ROUTES TO TRANSITION IN SHEAR FLOWS

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with contributions from:

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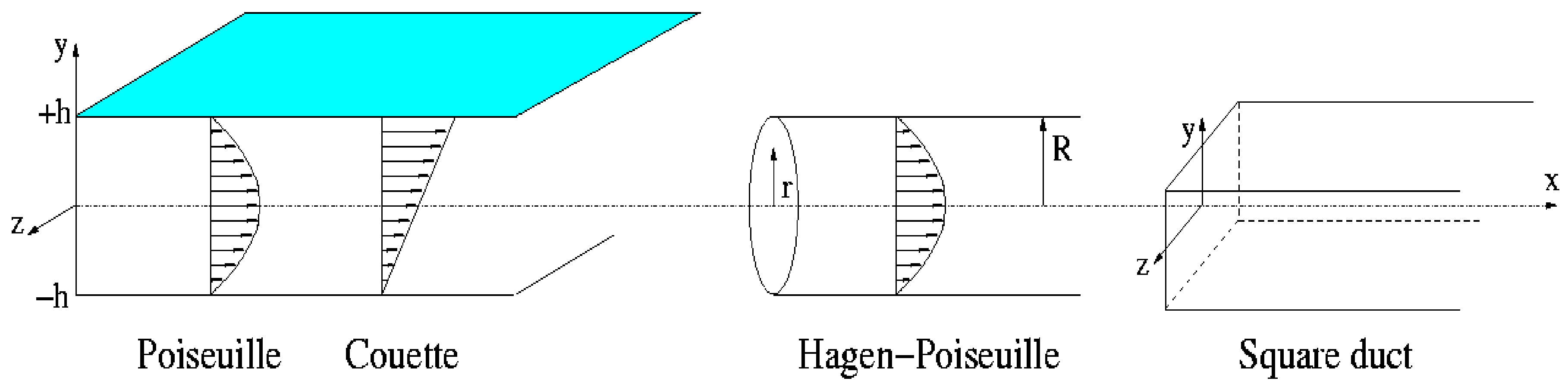
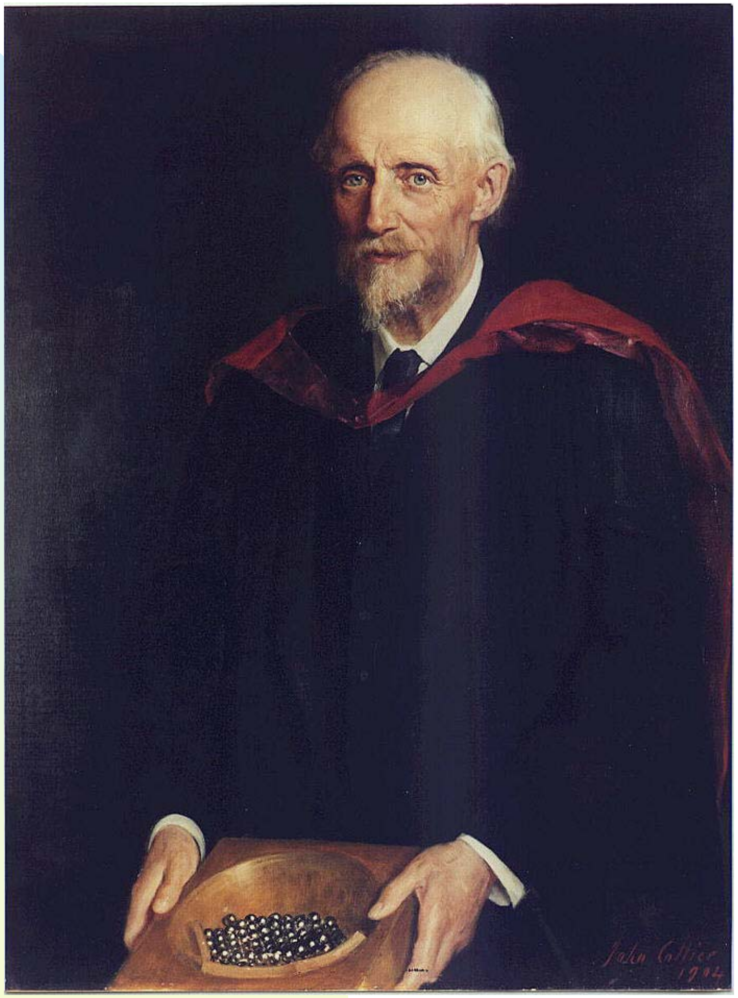
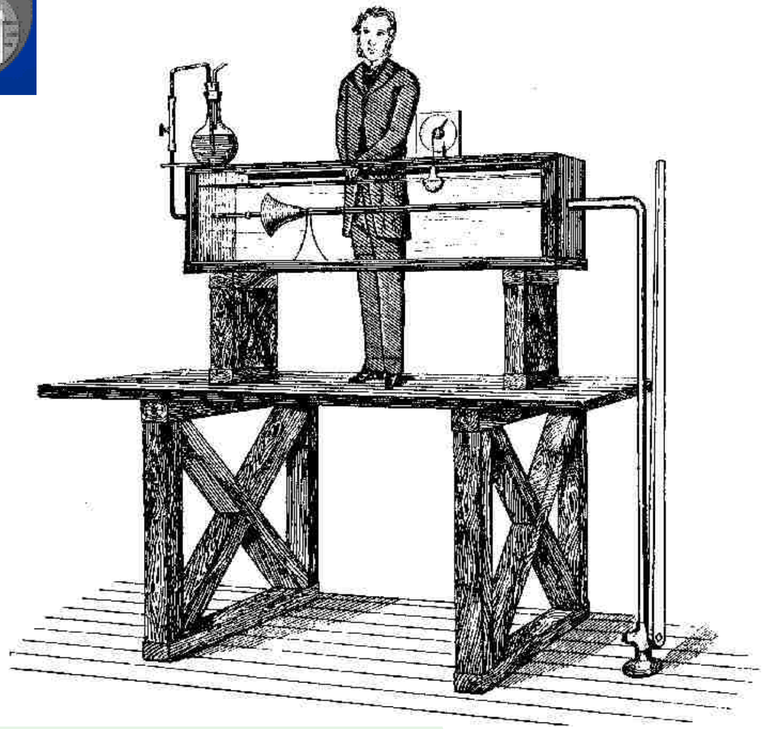


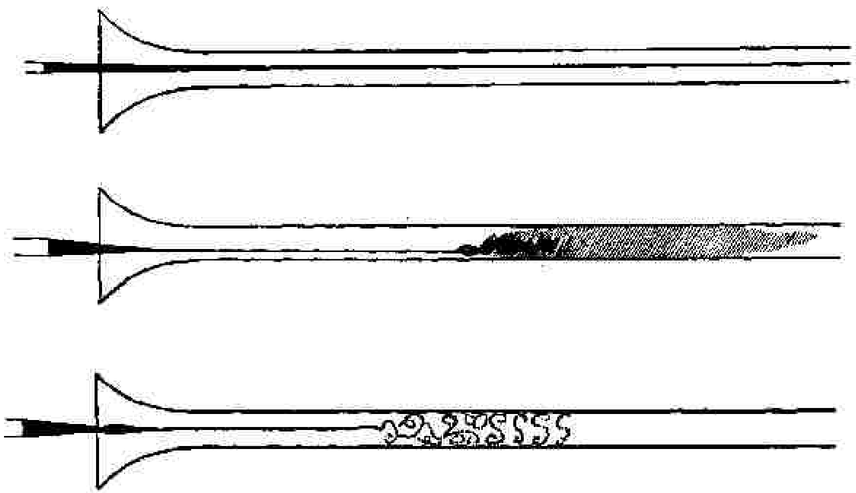
Osborne Reynolds, 1842-1912



"An experimental investigation of the circumstances which determine whether motion of water shall be direct or sinuous and of the law of resistance in parallel channels", *Royal Society, Phil. Trans*. 1883



`the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water ... On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less distinct curls, showing eddies.'



$Re_{trans} \approx 13000$

Hydrodynamic stability theory followed a few years later:

W. M'F.Orr, 'The stability or instability of the steady motions of a perfect liquid and of a viscous liquid', *Proc. Roy. Irish Academy*, 1907

A. Sommerfeld, 'Ein Beitrag zur hydrodynamischen Erklaerung der turbulenten Fluessigkeitsbewegungen', *Proc. 4th International Congress of Mathematicians*, Rome, 1908

Hints on the solution of the stability equations for the flow in a pipe arrived only much later (C.L. Pekeris, 1948), just to show that

$$Re_{crit} \to \infty$$

(!!)

STILL TODAY, TRANSITION IN SHEAR FLOWS IS STILL NOT FULLY UNDERSTOOD. For the simplest parallel flows there is poor agreement between predictions from the classical linear stability theory (Re_{crit}) and experimentals results (Re_{trans})

	Poiseuille	Couette	Hagen-Poiseuille	Square duct
Re_{crit}	5772	∞	∞	∞
Re_{trans}	~2000	~400	~2000	~2000

THE TRANSITION PROCESS

- Receptivity phase: the flow filters environmental disturbances

- Initial phase
  - **ROUTE 1: TRANSIENT GROWTH**
  - **ROUTE 2: EXPONENTIAL GROWTH**

- Late, nonlinear stages of transition

SMALL AMPLITUDE DISTURBANCES

- Linearized Navier-Stokes equations
  [to make things simple: cartesian coordinates, U=U(y)]

$$u_x + v_y + w_z = 0$$

$$u_t + Uu_x + vU' = -\frac{1}{\rho} p_x + \nu \Delta u$$

$$v_t + Uv_x = -\frac{1}{\rho} p_y + \nu \Delta v$$

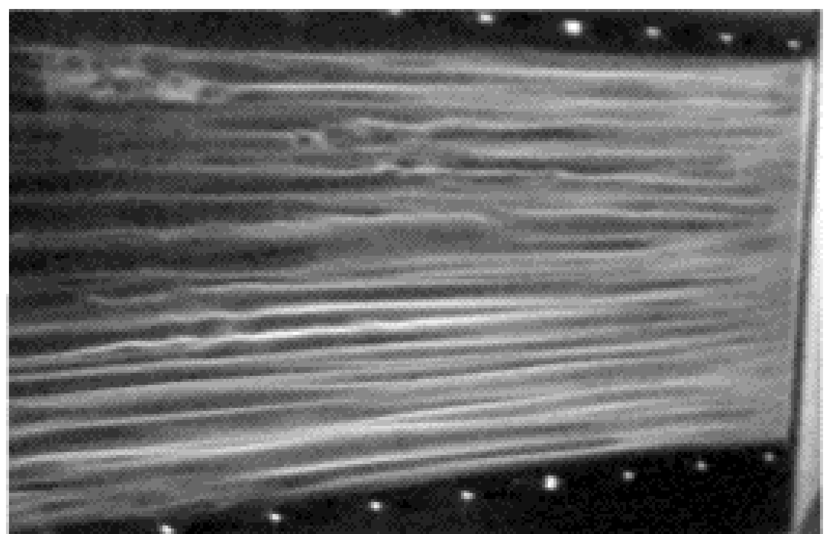
$$w_t + Uw_x = -\frac{1}{\rho} p_z + \nu \Delta w$$

  with homogeneous boundary conditions for (u, v, w) at $y=\pm h$

- Goal: SPATIAL EVOLUTION OF DISTURBANCES

ROUTE 1: TRANSIENT GROWTH

- **<u>THE MECHANISM</u>**: a stationary algebraic instability exists in the inviscid system ("lift-up" effect). In the viscous case the growth of the disturbance energy is hampered by diffusion $\Rightarrow$ <u>transient growth</u>



P.H. Alfredsson and M. Matsubara (1996): streaky structures in a boundary layer. Free-stream speed: 2 [m/s], free-stream turbulence level: 6%

INVISCID ANALYSIS: the physics suggests to employ different length and velocity scales along different directions (introducing a parameter $\varepsilon << 1$)

$$x \to h/\varepsilon \qquad y, z \to h$$

$$U, u \to U_{max} \qquad v, w \to \varepsilon\, U_{max}$$

$$p \to \rho\, (\varepsilon\, U_{max})^2$$

so that the steady inviscid equations to leading order are:

$$u_x + v_y + w_z = 0$$

$$U u_x + v U' = 0$$

$$U v_x = -p_y$$

$$U w_x = -p_z$$

Look for similarity solutions of the form:

$$u(x,y,z) = x^{\lambda}\hat{u}(y)\ e^{i\beta z} + c.c.$$

$$v(x,y,z) = x^{\lambda-1}\hat{v}(y)\ e^{i\beta z} + c.c.$$

$$w(x,y,z) = x^{\lambda-1}\hat{w}(y)\ e^{i\beta z} + c.c.$$

$$p(x,y,z) = x^{\lambda-2}\hat{p}(y)\ e^{i\beta z} + c.c.$$

By substituting into the equations it is easy to find that $\lambda = 1$ is a solution, so that:

$$u(x,y,z) = -\underline{x}\frac{U'}{U}\hat{v}_0(y)\ e^{i\beta z}$$

ALGEBRAIC GROWTH

$$v(x,y,z) = \hat{v}_0(y)\ e^{i\beta z}$$

$$w(x,y,z) = \frac{i}{\beta}\left[\hat{v}_{0\ y} - \hat{v}_0\frac{U'}{U}\right]\ e^{i\beta z}$$

VISCOUS ANALYSIS

- Classical approach: use ($U_{max}, \; h, \; \frac{h}{U_{max}}, \; \rho U_{max}^2$) as scales and find the Orr-Sommerfeld/Squire system, with $Re = \frac{U_{max} h}{\nu}$ and $\eta = \partial_z u - \partial_x w$

$$\left[\left(\partial_t + U\partial_x - Re^{-1}\Delta \right)\Delta - U''\partial_x \right] v = 0$$
$$\left(\partial_t + U\partial_x - Re^{-1}\Delta \right)\eta = -U'\partial_z v$$

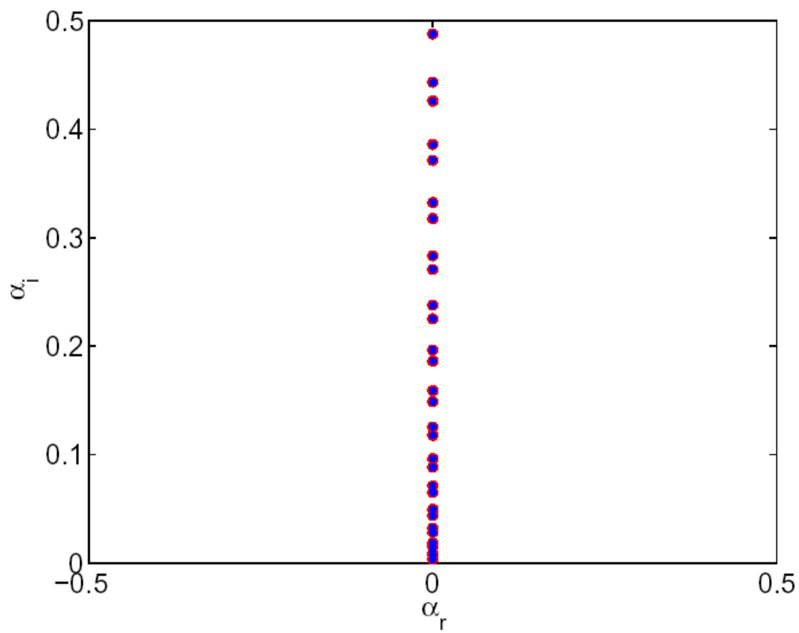
- Two-scale approach: use ($\frac{h}{\varepsilon U_{max}}$) as time scale and find a reduced Os/Squire system

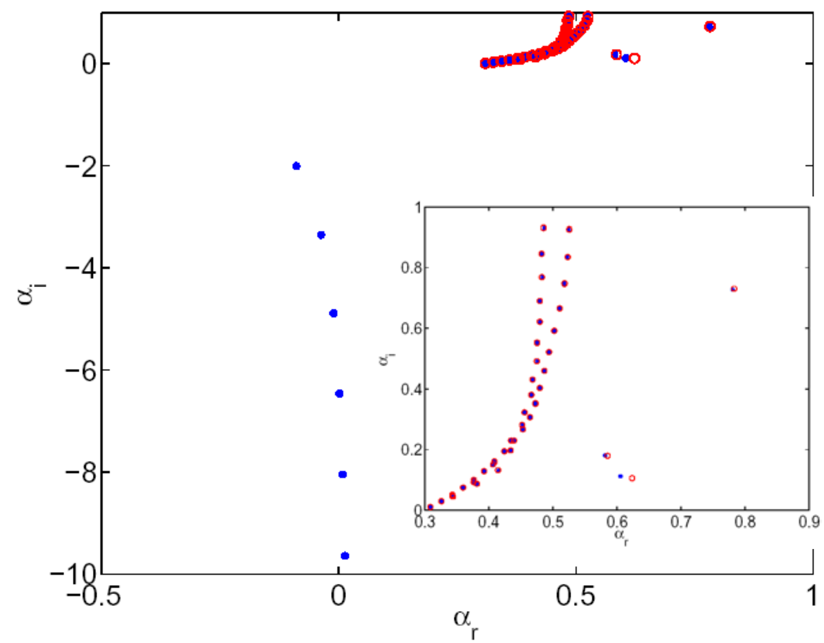
$$\left[\left(\partial_t + U\partial_x - \Delta_2 \right)\Delta_2 - U''\partial_x \right] v = 0$$
$$\left(\partial_t + U\partial_x - \Delta_2 \right)\eta = -U'\partial_z v$$

ROUTE 1: TRANSIENT GROWTH

Normal modes: $[v,\eta]=[\hat{v},\hat{\eta}](y)e^{i(\alpha_{(p)}x+\beta z-\omega_{(p)}t)}+c.c.$, $\beta,\omega\in\Re$

Poiseuille flow: eigenmodes full system *vs* parabolic model





$Re = 2000,\ \omega = 0,\ \beta = 1.91$

$Re = 2000,\ \omega = 0.3,\ \beta = 1.91$

Parabolic: $\alpha_p = \alpha\, Re,\ \omega_p = \omega\, Re$

Full: $\alpha,\ \omega$

ROUTE 1: TRANSIENT GROWTH

"Optimal" disturbances: most dangerous initial conditions at $x = 0$, *i.e.* those that maximize the output disturbance energy

$$G(x) = \frac{E(x)}{E(0)} = \frac{\iint_S \bar{u}u + \bar{v}v + \bar{w}w \big|_x \; dy \; dz}{\iint_S (\bar{u}_0 u_0 + \bar{v}_0 v_0 + \bar{w}_0 w_0) \; dy \; dz}$$

In boundary layer scalings it is obvious that, since $u = \mathcal{O}(U_{max})$ and $(v,w) = \mathcal{O}(U_{max}/Re)$, G is rendered maximum when $u_0 = 0$. Then:

$$G \approx Re^2 \frac{\iint_S \bar{u}u \big|_x \; dy \; dz}{\iint_S (\bar{v}_0 v_0 + \bar{w}_0 w_0) \; dy \; dz}$$

Decompose the generic disturbance vector $\mathbf{q}$ as a sum of normal modes:

$$\mathbf{q}(x,y,z,t) = \kappa_n \hat{\mathbf{q}}_n(y)\; e^{i(\alpha_n x + \beta_n z - \omega_n t)}$$

Then:

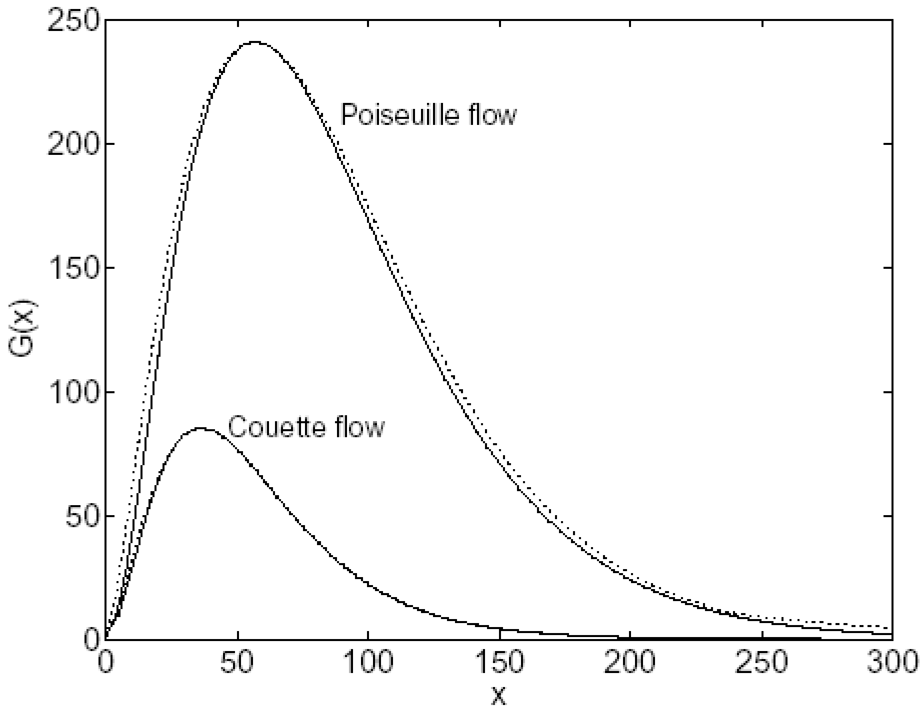
$$G(x) \approx Re^2 \frac{\int_{-1}^{1} \bar{\kappa}_n \bar{\hat{u}}_n e^{-i\bar{\alpha}_n x + i\alpha_m x} \hat{u}_m \kappa_m \; dy}{\int_{-1}^{1} \bar{\kappa}_n [\bar{\hat{v}}_n \hat{v}_m + \bar{\hat{w}}_n \hat{w}_m] \kappa_m \; dy} = Re^2 \frac{\bar{\kappa}_n A_{nm}(x) \kappa_m}{\bar{\kappa}_n B_{nm} \kappa_m}$$

Rayleigh quotient. The largest gain at each x is the largest eigenvalue $\frac{\lambda}{Re^2}$ solution of

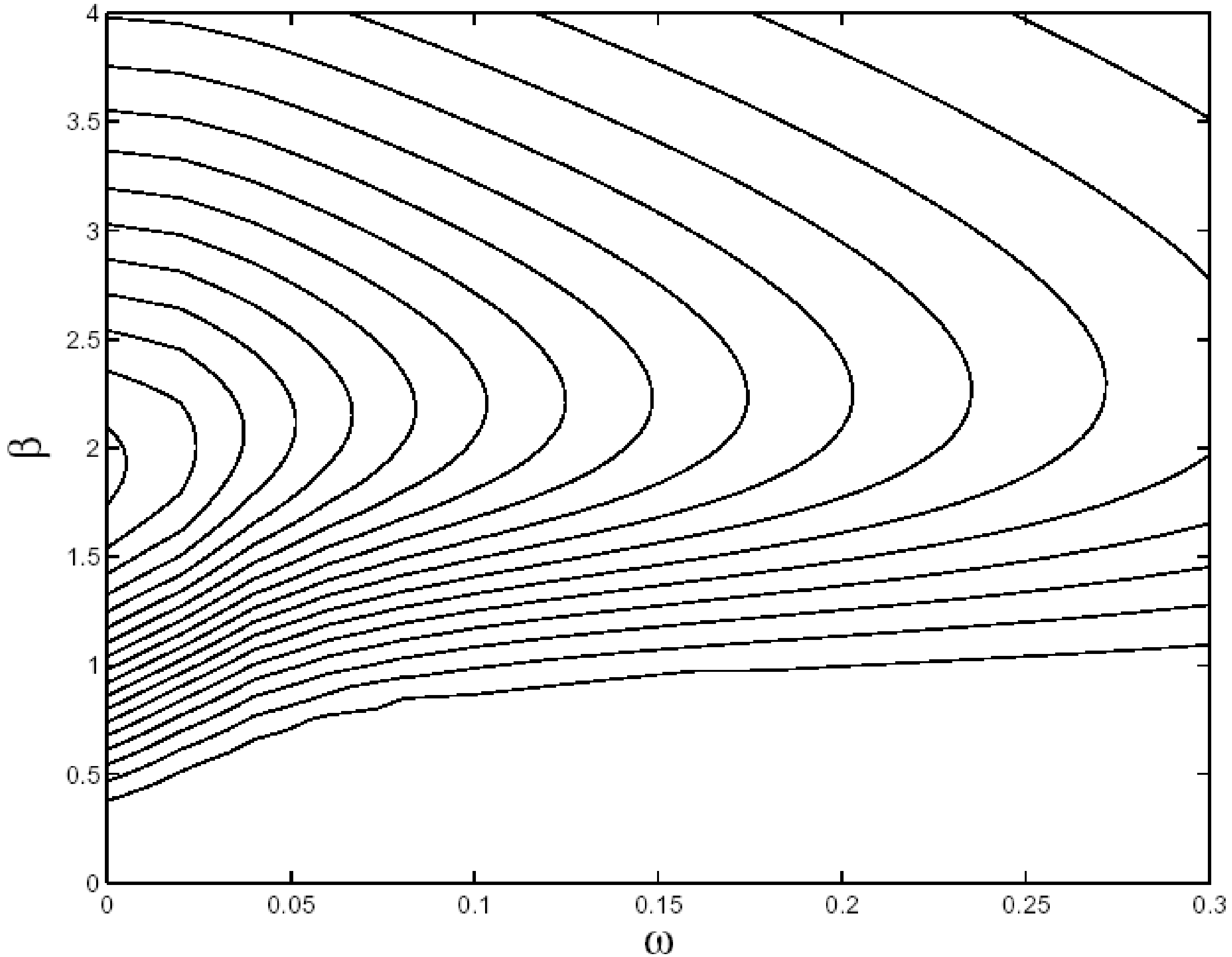
$$A_{nm}(x)\; \kappa_m = \frac{\lambda}{Re^2} B_{nm}\; \kappa_m$$

Optimal growth, $Re = 1000$, $\omega = 0$, $\beta = 1.91$ (Poiseuille flow)

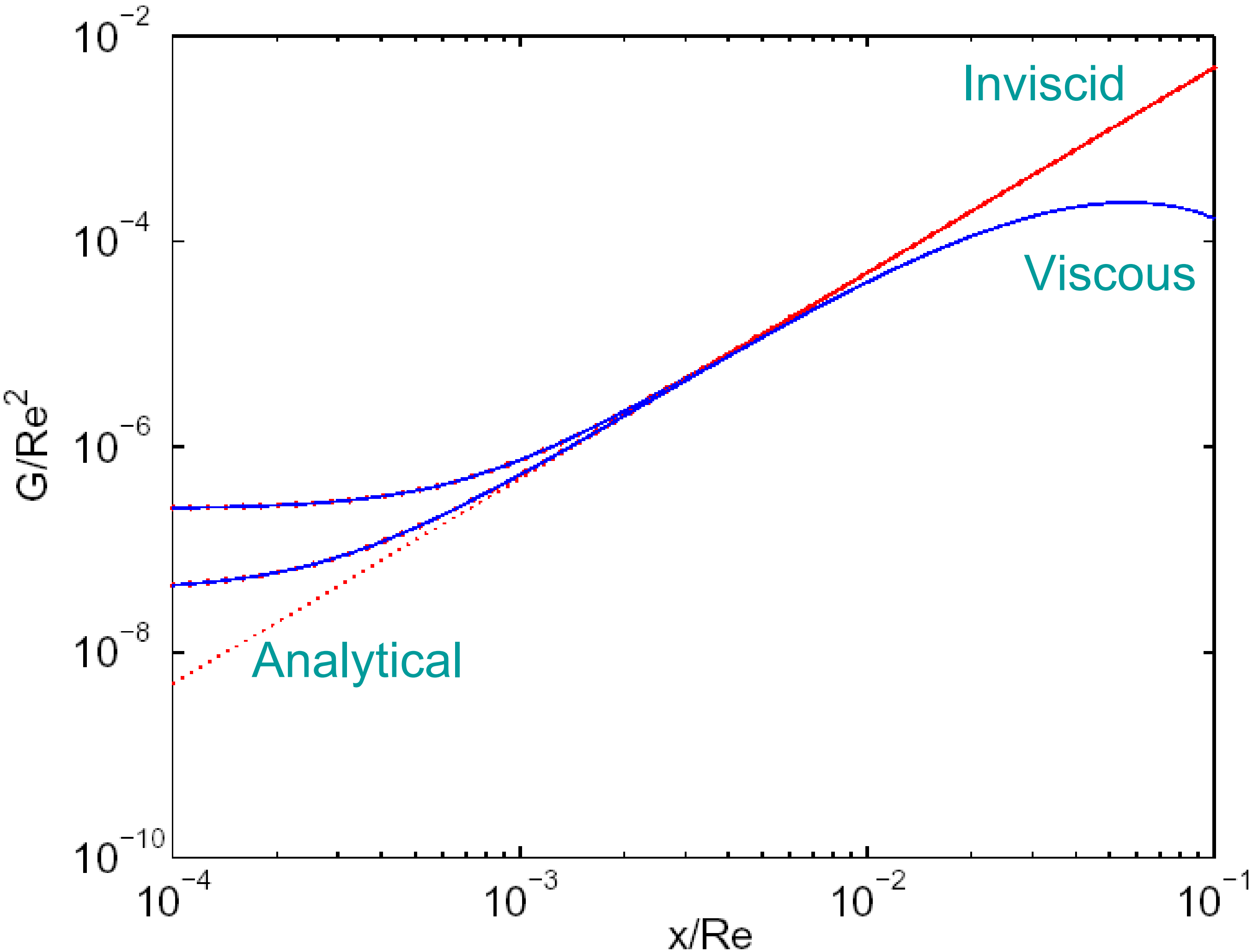
$Re = 500$, $\omega = 0$, $\beta = 1.58$ (Couette flow)



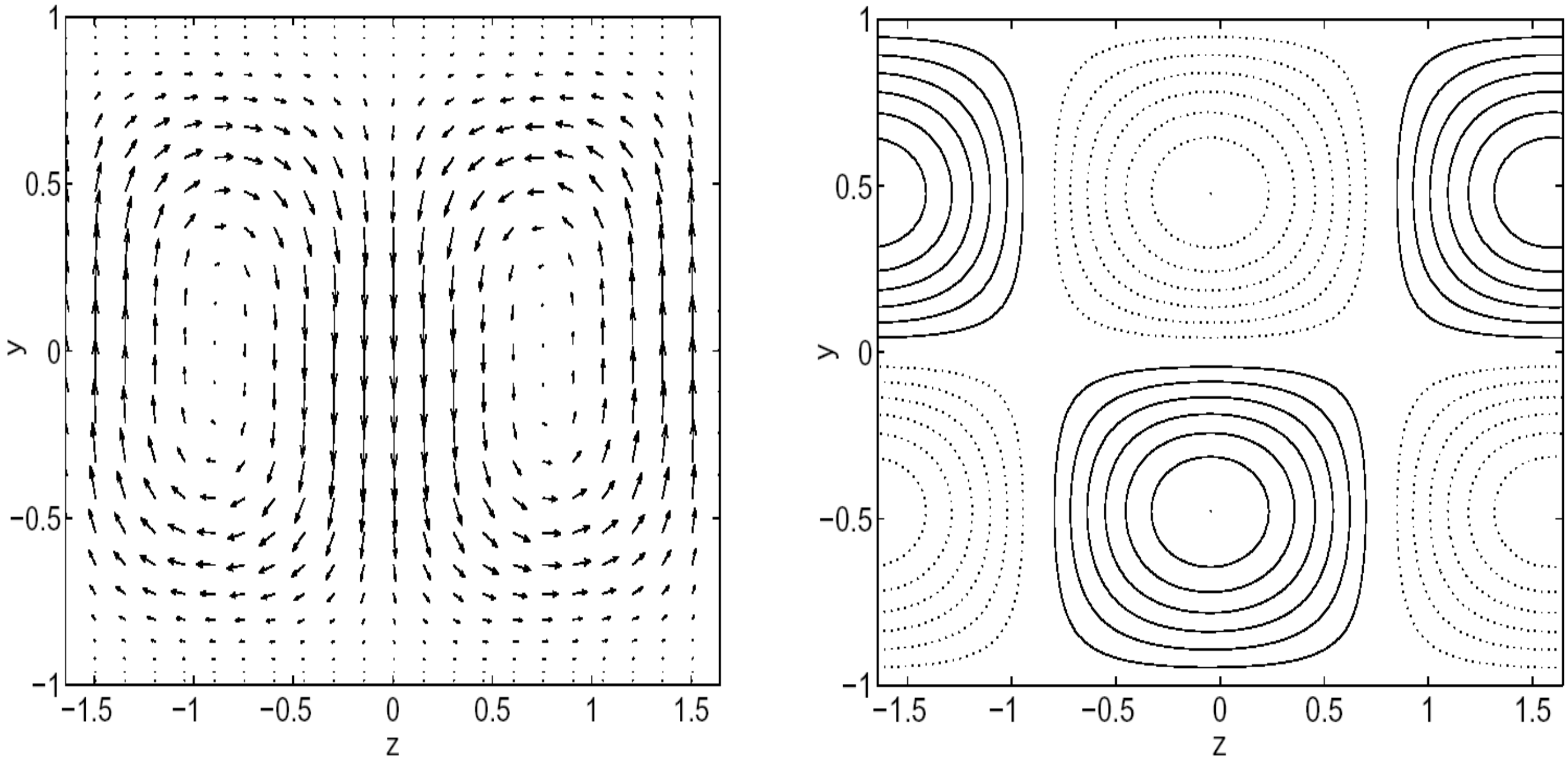
Iso-G, Poiseuille flow, $Re = 2000$



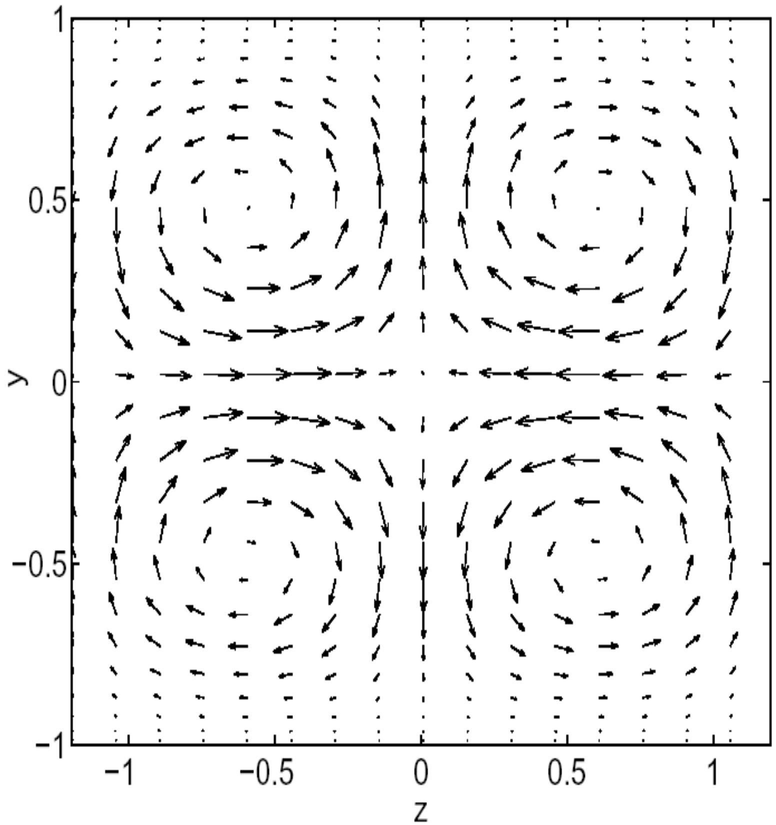
Optimal growth, Poiseuille flow, Re = 2000 & 5000, $\omega = 0$, $\beta = 1.91$

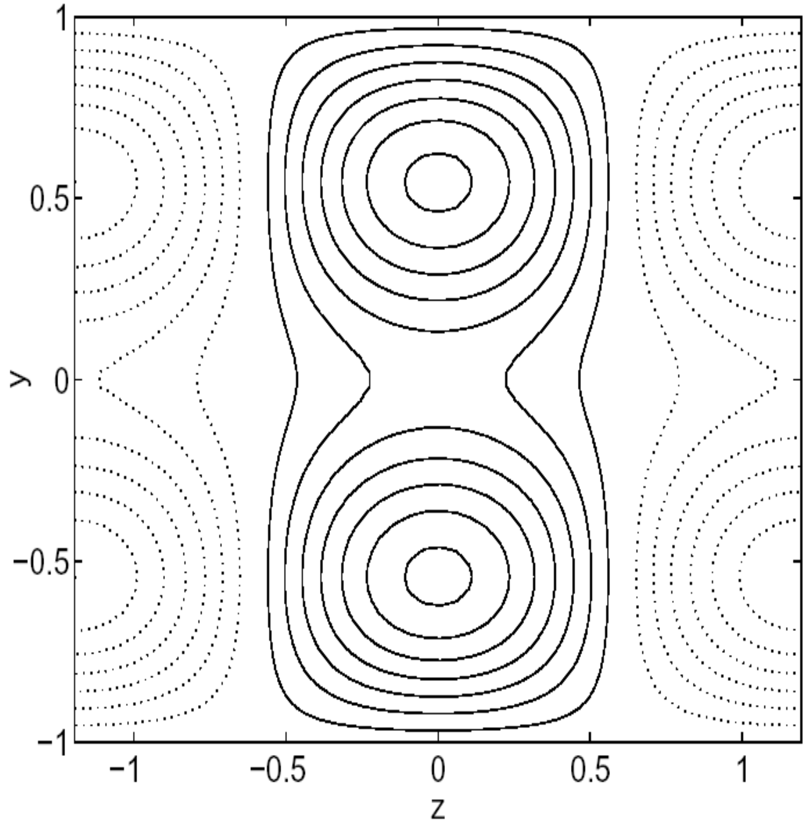


Optimal symmetric disturbance and optimal streak at x_{opt}, Poiseuille flow, $Re = 2000$, $\omega = 0$, $\beta = 1.91$

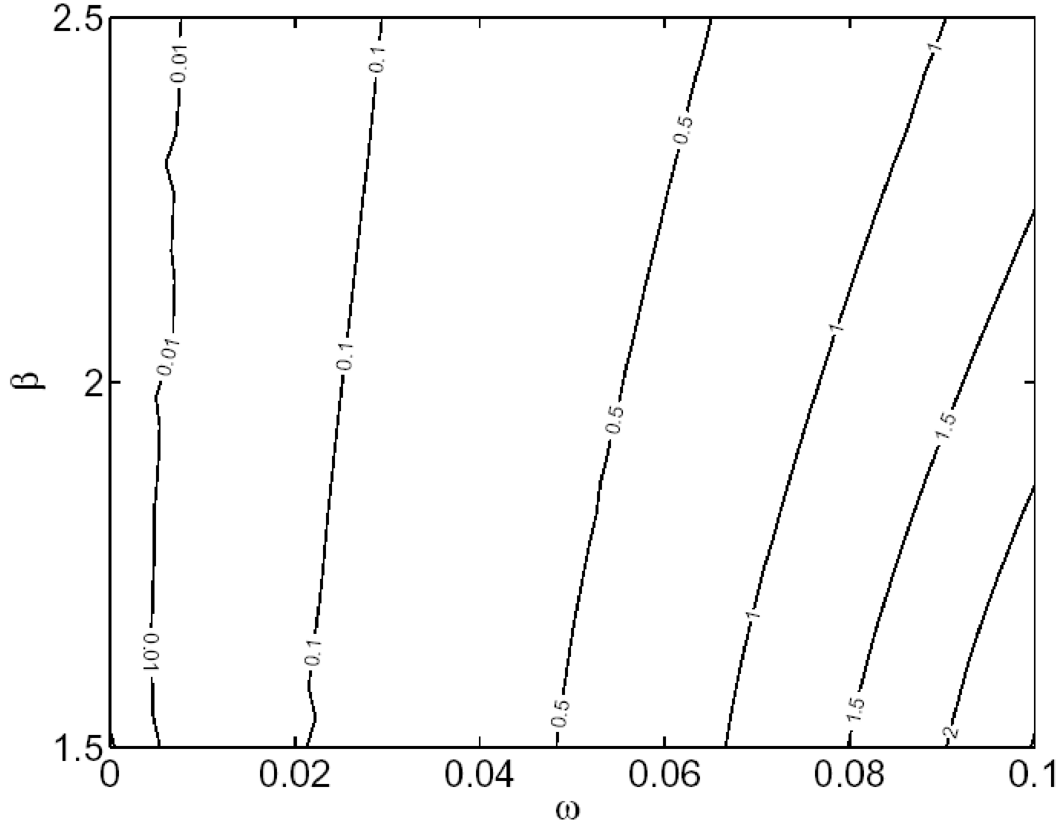


Optimal antisymmetric disturbance and optimal streak at x_{opt}, Poiseuille flow, $\mathrm{Re} = 2000$





The accuracy of the parabolic system degrades with ω



Percentage error, $\frac{G_{parab} - G_{ellipt}}{G_{ellipt}}$, Poiseuille flow, $Re = 2000$

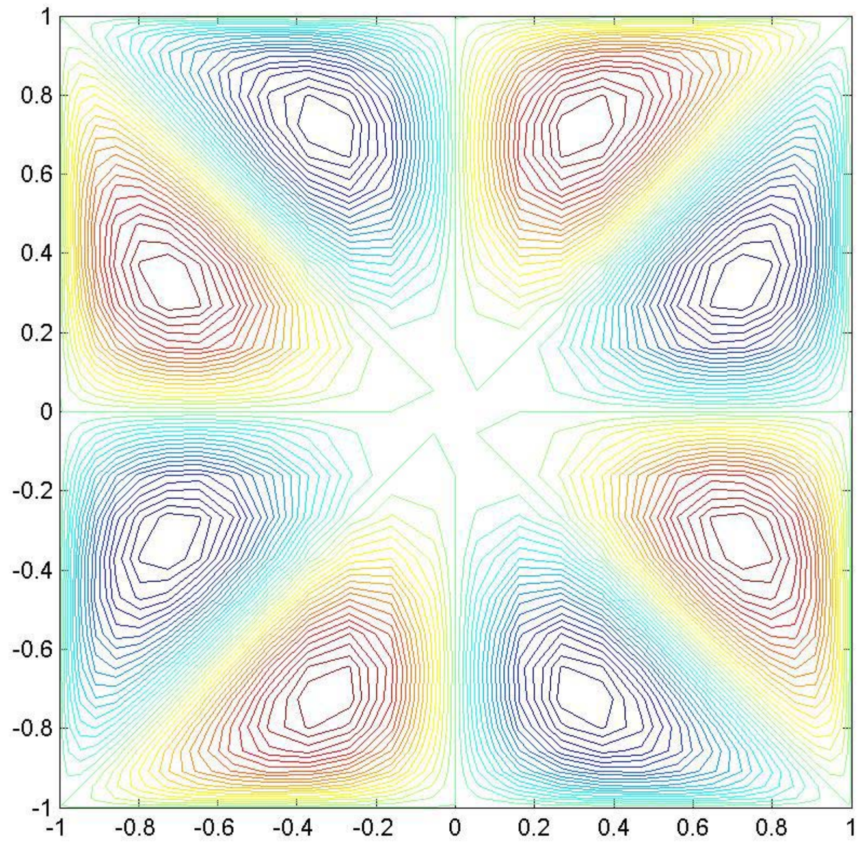
Optimal disturbances, spatial results ...

Flow	G_{max}/Re^2	x_{opt}/Re	β_{opt}	ω_{opt}
Poiseuille S	2.41×10^{-4}	0.057	1.91	0
Poiseuille AS	1.20×10^{-4}	0.036	2.63	0
Couette S	3.39×10^{-4}	0.0728	1.58	0
Couette AS	2.67×10^{-5}	0.0236	2.65	0

… *vs* corresponding temporal results

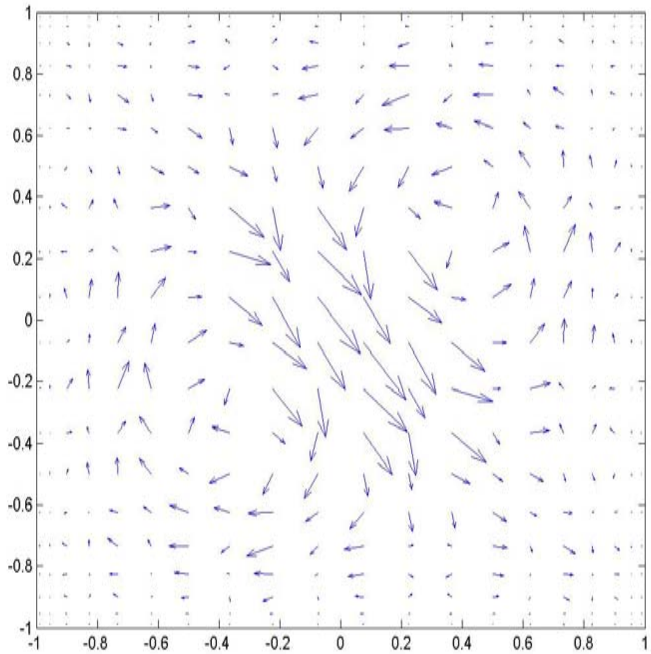
Flow	G_{max}/Re^2	t_{opt}/Re	β_{opt}	α_{opt}
Poiseuille S	1.96×10^{-4}	0.0759	2.04	0
Poiseuille AS	1.13×10^{-4}	0.0541	2.64	0
Couette S	2.96×10^{-4}	0.117	1.6	0.0139
Couette AS	3.52×10^{-5}	0.0329	2.08	0.143

Square duct, exit result, no optimization

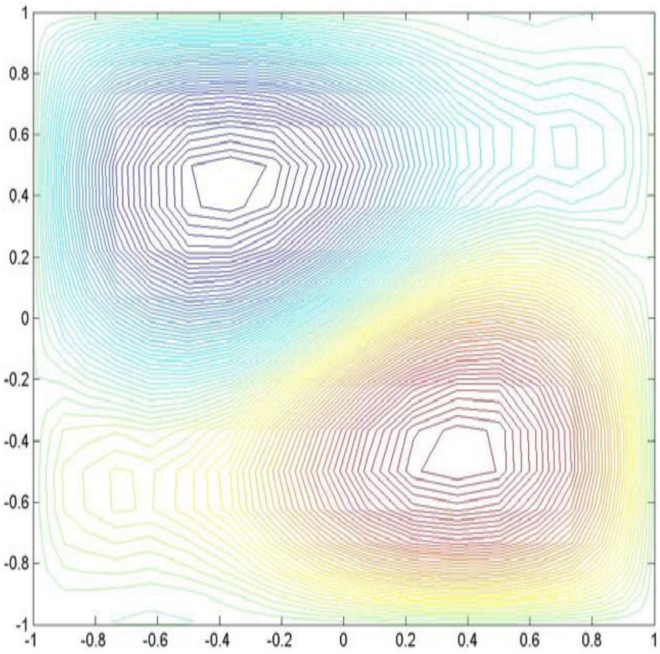


Square duct

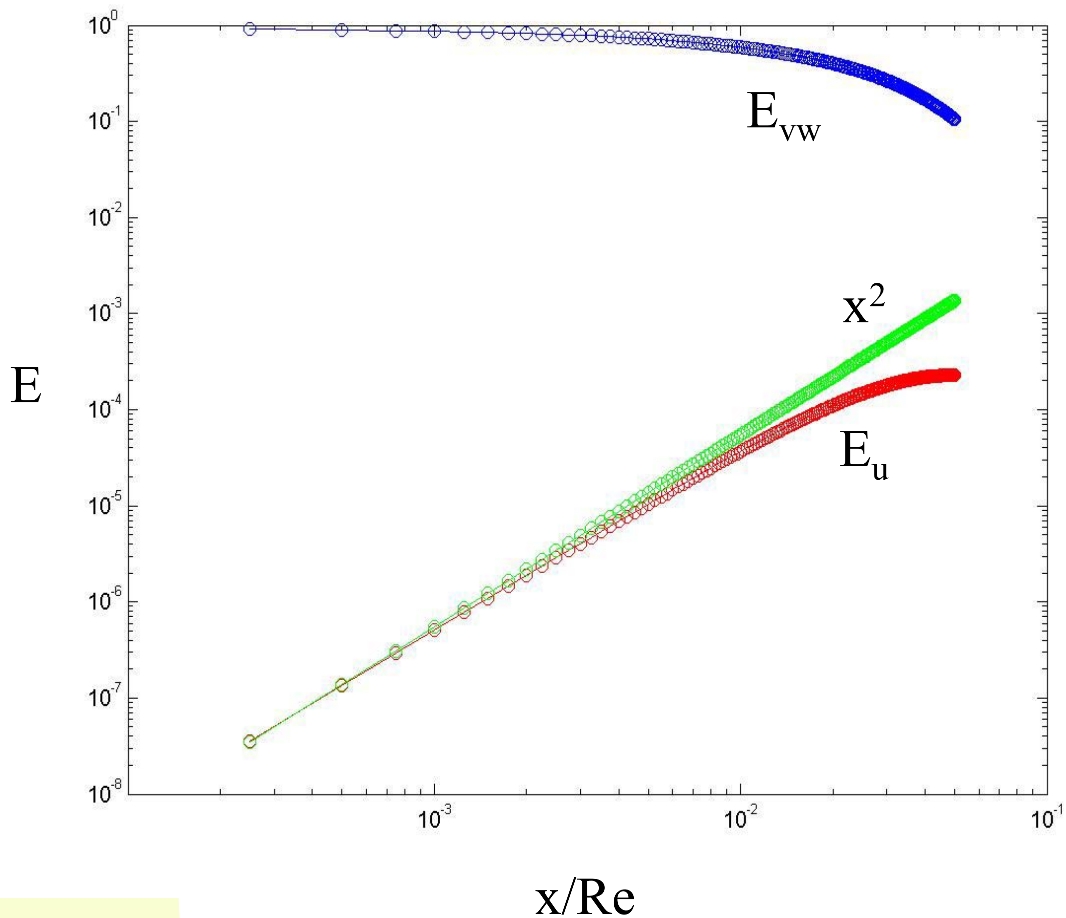
Optimal inlet flow



Outlet streaks



Square duct



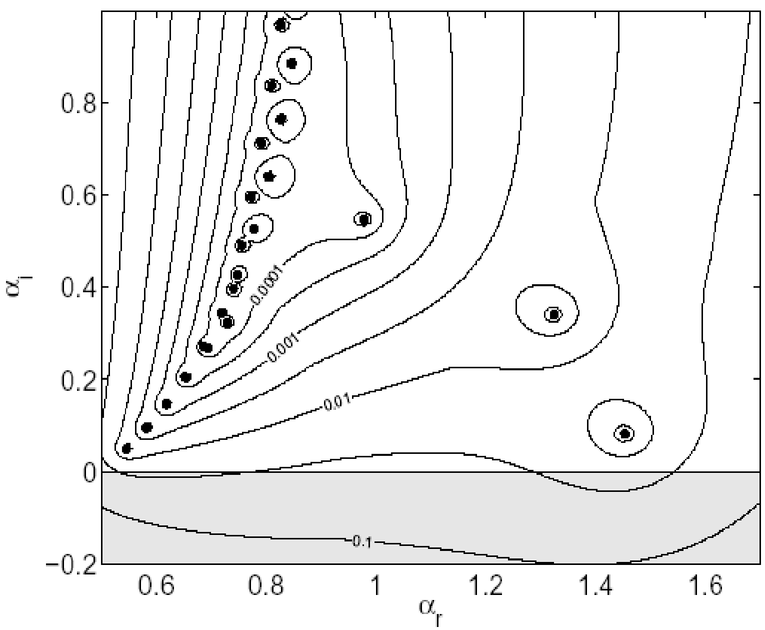
Optimal perturbations

	G_{max}/Re^2	x_{opt}/Re	
Poiseuille	$2.41*10^{-4}$	0.057	$\beta_{opt} = 1.91$
Couette	$3.39*10^{-4}$	0.073	$\beta_{opt} = 1.58$
Pipe	$1.03*10^{-4}$	0.033	$m_{opt} = 1$
Square duct	$1.12*10^{-4}$	0.039	

ROUTE 2: EXPONENTIAL GROWTH

Preliminary observation: eigenvalues of the OS/Squire system are very sensitive to operator perturbations E

$$\Lambda_\varepsilon(L) = \{\alpha \in C : \ \alpha \in \Lambda(L+E), \ \text{with } E \text{ such that } \|E\| \le \varepsilon\}$$



Consider a very particular operator perturbation, a distortion of the mean flow $U(y)$ (induced by whatever environmental forcing) $\rightarrow$

$$\Lambda_{\delta U}(L) = \left\{ \alpha \in C : \ \ \alpha \in \Lambda[L(U_{ref} + \delta U)], \ \text{with} \ \left\| \delta U \right\| \leq \varepsilon \right\}$$

OS equation: $L\,(U, \alpha; \omega, \beta, Re)\, v = 0$

With a base flow variation $\delta U(y)$:

$$\delta L\, v + L\, \delta v = 0$$

$$\delta U \frac{\partial L}{\partial U} v + \delta\alpha \frac{\partial L}{\partial \alpha} v + L\, \delta v = 0$$

Projecting on a, eigenfunction of the adjoint system ($L^* a = 0$) we find

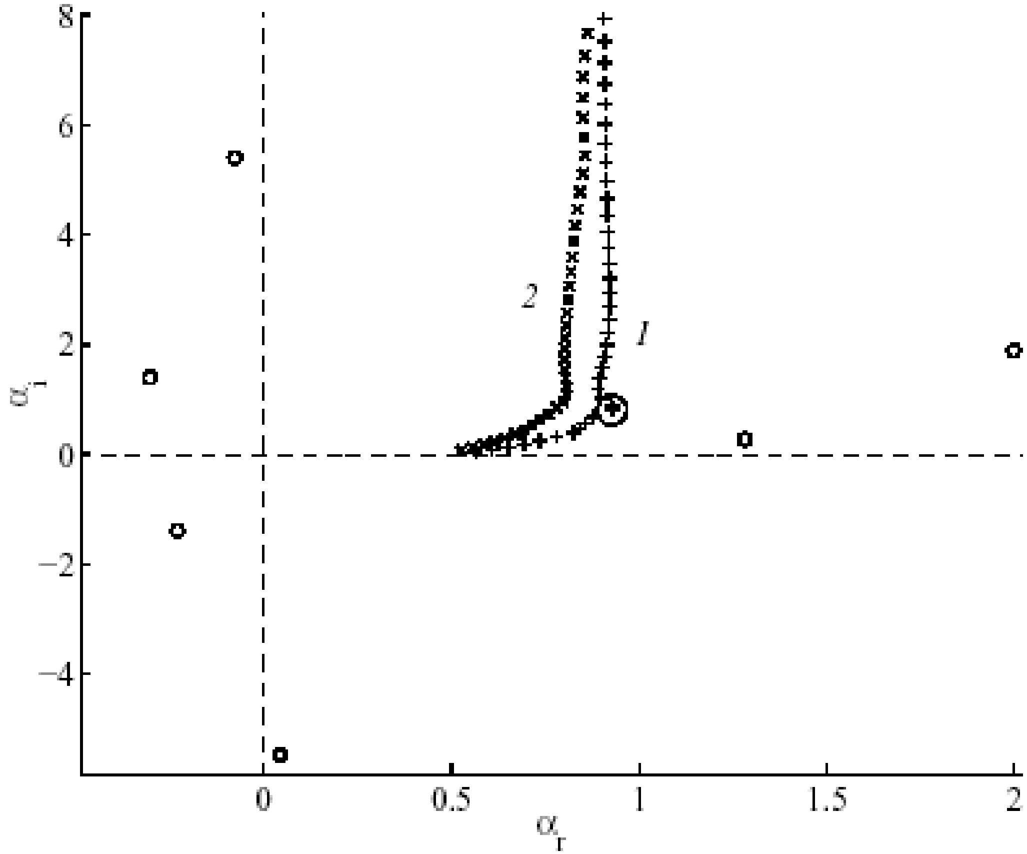
$$a \cdot \delta U \frac{\partial L}{\partial U} v + \delta\alpha \, a \cdot \frac{\partial L}{\partial \alpha} v + a \cdot L \, \delta v = 0$$

and hence,

$$\delta\alpha = - \frac{a \cdot \delta U \dfrac{\partial L}{\partial U} v}{a \cdot \dfrac{\partial L}{\partial \alpha} v} = \ldots = \int_{-1}^{1} G_U \, \delta U \, dy$$

In practice, for each eigenvalue α_n we can tie the base flow variation δU to the ensuing variation $\delta\alpha$ via a sensitivity function G_U

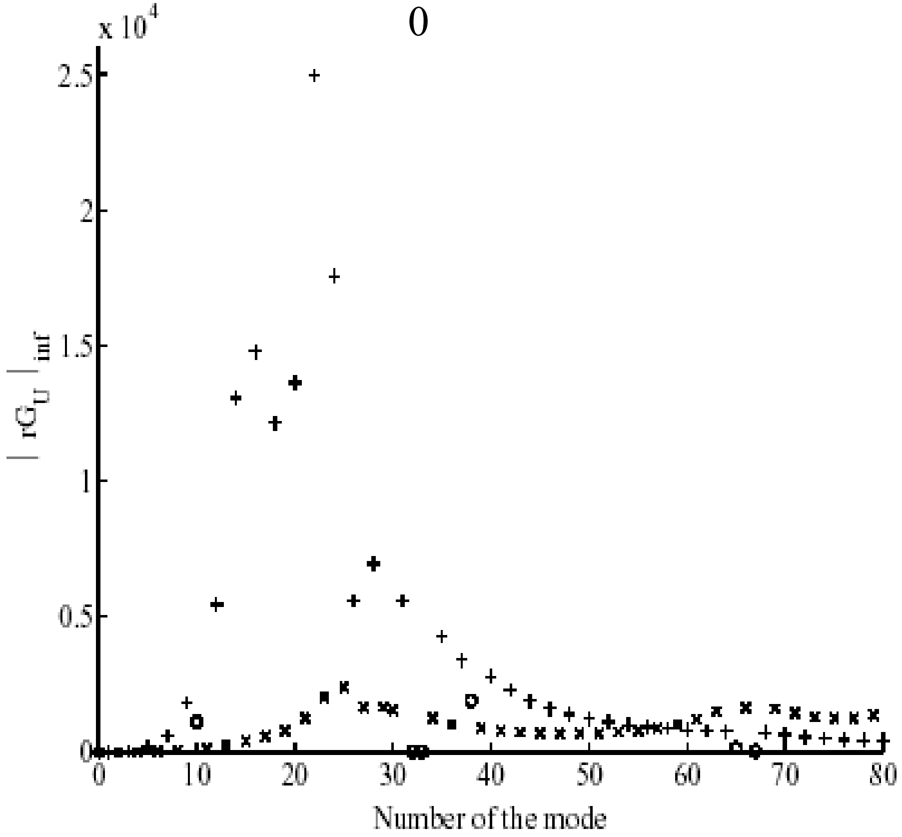
HAGEN-POISEUILLE FLOW



Spectrum of eigenvalues at $\mathrm{Re} = 3000$, $m = 1$, $\omega = 0.5$.
The circle includes the two most receptive eigenvalues

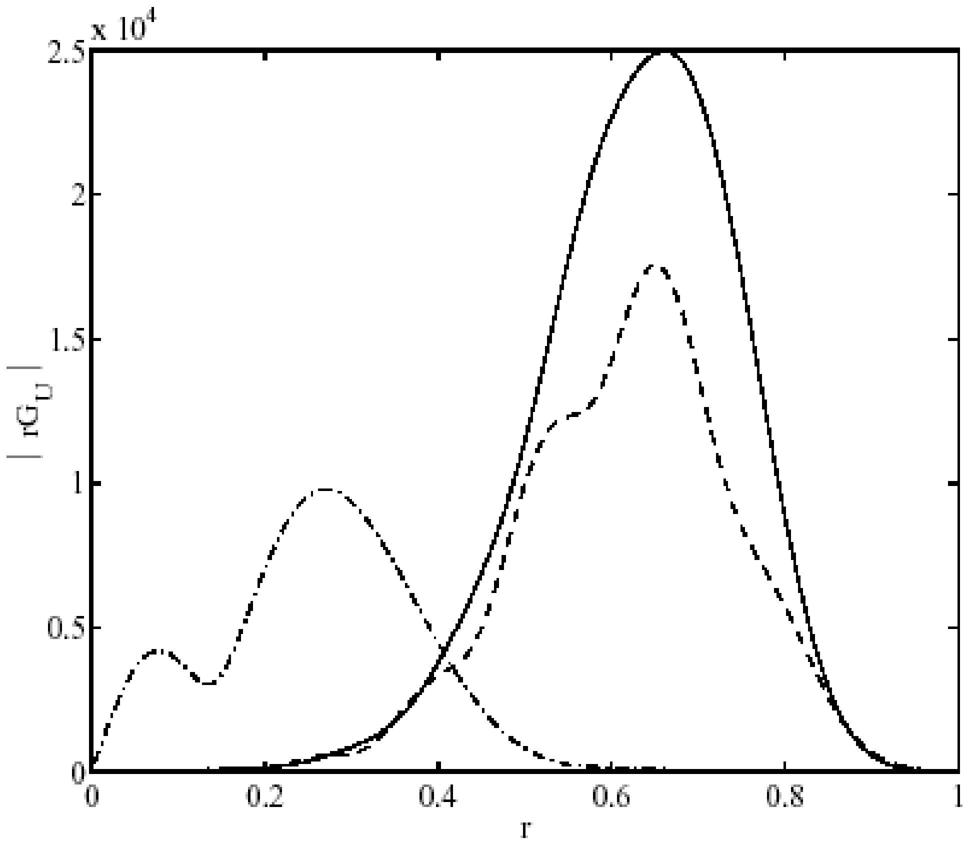
ROUTE 2: EXPONENTIAL GROWTH

$$\delta\alpha = \int_0^1 rG_U \delta U \, dr$$



Corresponding ∞ norm of rG_u. Modes arranged in order of increasing $|\alpha_i|$

SENSITIVITY FUNCTIONS



Mode 22 (solid); mode 24 (dashed); 10^3*mode 1 (dash-dotted)

“Optimization”

Look for *optimal* base flow distortion of given norm ε, so that the growth rate of the instability ($-\alpha_i$) is maximized:

$$\mathrm{Min}(\alpha_i) = \mathrm{Min}\left\{ \alpha_i + \gamma\left[\int_{-1}^{1} (U - U_{ref})^2 \, dy - \varepsilon \right] \right\}$$

Necessary condition is that:

$$\delta\alpha_i + \gamma\left[\int_{-1}^{1} 2(U - U_{ref}) \; \delta U \; dy \right] = 0$$

Employing the previous result:

$$\int_{-1}^{1}\left[\mathrm{Im}(G_U)+2\gamma(U-U_{ref})\right]\,\delta U\;dy=0$$

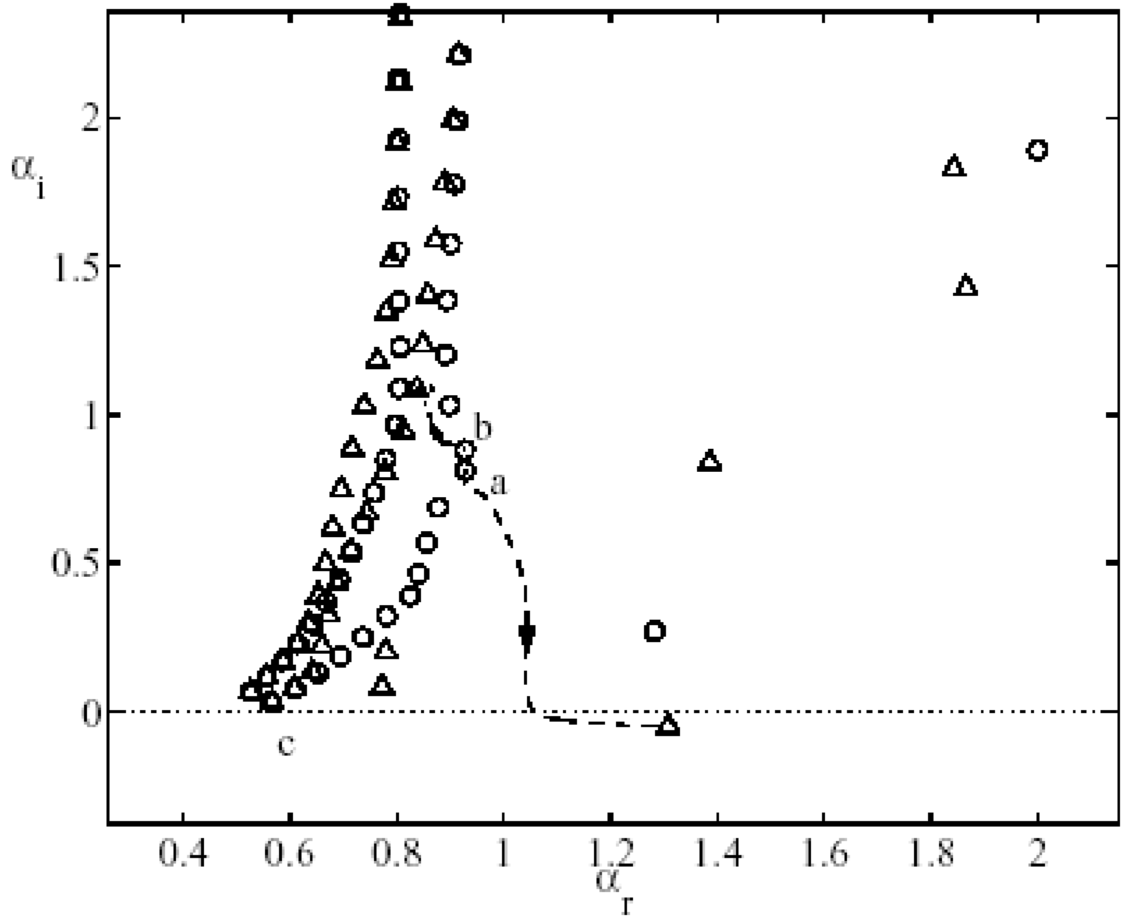
A simple gradient algorithm can be used to find the new base flow that maximizes the growth rate, for any α_n and for any given base flow distortion norm ε:

$$U^{(i+1)}=U^{(i)}+\vartheta\,\left[\mathrm{Im}(G_U{}^{(i)})+2\gamma^{(i)}(U^{(i)}-U_{ref})\right]$$

with

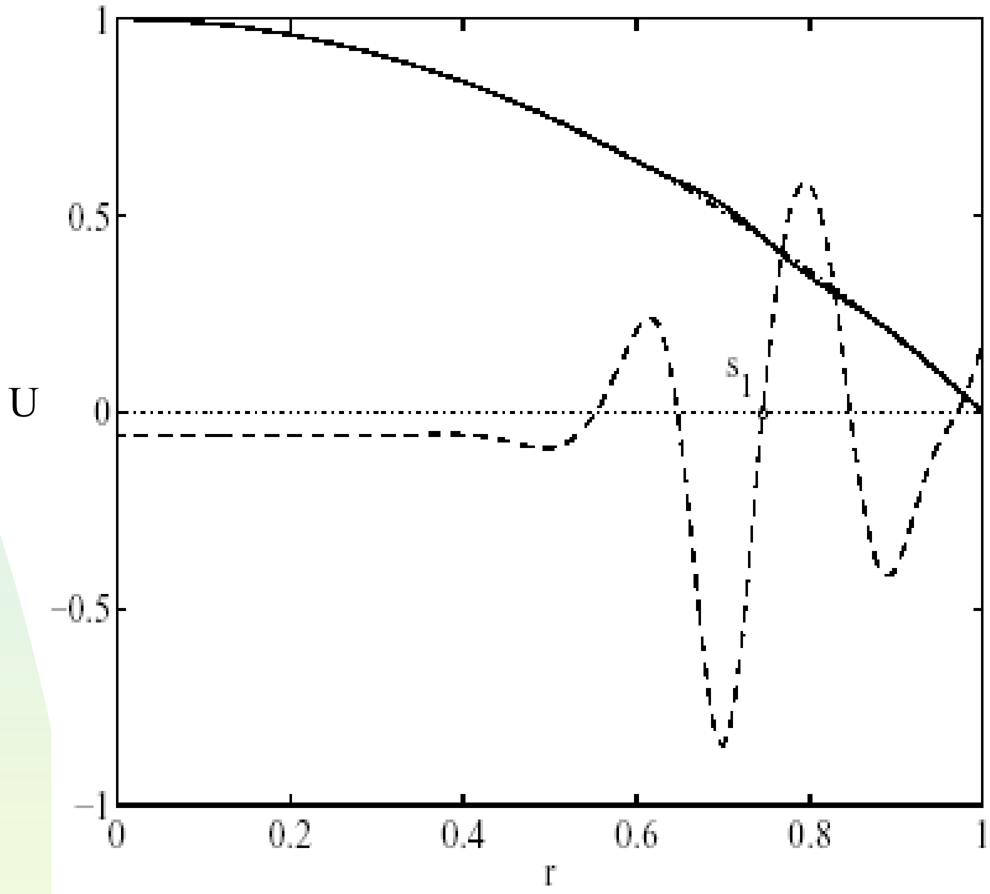
$$\gamma^{(i)}=\mu\left\{\frac{\int_{-1}^{1}\left[\mathrm{Im}(G_U{}^{(i)})^2\right]}{4\varepsilon}\right\}^{\frac{1}{2}}$$

ROUTE 2: EXPONENTIAL GROWTH



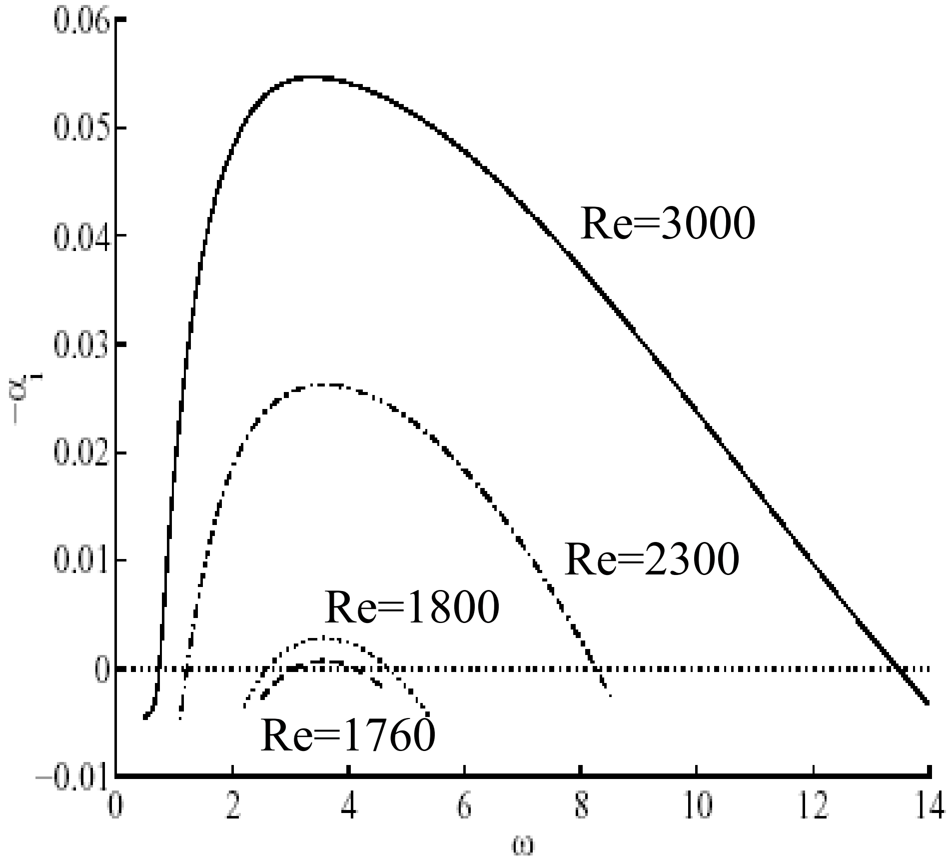
Re = 3000, m = 1, ω = 0.5. HP flow (circles), OD flow (triangles) with $\varepsilon = 2.5 * 10^{-5}$ which minimizes α_i of mode 22.

ROUTE 2: EXPONENTIAL GROWTH



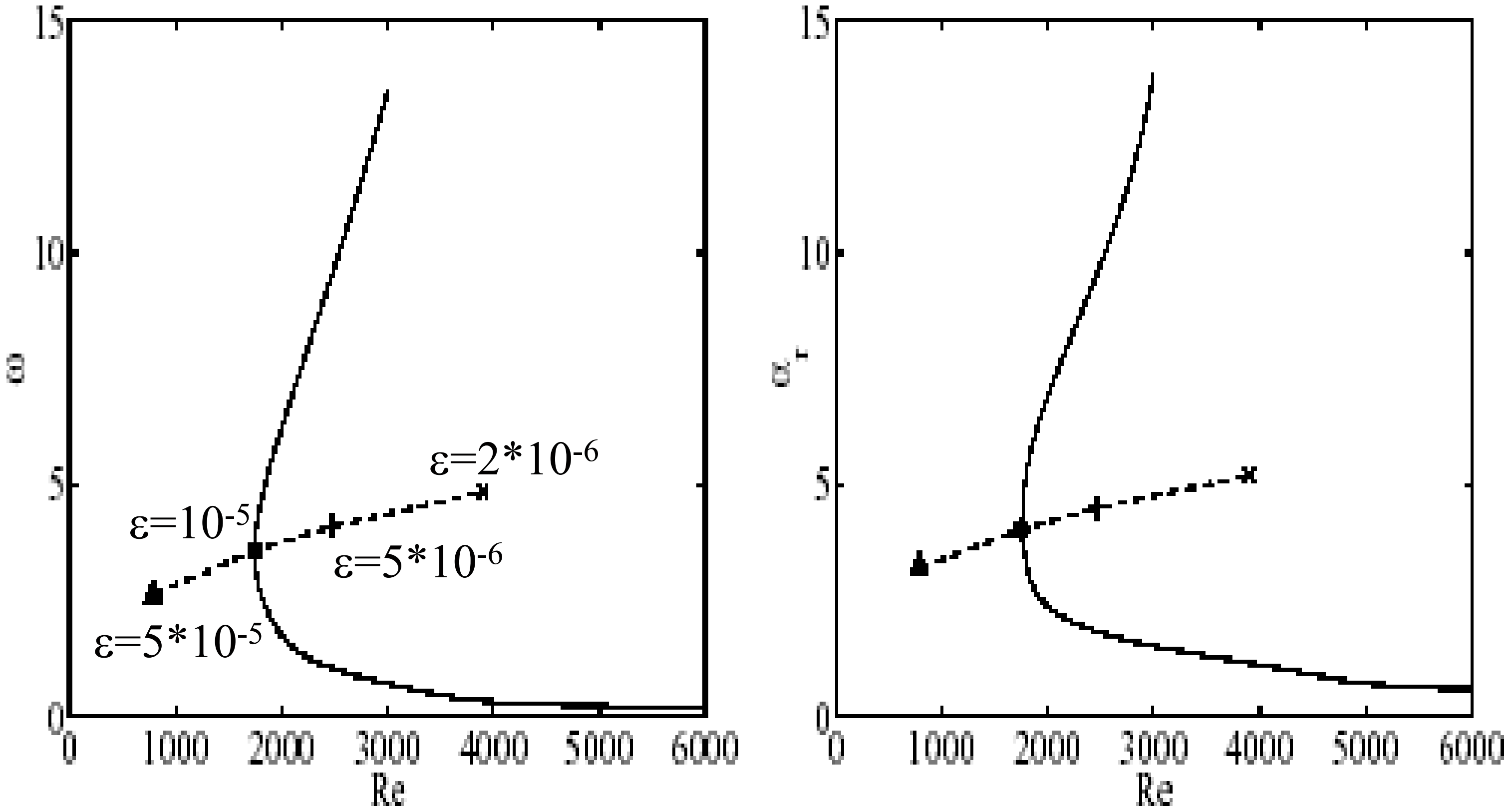
Optimally distorted base flow *vs* Hagen-Poiseuille flow.
The curve of $(U'/r)'/20$ indicates an inflectional instability

ROUTE 2: EXPONENTIAL GROWTH



Growth rate as function of ω for $m = 1$ and $\varepsilon = 10^{-5}$

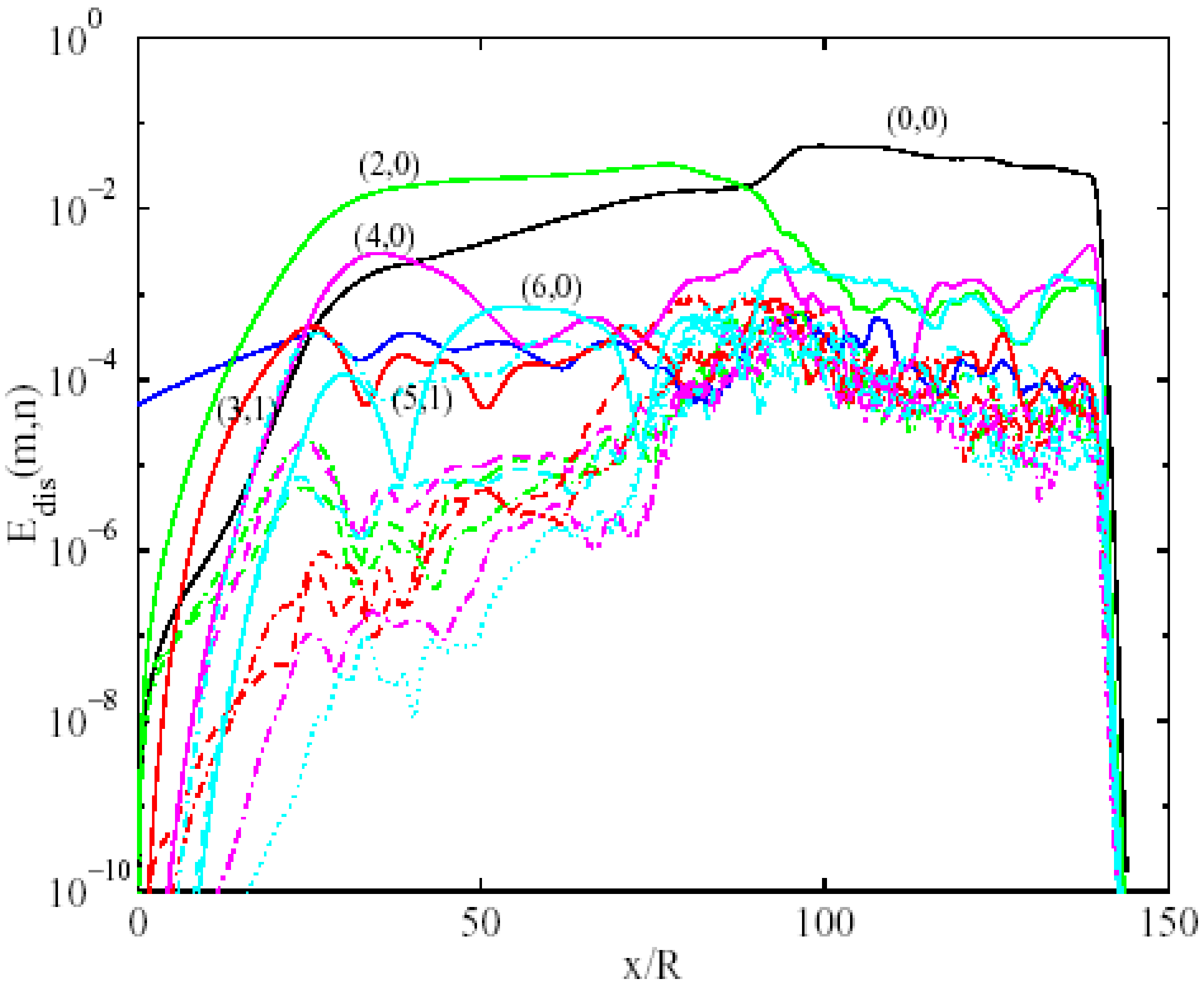
ROUTE 2: EXPONENTIAL GROWTH



Neutral curves for $m = 1$ and $\varepsilon = 10^{-5}$. Symbols give Re_{crit}

FULL NONLINEAR SIMULATIONS

$$E_{dis}^{(m,n)}(x) = \sum_{j=\pm m} \sum_{k=\pm n} \frac{1}{2T} \int_{\tau}^{\tau+T} \int_{0}^{2\pi} \int_{0}^{1} \left| \hat{\mathbf{u}}\ e^{i(j\vartheta - k\omega t)} \right|^2 r dr d\vartheta dt$$



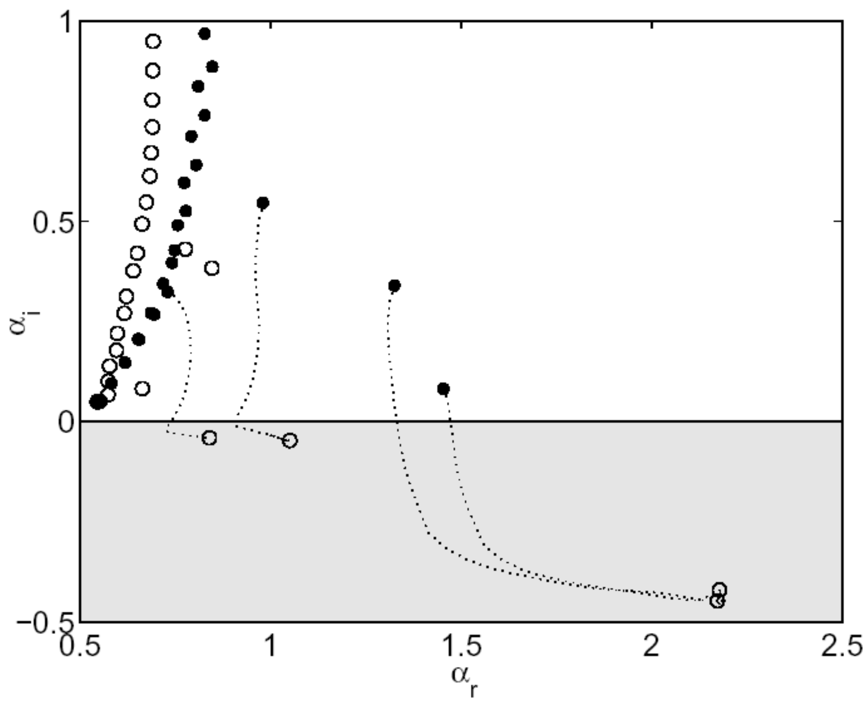
Spatial evolution of the disturbance energy for the Fourier modes (m, n), with ω = 0.5. Initial amplitude of the (1,1) eigenmode (shown with thick blue line) is 0.002. Re = 3000, m = 1, n = 1, ε = $2.5*10^{-5}$.

CONCLUSIONS

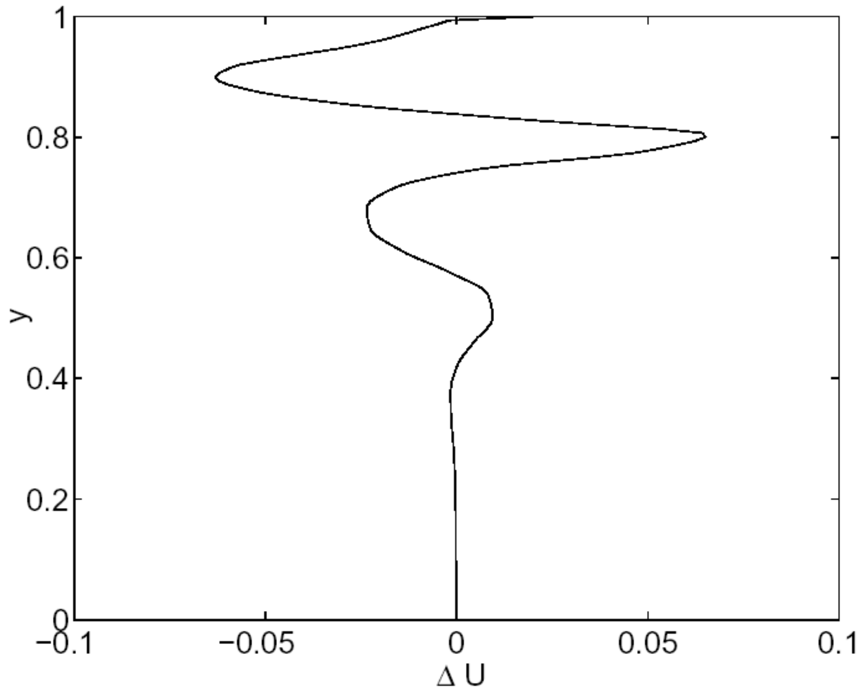
- Transient growth in space can be described by a parabolic system
- Transient growth is related to the non-normality of OS-Squire eigenmodes (L. N. Trefethen *et al.*, 1993)
- OS eigenmodes are very sensitive to base flow variations (δU-pseudospectrum, the growth is less than for the ε-pseudospectrum since two-way (possibly *unphysical?)* coupling between OS and Squire equations is not allowed)

CONCLUSIONS

- Exponential growth can take place even in nominally subcritical conditions for mild distortions of the base flow



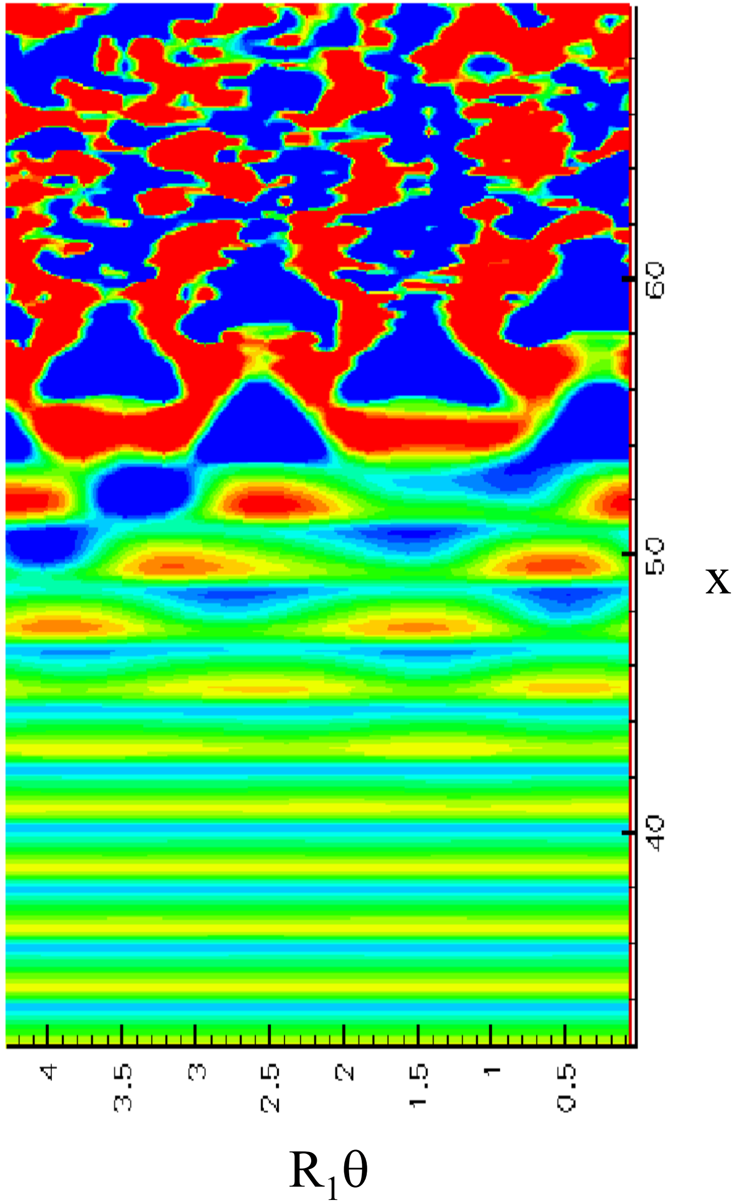
Poiseuille flow, Re = 3000, ω = 0.5



Optimal distortion

CONCLUSIONS

- Transition is likely to be provoked by the combined effect of algebraic and exponentially growing disturbances, the ultimate fate of the flow being decided by the prevailing receptivity conditions of the flow


60
50
40
x
4
3.5
3
2.5
2
1.5
1
0.5
R1θ