.33091	0.16589	0.04149	4.0	0.06423	0.95552	2.30574
0.6	0.33008	0.19894	0.05973	4.5	0.03398	0.97951	2.79013
0.8	0.32739	0.26471	0.10611	5.0	0.01591	0.99154	3.28327
1.0	0.32301	0.32978	0.16557	5.5	0.00658	0.99688	3.78057
1.2	0.31659	0.39378	0.23795	6.0	0.00240	0.99897	4.27962
1.4	0.30787	0.45626	0.32298	6.5	0.00077	0.99970	4.77932
1.6	0.29666	0.51676	0.42032	7.0	0.00022	0.99992	5.27923
1.8	0.28293	0.57476	0.52952	8.0	0.00001	1.00000	6.27921
2.0	0.26675	0.62977	0.65002	9.0	0.00000	1.00000	7.27921
2.2	0.24835	0.68131	0.78119	10.0	0.00000	1.00000	8.27921

* η is the similarity variable defined in Eq. 4 above, and function $f(\eta)$ is solved using the Runge–Kutta numerical technique. Note that f'' is proportional to the shear stress τ, f' is proportional to the x-component of velocity in the boundary layer ($f' = u/U$), and f itself is proportional to the stream function. f' is plotted as a function of η in Fig. 10–99.

Blasius Similarity Solution

- Boundary layer thickness can be computed by assuming that δ_{99} corresponds to point where $u/U_e = 0.990$. At this point, $\eta = 4.91$, therefore

$$\eta = 4.91 = \sqrt{\frac{U_e}{\nu x}}\,\delta_{99} \quad \Rightarrow \quad \frac{\delta_{99}}{x} = \frac{4.91}{\sqrt{Re_x}}$$

Recall

$$Re_x = U_e\, x / \nu$$

- Wall shear stress τ_w and friction coefficient $C_{f,x}$ can be directly related to Blasius solution

$$\tau_w = \mu \left.\frac{\partial U}{\partial y}\right|_{y=0} = f''(0)\frac{\rho U_e^2}{\sqrt{Re_x}} = 0.332\frac{\rho U_e^2}{\sqrt{Re_x}} \qquad C_{f,x} = \frac{\tau_w}{\frac{1}{2}\rho U_e^2} = \frac{0.664}{\sqrt{Re_x}}$$

Displacement Thickness

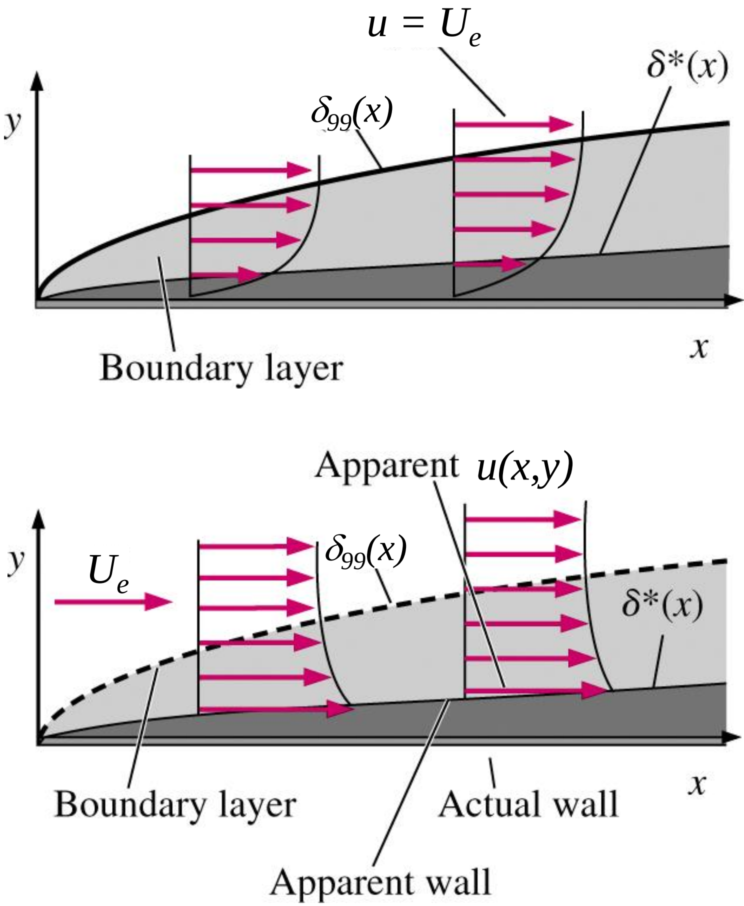
- Displacement thickness δ^* is the imaginary increase in thickness of the wall (or body), as seen by an ideal inviscid flow of same flow rate, and is due to the effect of a growing BL.
- Expression for δ^* is based upon control volume analysis of conservation of mass

$$\delta^\star = \int_0^\infty \left(1 - \frac{u}{U_e}\right) \mathrm{d}y$$

- Blasius profile for laminar BL can be integrated to give

$$\frac{\delta^\star}{x} = \frac{1.72}{\sqrt{Re_x}}$$

(≈1/3 of δ_{99})



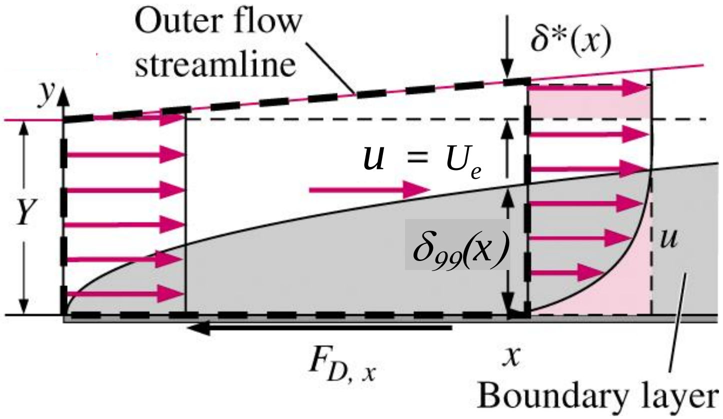
Momentum Thickness

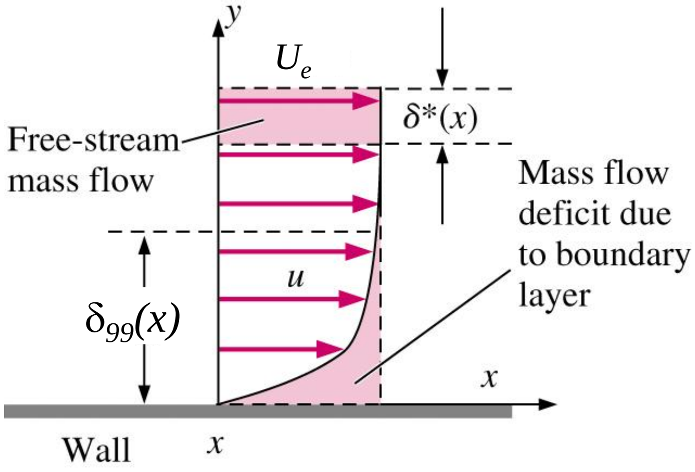
- Momentum thickness θ is another measure of boundary layer thickness.
- Defined as the loss of momentum flux per unit width divided by $\rho U_e^{\,2}$ due to the presence of the growing BL.
- Derived using CV analysis (*Karman integral equation*).

$$\theta = \int_0^\infty \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right)\, \mathrm{d}y = \frac{F_{D,x}}{\rho U_e^2 w}$$

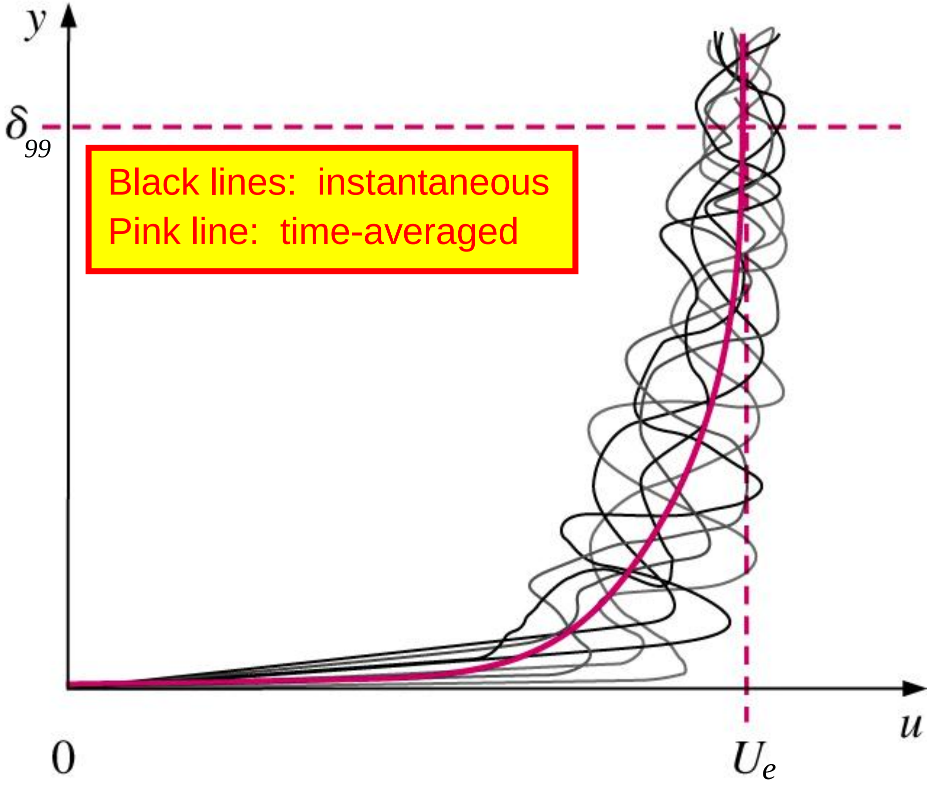
$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

θ for Blasius solution, identical to $C_{f,x}$



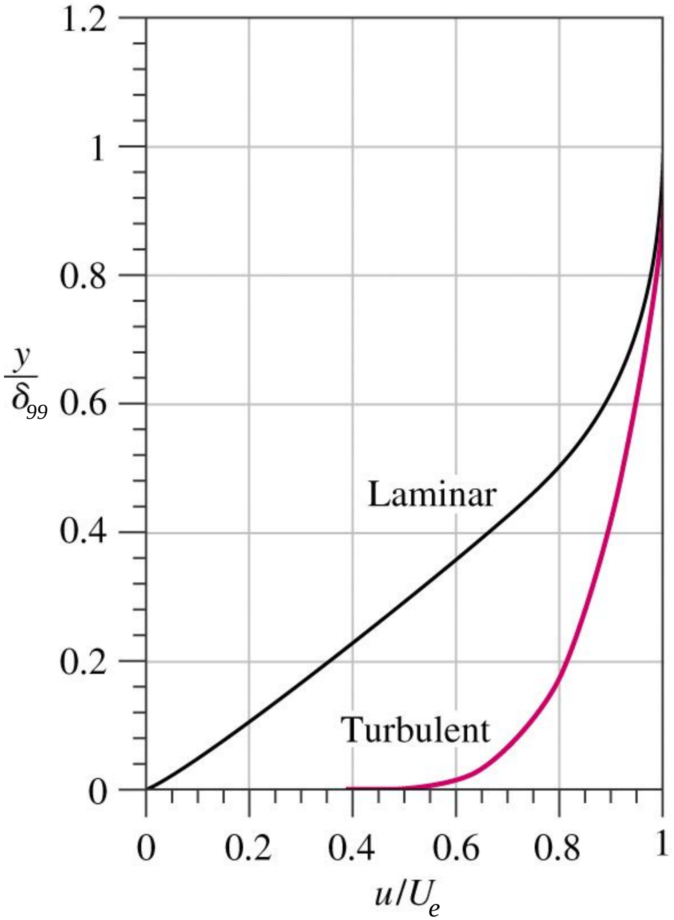


Turbulent Boundary Layer



Black lines: instantaneous
Pink line: time-averaged

Illustration of unsteadiness of a turbulent BL



Comparison of laminar and turbulent BL profiles

Turbulent Boundary Layer

- All BL variables [$\overline{u}(x,y)$, δ_{99}, δ^*, θ] are determined empirically.
- One common empirical approximation for the time-averaged velocity profile is the **one-seventh-power law**

$$\frac{\overline{u}}{U_e} = \left(\frac{y}{\delta_{99}}\right)^{1/7} \qquad y \le \delta_{99}$$

$$\frac{\overline{u}}{U_e} \cong 1 \qquad y > \delta_{99}$$

Turbulent Boundary Layer

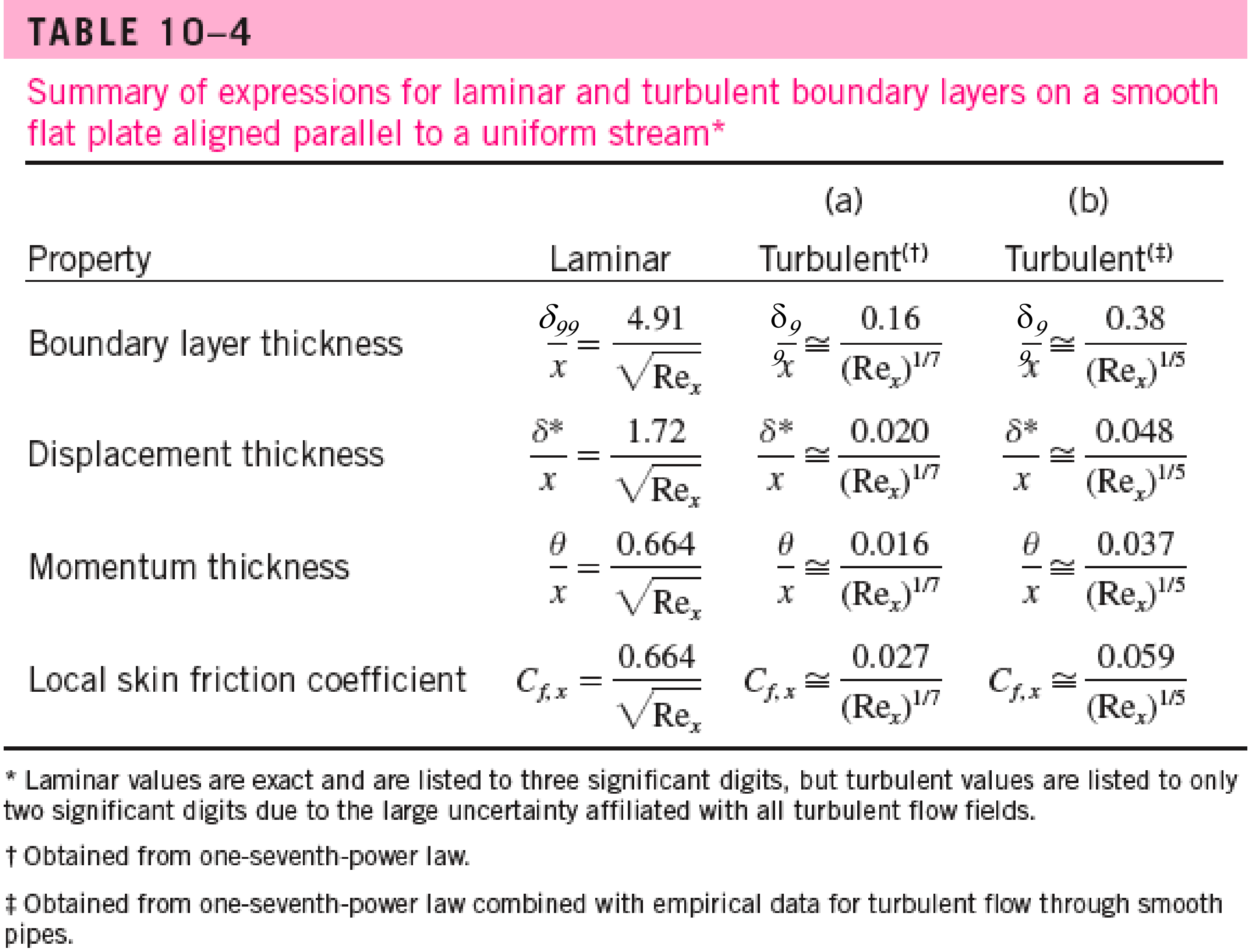


TABLE 10–4

Summary of expressions for laminar and turbulent boundary layers on a smooth flat plate aligned parallel to a uniform stream*

Property	Laminar	(a) Turbulent(†)	(b) Turbulent(‡)
Boundary layer thickness	$\frac{\delta_{99}}{x} = \frac{4.91}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta_{9}}{x} \cong \frac{0.16}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta_{9}}{x} \cong \frac{0.38}{(\mathrm{Re}_x)^{1/5}}$
Displacement thickness	$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{\mathrm{Re}_x}}$	$\frac{\delta^*}{x} \cong \frac{0.020}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\delta^*}{x} \cong \frac{0.048}{(\mathrm{Re}_x)^{1/5}}$
Momentum thickness	$\frac{\theta}{x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$\frac{\theta}{x} \cong \frac{0.016}{(\mathrm{Re}_x)^{1/7}}$	$\frac{\theta}{x} \cong \frac{0.037}{(\mathrm{Re}_x)^{1/5}}$
Local skin friction coefficient	$C_{f,x} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$	$C_{f,x} \cong \frac{0.027}{(\mathrm{Re}_x)^{1/7}}$	$C_{f,x} \cong \frac{0.059}{(\mathrm{Re}_x)^{1/5}}$

* Laminar values are exact and are listed to three significant digits, but turbulent values are listed to only two significant digits due to the large uncertainty affiliated with all turbulent flow fields.

† Obtained from one-seventh-power law.

‡ Obtained from one-seventh-power law combined with empirical data for turbulent flow through smooth pipes.

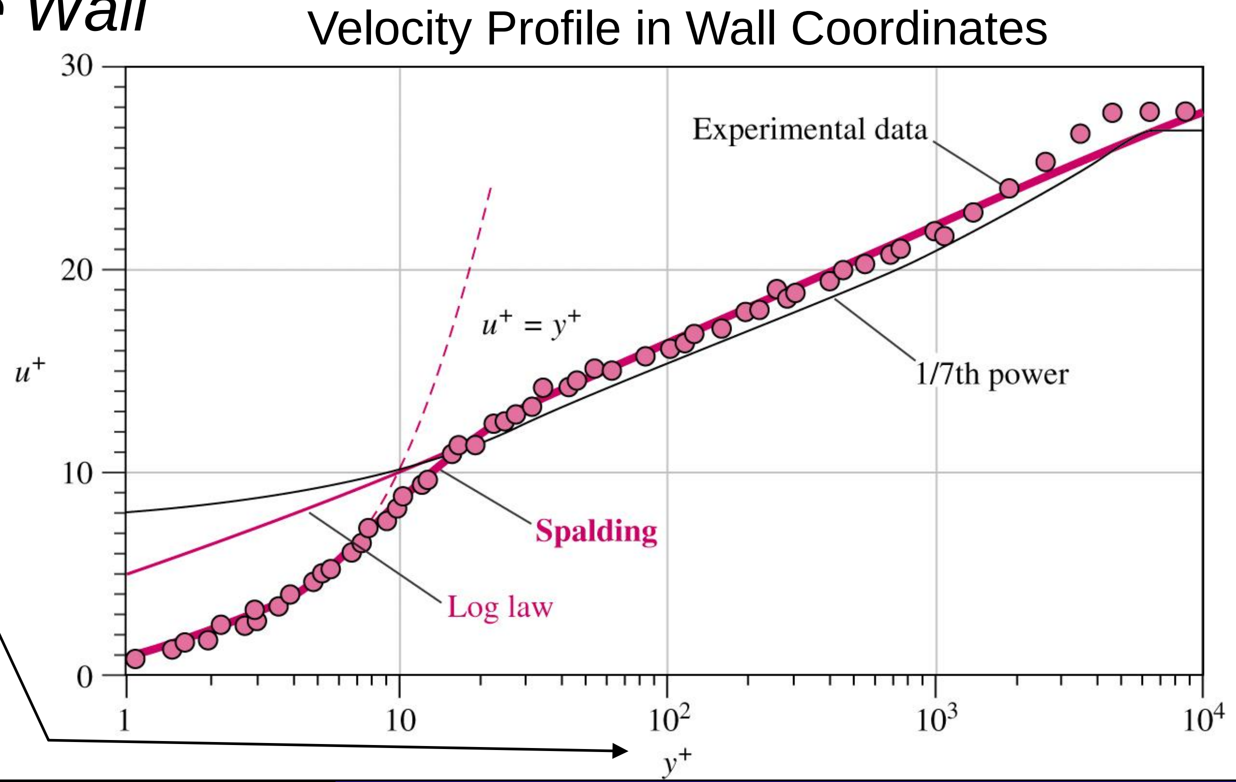
Turbulent Boundary Layer

- Flat plate zero-pressure-gradient TBL can be plotted in a universal form if a new velocity scale, called the friction velocity U_τ, is used. Sometimes referred to as the "*Law of the Wall*"

$$u^+ = \frac{\overline{u}}{U_\tau}$$

$$y^+ = \frac{U_\tau y}{\nu}$$

$$\boxed{U_\tau = \sqrt{\frac{\tau_w}{\rho}}}$$


Velocity Profile in Wall Coordinates

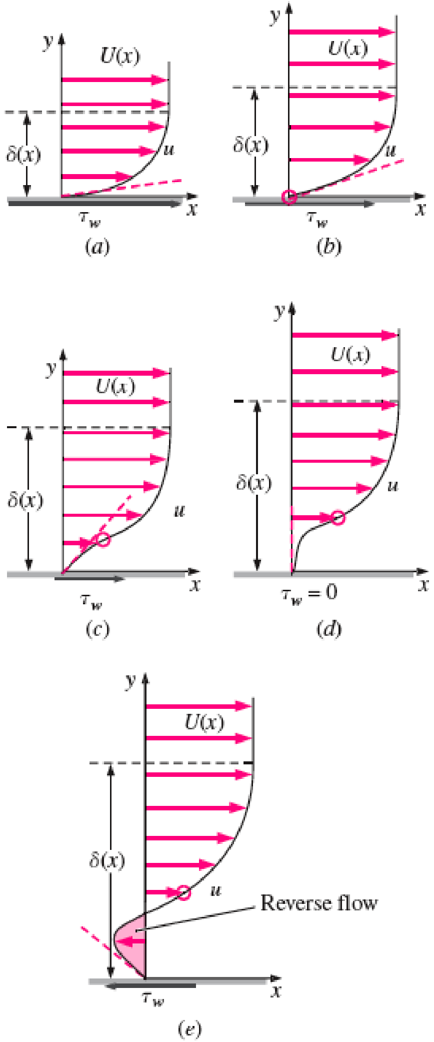
Turbulent Boundary Layer

- Despite its simplicity, the *Law of the Wall* is the basis for many CFD turbulence models.
- Spalding (1961) developed a formula which is valid over most of the boundary layer

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right]$$

  - κ, B are constants

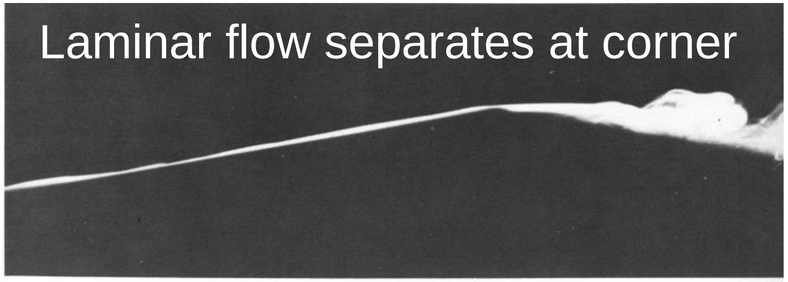
Pressure Gradients



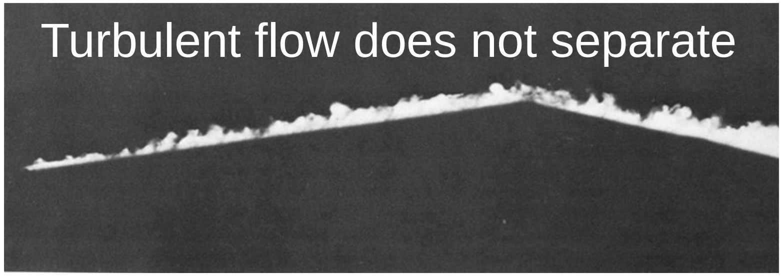
- Shape of the BL is strongly influenced by external pressure gradient
  - (a) favorable ($dp/dx < 0$)
  - (b) zero
  - (c) mild adverse ($dp/dx > 0$)
  - (d) critical adverse ($\tau_w = 0$)
  - (e) large adverse with reverse (or separated) flow

Pressure Gradients

- The BL approximation is not valid downstream of a separation point because of reverse flow in the separation bubble.
- Turbulent BL is more resistant to flow separation than laminar BL exposed to the same adverse pressure gradient



Laminar flow separates at corner



Turbulent flow does not separate