Optimal and robust control of streaks in wall-bounded shear flows

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with contributions from:

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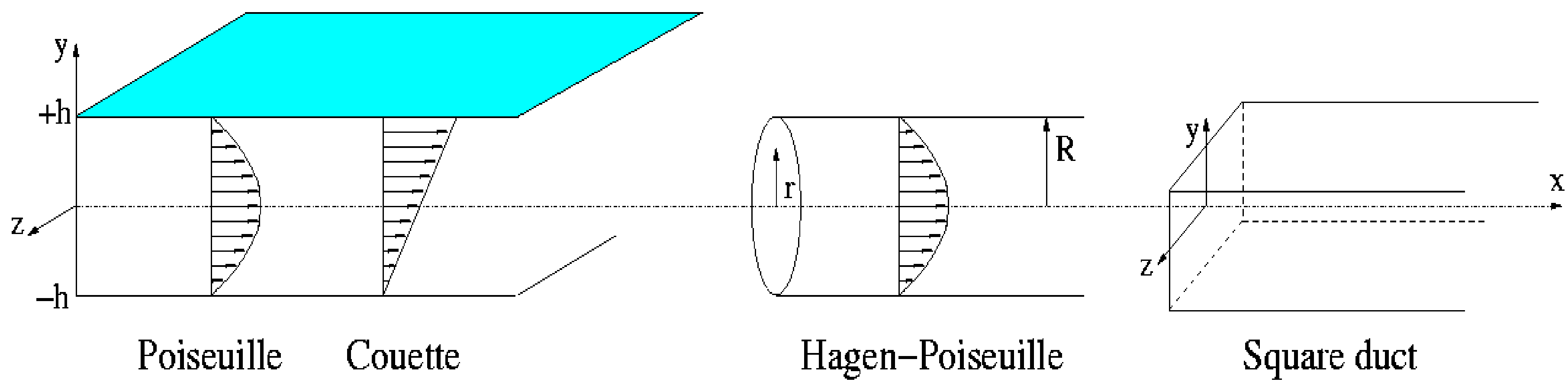
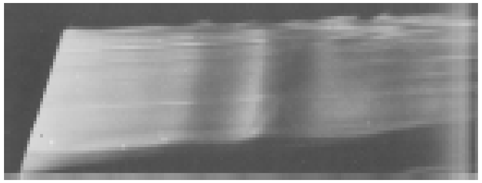


1. ***TRANSITION IN SHEAR FLOWS IS A PHENOMENON STILL NOT FULLY UNDERSTOOD***. For the simplest parallel or quasi-parallel flows there is poor agreement between predictions from the classical linear stability theory (Re_{crit}) and experimentals results (Re_{trans})

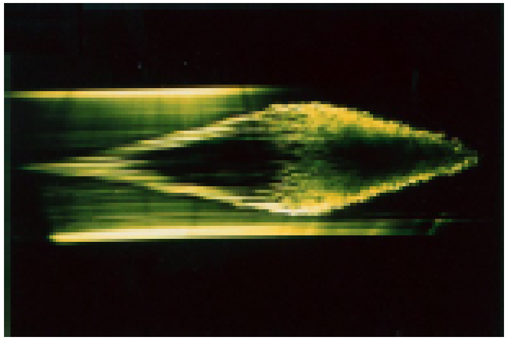
	Poiseuille	Couette	Hagen-Poiseuille	Blasius
Re_{crit}	5772	∞	∞	~ 500
Re_{trans}	~ 1000	~ 400	~ 2000	~ 400

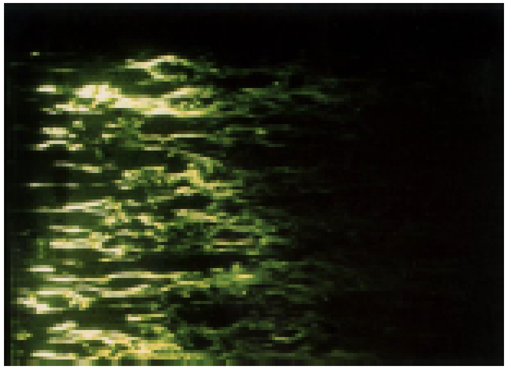
2. TRANSITION IN SHEAR FLOWS IS A PHENOMENON STILL NOT FULLY UNDERSTOOD.

- Classical theory predicts *Tollmien-Schlichting waves* in Poiseuille and boundary layer flows:



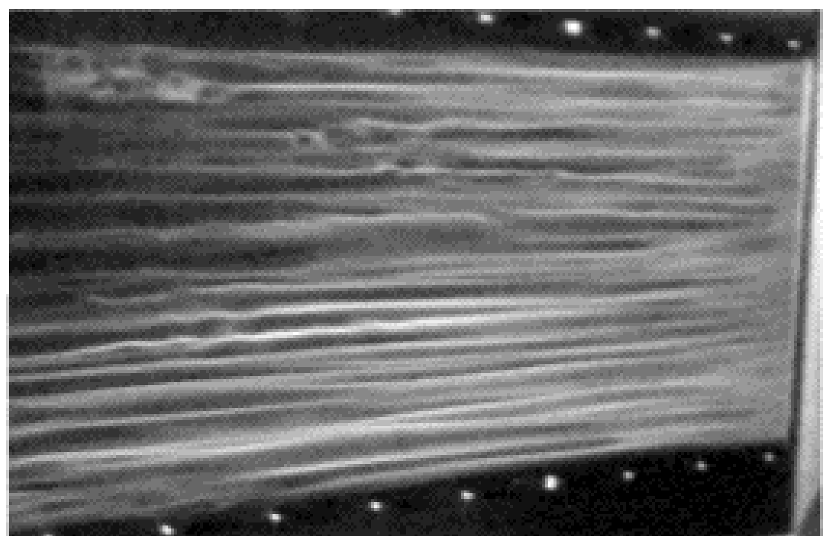
- Except in very noise-free and controlled experiments, flow structures in transition are more like turbulent spots and streaky boundary layers:





THE TRANSIENT GROWTH

- **<u>THE MECHANISM</u>**: a stationary algebraic instability exists in the inviscid system ("lift-up" effect). In the viscous case the growth of the streaks is hampered by diffusion $\Rightarrow$ <u>transient growth</u>



P.H. Alfredsson and M. Matsubara (1996): streaky structures in a boundary layer. Free-stream speed: 2 [m/s], free-stream turbulence level: 6%

- **Proposition:**
  Optimal and robust control of streaks during their initial development phase, in pipes and boundary layers by
  - acting at the level of the disturbances ("cancellation control")
  - acting at the level of the mean flow ("laminar flow control")

Cancellation Control

Known base flow U(r); disturbance O(ε) ⟹ control O(ε)

Control O(ε) ⟹ Disturbance field (system's state)

Laminar Flow Control

Compute base flow O(1) together with the disturbance field O(ε) $\Longrightarrow$ control O(1)

Control O(1) $\Longrightarrow$ Whole flow field (system's state)

OPTIMAL CONTROL: A PARABOLIC MODEL PROBLEM

$u_t + u u_x = u_{xx} + S$	$u(t,x)$:	state of the system
$u(0,x)=u_0(x)$	$u_0(x)$:	initial condition
$u(t,0)=u_w(t)$	$u_w(t)$:	boundary control
$u(t,\infty)=0$	$S(t,x)$:	volume control

Suppose $u_0(x)$ is known; we wish to find the controls, u_w and S, that minimize the functional:

$$I(u, u_w, S) = \underbrace{\int_0\int_0 u^2\, dt\, dx}_{\text{disturbance norm}} + \underbrace{\alpha^2\int_0 u_w^{\,2}\, dt + \beta^2\int_0\int_0 S^2\, dt\, dx}_{\text{energy needed to control}}$$

$$I(u,u_w,S) = \int_0 \int_0 u^2 \, dt \, dx + \alpha^2 \int_0 u_w^{\,2} \, dt + \beta^2 \int_0 \int_0 S^2 \, dt \, dx$$

$\alpha=\beta=0$	no limitation on the cost of the control
α and/or β small	cost of employing u_w and/or S is not important
α and/or β large	cost of employing u_w and/or S is important

For the purpose of minimizing I, let us introduce an augmented functional $L = L(u,u_w,S,a,b)$, with $a(t,x)$ and $b(t)$ Lagrange multipliers, and let us minimize the new objective functional

$$L(u,\, u_w,\, S,\, a,\, b) = I + \int_0 \int_0 a\,(u_t + u u_x - u_{xx} - S)\, dt\, dx\, +$$

$$+ \int_0 b(t)\,[u(t,0) - u_w(t)]\, dt$$

Constrained minimization of I

$\Downarrow$

Unconstrained minimization of L

Each directional derivative must independently vanish for a relative minimum of L to exist.

For example it must be:

$$\frac{dL}{du}\delta u = \frac{dI}{du}\delta u + \iint a\ \delta u_t + a\ u\ \delta u_x + a\ \delta u\ u_x - a\ \delta u_{xx}\ dtdx +$$

$$+\int b\ \delta u(t,0)\ dt = \iint 2u\ \delta u - a_t\ \delta u - a_x\ u\ \delta u - a_{xx}\ \delta u\ dtdx +$$

$$+\int b\ \delta u(t,0)\ dt\ +\ \text{boundary and initial terms}$$

Since δu is an arbitrary variation, the double integral vanish if and only if the linear adjoint equation

$$a_t + u a_x = - a_{xx} + 2u$$

is satisfied, together with:

$a(T,x)=0$ terminal condition

$a(t,0)=a(t,\infty)=0$ boundary conditions

The vanishing of the other directional derivatives is accomplished by letting:

$$\frac{dL}{du_w}\delta u_w = 0 \qquad \text{iff} \qquad 2\alpha^2 u_w(t) = b(t) = a_x(t,0)$$

$$\frac{dL}{dS}\delta S = 0 \qquad \text{iff} \qquad 2\beta^2 S(t,x) = a(t,x)$$

Resolution algorithm:

direct equations for state u (using u_w and S)

convergence test

adjoint equations for dual state a

optimality conditions:

$$u_w(t)=a_x(t,0)/2\alpha^2$$
$$S(t,x)=a(t,x)/2\beta^2$$

Optimality conditions are enforced by employing a simple gradient or a conjugate gradient method

ROBUST CONTROL:
A PARABOLIC MODEL PROBLEM

$u_t + uu_x = u_{xx} + S$	$u(t,x)$:	state of the system
$u(0,x)=u_0(x)$	$u_0(x)$:	initial condition
$u(t,0)=u_w(t)$	$u_w(t)$:	boundary control
$u(t,\infty)=0$	$S(t,x)$:	volume control

Now $u_0(x)$ is not known; we wish to find the controls, u_w and S, and the initial condition $u_0(x)$ that minimize the functional:

$$I_1(u, u_0, u_w, S) = \int_0\int_0 u^2\,dt\,dx + \alpha^2\int_0 u_w^{\,2}\,dt + \beta^2\int_0\int_0 S^2\,dt\,dx - \\ - \gamma^2\int_0 u_0^{\,2}\,dx$$

i.e. we want to maximize over all u_0.

This non-cooperative strategy consists in finding the worst initial condition in the presence of the best possible control.

Procedure: like previously we introduce a Lagrangian functional, including a statement on u_0

$$L_1(u, u_0, u_w, S, a, b, c) = I_1 + \int_0 \int_0 a\,(u_t + u u_x - u_{xx} - S)\, dt\, dx + \int_0 b(t)\,[u(t,0)-u_w(t)]\, dt + \int_0 c(x)\,[u(0,x)-u_0(x)]\, dx$$

Vanishing of $\frac{dL}{du_0}\delta u_0$ yields: $\boxed{2\gamma^2 u_0(x)=a(0,x)}$

⇒ Robust control algorithm requires alternating ascent iterations to find u_0 and descent iterations to find u_w and/or S.

Convergence to a saddle point in the space of variables.

Optimal control of streaks in pipe flow

Spatially parabolic model for the streaks: structures elongated in the streamwise direction x

Long scale for x:	$R\,Re$
Short scale for r:	R
Fast velocity scale for u:	U_{max}
Slow velocity scale for v, w:	U_{max}/Re
Long time:	$R\,Re/U_{max}$

Optimal control of streaks in pipe flow

$$\left\{\begin{aligned}
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial r} + \left(\nabla^2 v - \frac{v}{r^2} - \frac{2}{r^2}\frac{\partial w}{\partial \theta}\right), \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{dU}{dr}v &= \nabla^2 u, \\
\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \theta} &= 0, \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} &= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left(\nabla^2 w - \frac{w}{r^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right),
\end{aligned}\right.$$

with $$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Optimal cancellation control of streaks

$$\mathbf{v} = (u, v, w) \qquad \mathbf{v}(r, \theta, x, t) = \widehat{\mathbf{v}}(r, x; m, \omega)\, e^{i(m\theta - \omega t)}$$

$$p(r, \theta, x, t) = \widehat{p}(r, x; m, \omega)\, e^{i(m\theta - \omega t)}$$

$$\left\{ \begin{aligned} -i\omega v + U\frac{\partial v}{\partial x} &= -\frac{\partial p}{\partial r} + \left(\nabla^2 v - \frac{v}{r^2} - \frac{2im}{r^2} w\right), \\ -i\omega u + U\frac{\partial u}{\partial x} + \frac{dU}{dr} v &= \nabla^2 u, \\ \frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial (rv)}{\partial r} + \frac{im}{r} w &= 0, \\ -i\omega w + U\frac{\partial w}{\partial x} &= -\frac{im}{r} p + \left(\nabla^2 w - \frac{w}{r^2} + \frac{2im}{r^2} v\right), \end{aligned} \right\}$$

with $\nabla^2 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) - \dfrac{m^2}{r^2}.$

Optimal disturbance at $x=0$

Initial condition for the state: optimal perturbation, i.e. the disturbance which maximizes the gain G in the absence of control.

$$G(x_{fin}) = \frac{E_{x_{fin}}}{E_0} = \frac{\left\{ \frac{1}{2} \int_0^1 \left[u^* u + Re^{-2} \left(v^* v + w^* w \right) \right] \, r dr \right\}_{x = x_{fin}}}{\left\{ \frac{1}{2} \int_0^1 \left[u^* u + Re^{-2} \left(v^* v + w^* w \right) \right] \, r dr \right\}_{x = 0}}.$$

$$\Rightarrow \quad G(x_{fin}) = \frac{E_u(x_{fin})}{E_0} = Re^2 \frac{\left\{ \frac{1}{2} \int_0^1 u^* u \, r dr \right\}_{x = x_{fin}}}{\left\{ \frac{1}{2} \int_0^1 \left(v^* v + w^* w \right) \, r dr \right\}_{x = 0}},$$

Optimal disturbance at $x=0$

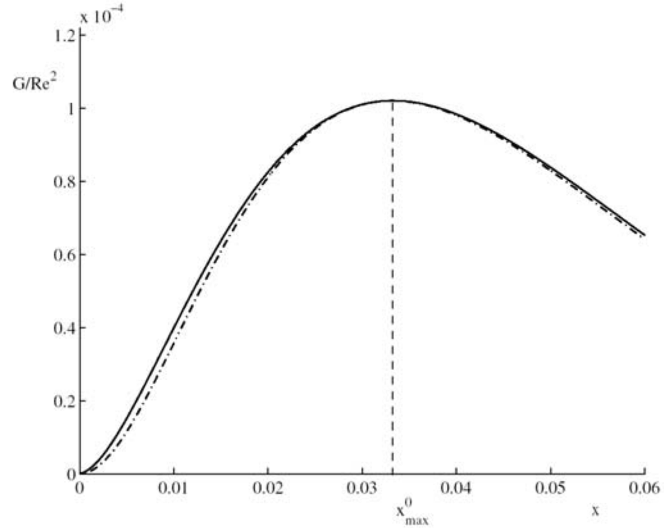


FIGURE 2. Envelope of the curves of gain *versus* x for $m=1$, $\omega=0$: elliptic system at Re=3000 (dashed), parabolic system (solid). The two solutions coincide to plotting accuracy. The growth curve for the specific inflow condition that maximizes the energy at $x=x^0_{max}$ is also plotted by the dash-dotted line.

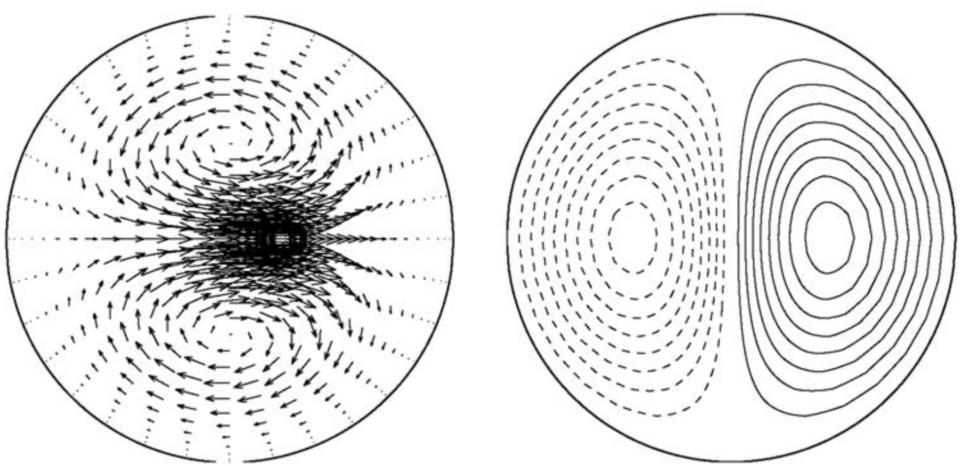


FIGURE 3. Vector field in the (r,θ) plane, of the optimal perturbation with $m=1$, $\omega=0$, that maximizes the gain G at $x=x^0_{max}$ (left). Contour plot of the resulting streaks at $x=x^0_{max}$ (right): continuous and dashed lines are used for positive and negative values respectively.

Optimal control

$$\mathcal{I}(\mathbf{v}, v_w) = \zeta E_c + \chi E_u(x_{fin}) + \psi \int_{x_{in}}^{x_{fin}} E_u(x)\, dx$$

with
$$E_c = \frac{1}{2}\int_{x_{in}}^{x_{fin}} v_w^* v_w \, dx$$

where $v(x,1) = v_w(x)$ is the control statement

Order of magnitude analysis:

$\zeta \sim \mathcal{O}(1),\ \chi \sim \mathcal{O}(Re),\ \psi \sim \mathcal{O}(Re^2)$ *Equilibrium*

ζ increases $\Longrightarrow$ *Cheap*

χ increases $\Longrightarrow$ *Small*

ψ increases $\Longrightarrow$ *Flat*

As usual we introduce a Lagrangian functional, we impose stationarity with respect to all independent variables and recover a system of direct and adjoint equations, coupled by transfer and optimality conditions. The system is solved iteratively; at convergence we have the optimal control.

Optimal cancellation control of streaks

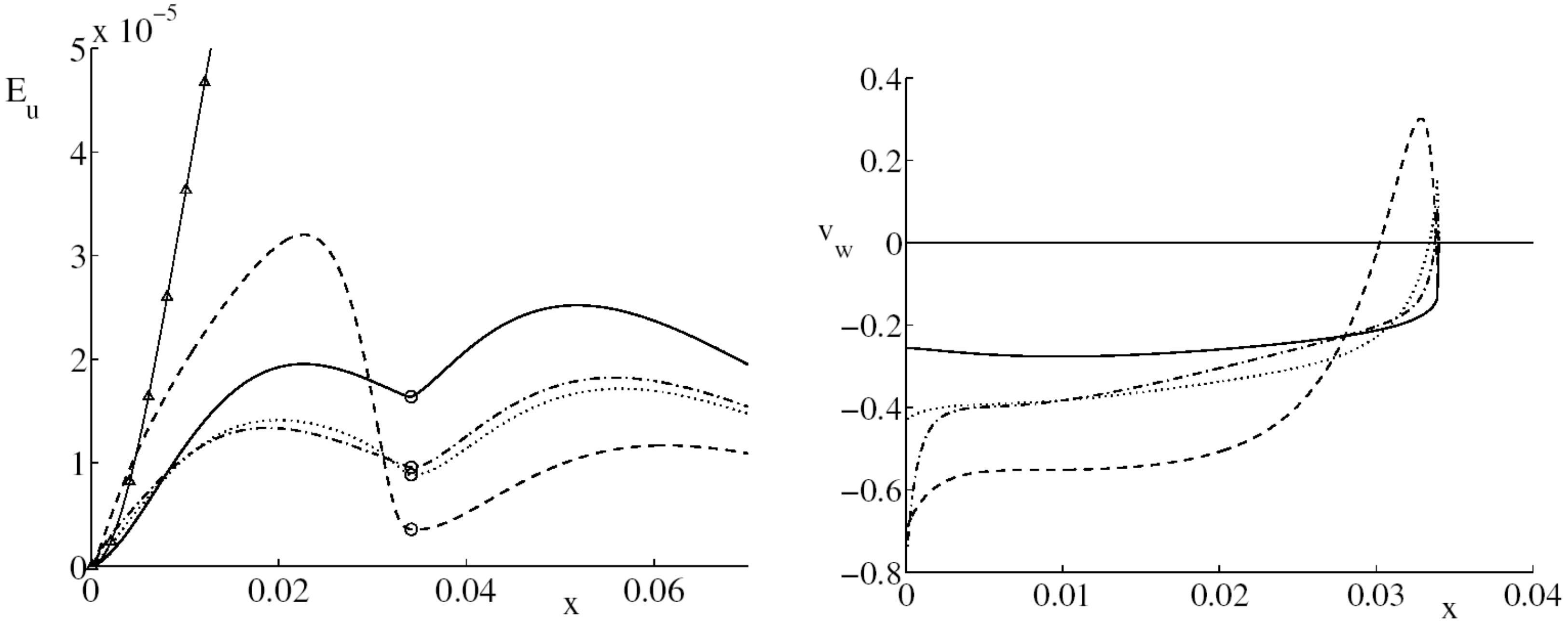


FIGURE 7. Growth curves in presence of different control strategies, with circles indicating the end of the control interval $x = x_{fin}$ (left), and corresponding wall velocity distributions (right), for $m = 1$, $\omega = 0$: 'equil' (dotted); 'cheap' (solid); 'flat' (dash-dotted); 'small' (dashed); uncontrolled (solid with triangles). For all the strategies, the control function has zero imaginary part.

Optimal cancellation control of streaks

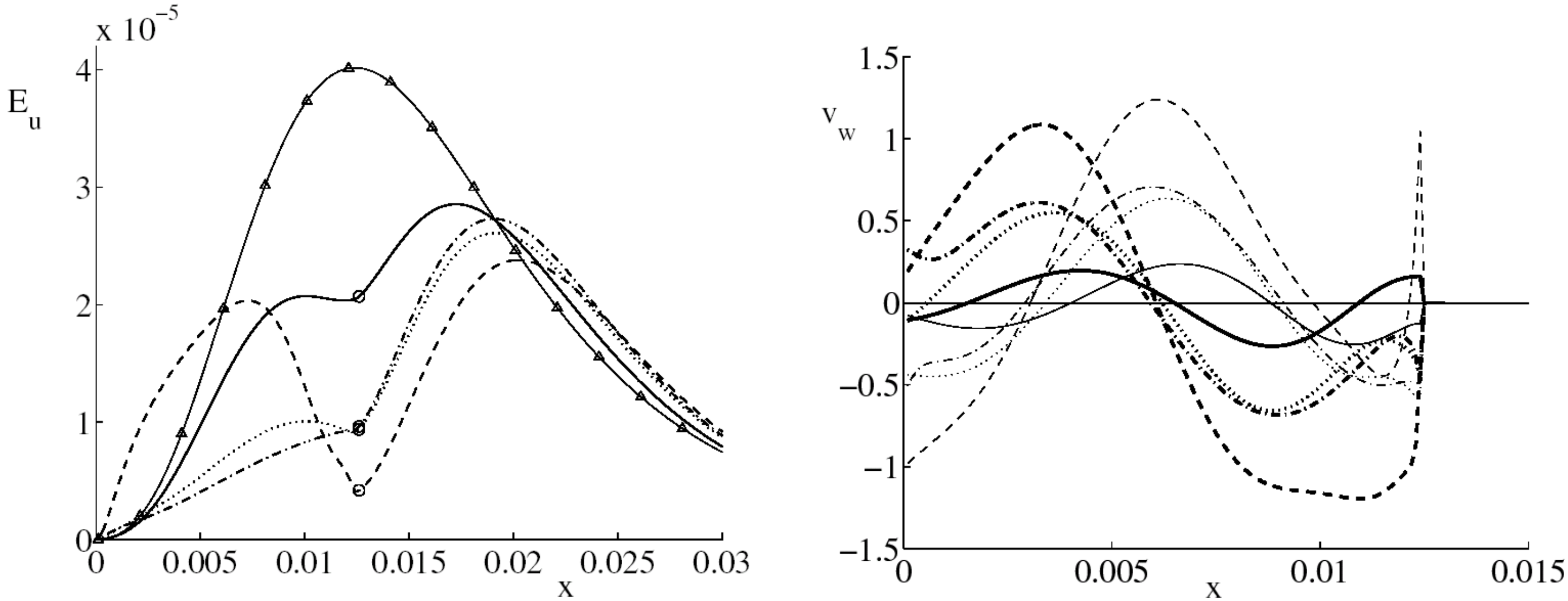


FIGURE 9. Growth curves in presence of different control strategies, with circles indicating the end of the control interval $x = x_{fin}$ (left), and corresponding wall velocity distributions (right), for $m = 1$, $\omega = 300$ (line styles as in Figure 7). In the figure on the right, thick and thin lines are used respectively for the real and imaginary parts of the control function.

Optimal cancellation control of streaks

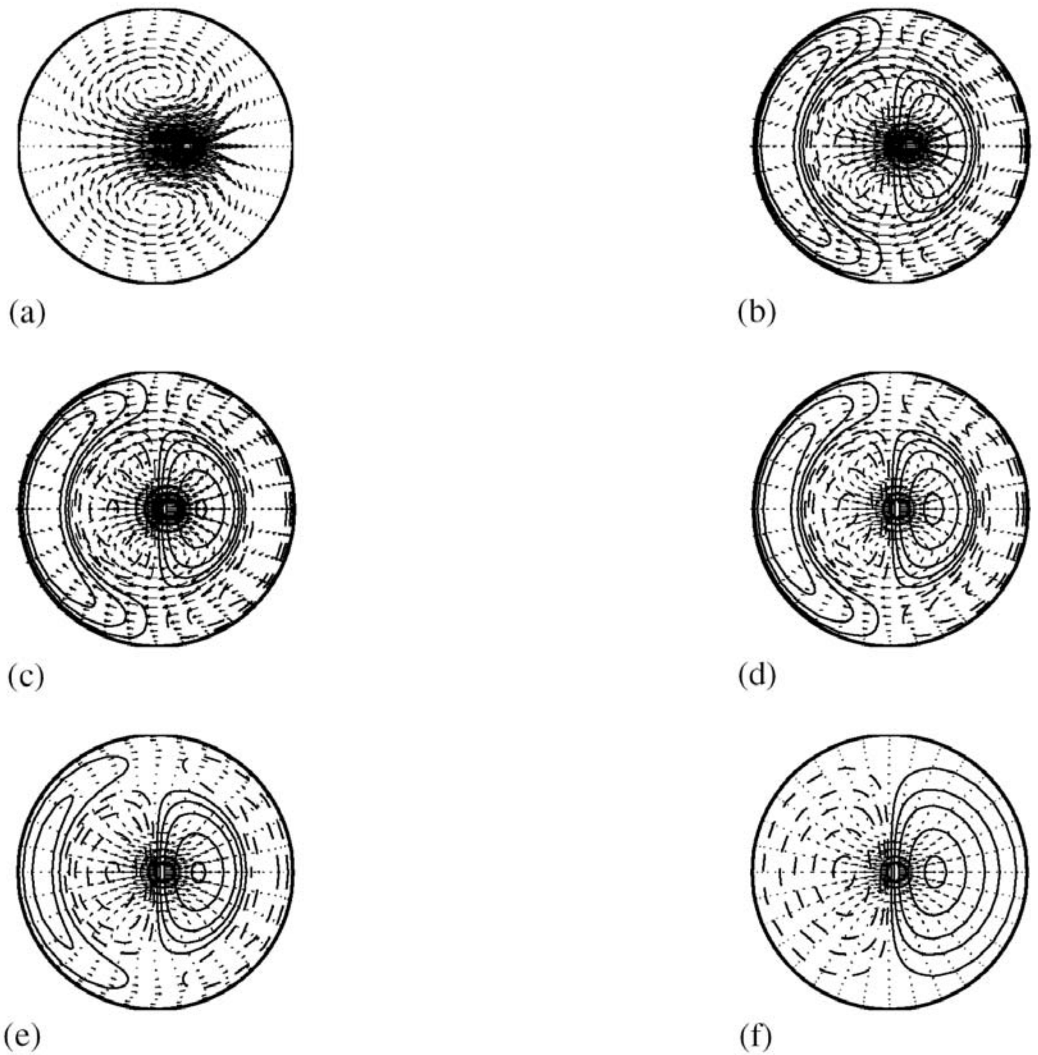


FIGURE 12. Spatial evolution of the optimal steady disturbance subject to the 'flat' control strategy: (a) $x = 0$, (b) $x = x_{fin}/4$, (c) $x = x_{fin}/2$, (d) $x = 3/4\,x_{fin}$, (e) $x = x_{fin}$, (f) $x = 5/4\,x_{fin}$. Shown are contours of u (dashed lines are negative) and vector plots of the cross-flow velocity components. Contour levels and vector scaling are identical in all cases.

Optimal cancellation control of streaks

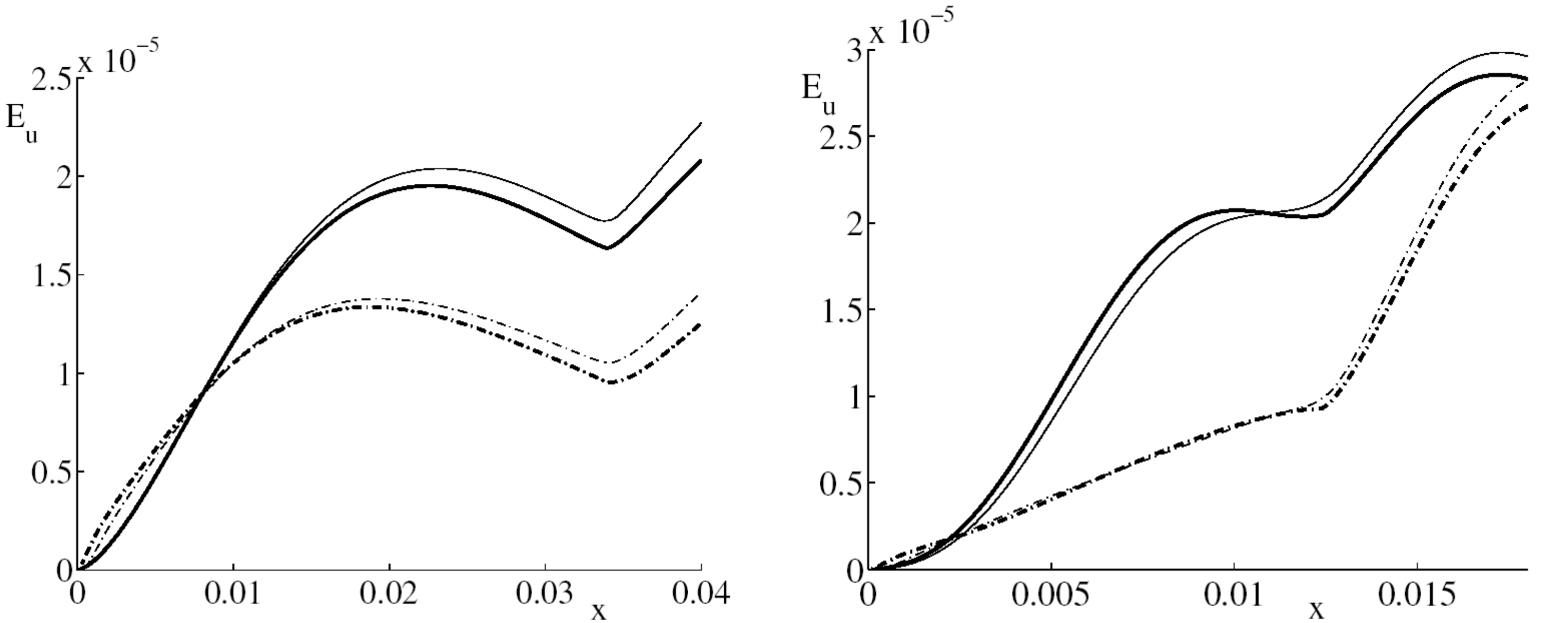
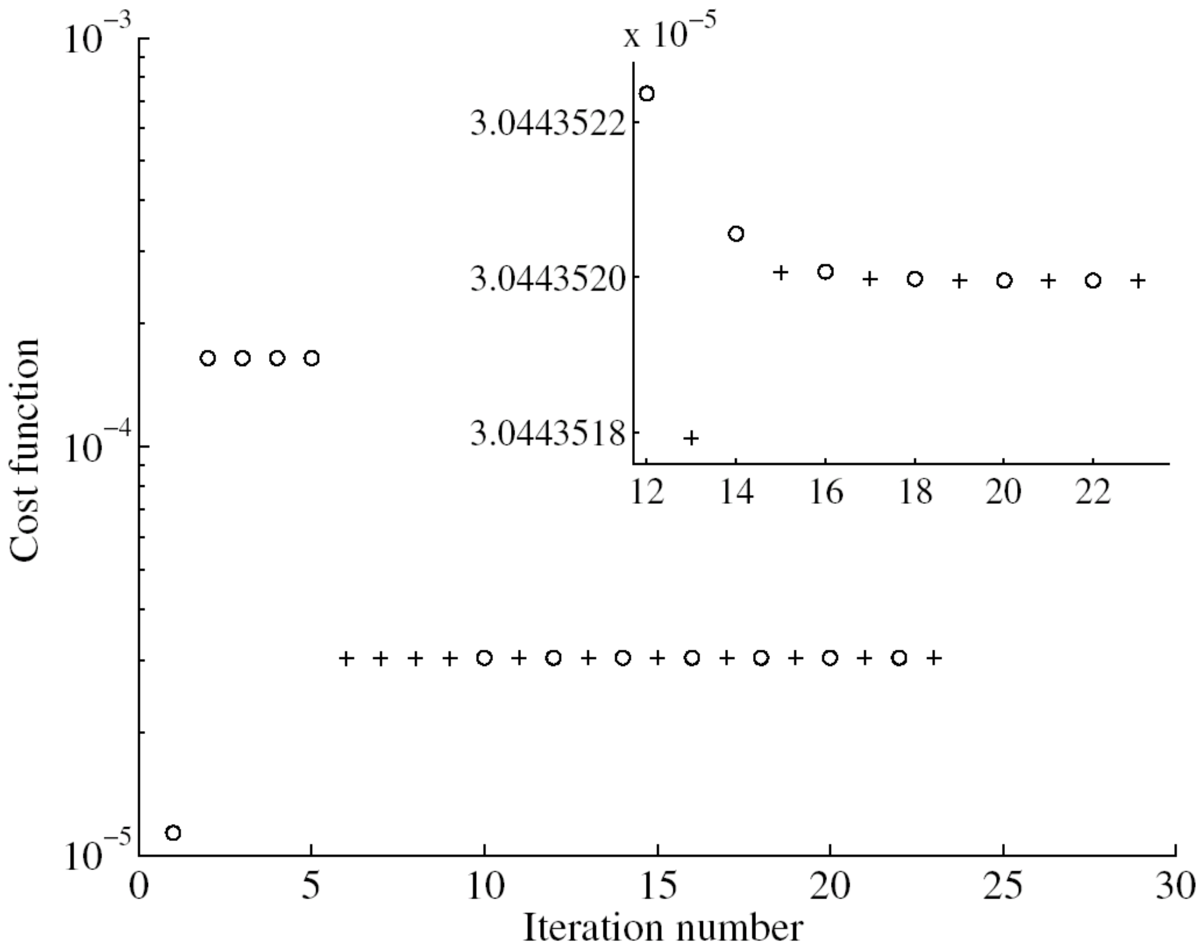


FIGURE 16. Transient growth curves for $m = 1$, $\omega = 0$ (left) and $\omega = 300$ (right), under the action of control: comparison between the parabolic approach (thick lines) and Navier-Stokes computations (thin lines). Solid lines: 'cheap' strategy; dash-dotted lines: 'flat' strategy.

Robust cancellation control of streaks



19. Value of the cost $\mathcal{I}$ as a function of the iteration number for the 'cheap' strategy. Circles: ascent iterations, crosses: descent iterations.

Robust cancellation control of streaks

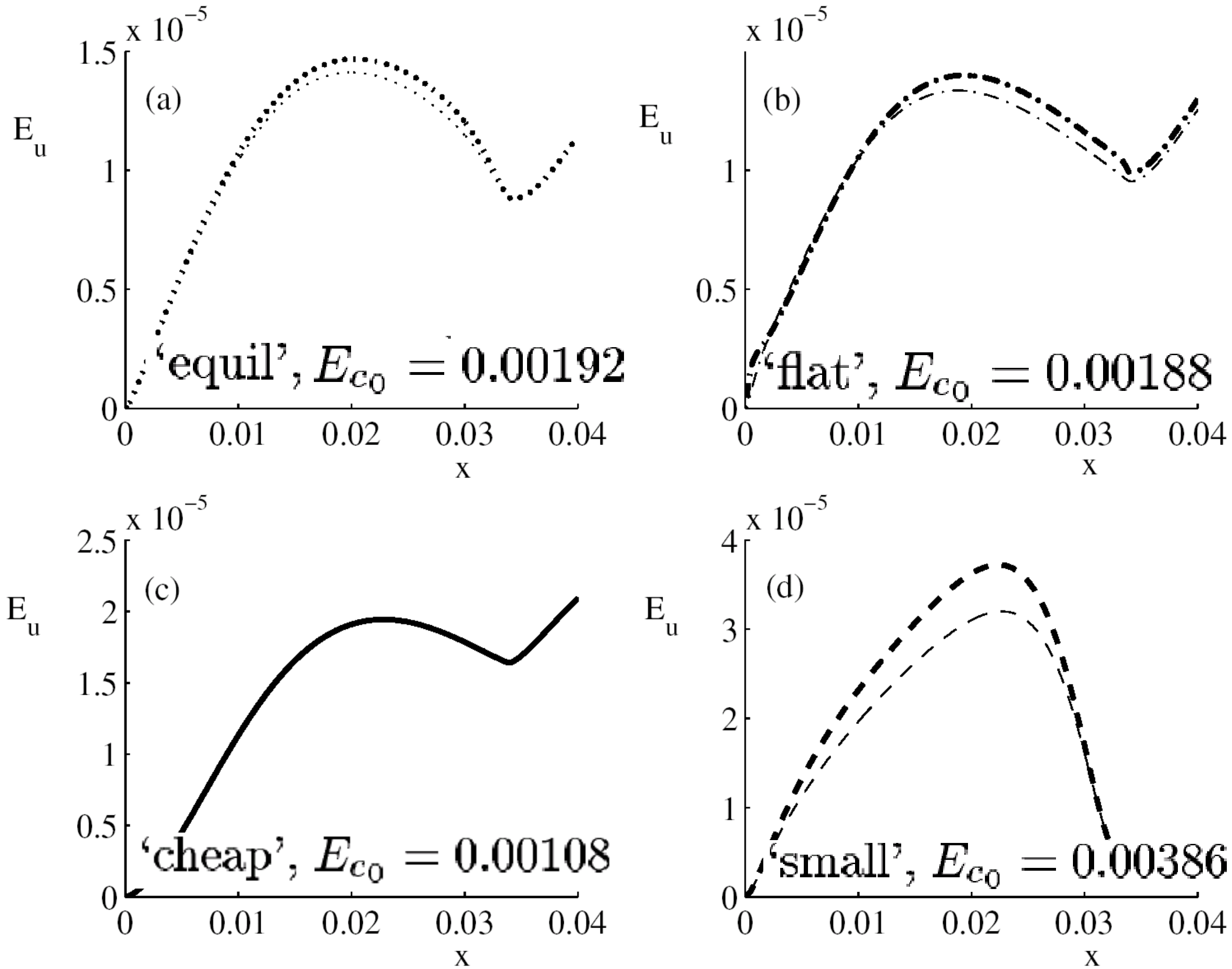


FIGURE 20. Optimal and robust growth curves for $m = 1$, $\omega = 0$, in presence of different control strategies.

Optimal and robust cancellation control of streaks

- In theory it is possible to optimally counteract disturbances propagating downstream of an initial point
  (trivial to counteract a mode)
- Physics: role of buffer streaks
- Robust control laws are available
- Next:
  - Feedback control, using the framework recently proposed by Cathalifaud & Bewley (2004)
- Is it technically feasible?

Optimal control of streaks in boundary layer flow

Spatially parabolic model for the streaks: steady structures elongated in the streamwise direction x

Long scale for x: L
Short scale for y and z: $\delta = L/Re$

Fast velocity scale for u: U_∞
Slow velocity scale for v, w: U_∞ /Re

Optimal control of streaks in boundary layer flow

$$\left\{\begin{array}{rcl} u_x + v_y + w_z &=& 0, \\ (uu)_x + (uv)_y + (uw)_z - u_{yy} - u_{zz} &=& 0, \\ (uv)_x + (vv)_y + (vw)_z + p_y - v_{yy} - v_{zz} &=& 0, \\ (uw)_x + (vw)_y + (ww)_z + p_z - w_{yy} - w_{zz} &=& 0. \end{array}\right\}$$

with

$$\begin{array}{lll lll} u = 0 & \text{at} & y = 0, & u = 1 & \text{for} & y \to \infty \\ v = v_w & \text{at} & y = 0, & w = 0 & \text{for} & y \to \infty \\ w = 0 & \text{at} & y = 0, & p = 0 & \text{for} & y \to \infty \end{array}$$

objective function : :

$$\mathcal{J} = \alpha_1 E_{\text{out}} + \alpha_2 E_{\text{mean}}$$

with the gain given by:

$$G_{\text{mean}} = \frac{\alpha_1 E_{\text{out}} + \alpha_2 E_{\text{mean}}}{E_{\text{in}}} = Re \frac{\alpha_1 E_u|_{x=1} + \alpha_2 \int_0^1 E_u \, dx}{\left[\frac{1}{2Z} \int_{-Z}^{Z} \int_0^{\infty} (|v_0|^2 + |w_0|^2) \, dy \, dz \right]_{x=0}}$$

with

$$E_u(x) = \frac{1}{2Z} \int_{-Z}^{Z} \int_0^{\infty} |u'|^2 \, dy \, dz.$$

Lagrangian functional:

$$
\begin{aligned}
\mathcal{L} = \quad \mathcal{J} \quad & + \frac{1}{2Z}\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{1} a[u_x + v_y + w_z]\,dx\,dy\,dz \\
& + \frac{1}{2Z}\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{1} b[(uu)_x + (uv)_y + (uw)_z - u_{yy} - u_{zz}]\,dx\,dy\,dz \\
& + \frac{1}{2Z}\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{1} c[(uv)_x + (vv)_y + (vw)_z + p_y - v_{yy} - v_{zz}]\,dx\,dy\,dz \\
& + \frac{1}{2Z}\int_{-Z}^{Z}\int_{0}^{\infty}\int_{0}^{1} d[(uw)_x + (vw)_y + (ww)_z + p_z - w_{yy} - w_{zz}]\,dx\,dy\,dz \\
& + \lambda_0[E_{\mathrm{in}}(v_0) - E_0] + \lambda_{\mathrm{w}}[E_{\mathrm{w}}(v_{\mathrm{w}}) - E_{\mathrm{w}0}],
\end{aligned}
$$

Stationarity of $\mathcal{L}$ $\Longrightarrow$ Optimality system

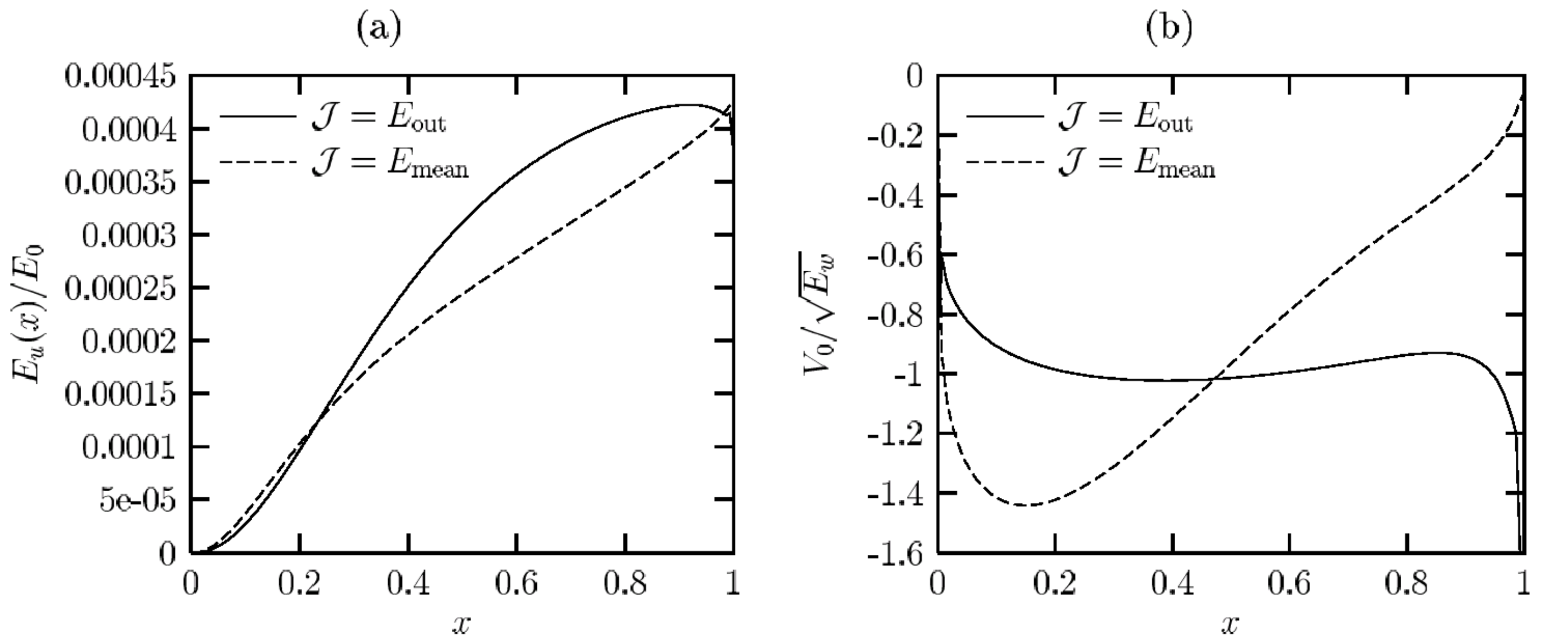


FIGURE 1. Optimal control: comparison between two different objective functions, linear behaviour. $E_0 = 10^{-7}$, $E_{\rm w} = 1$, $\beta = 0.45$ for $\mathcal{J} = E_{\rm out}$, $\beta = 0.547$ for $\mathcal{J} = E_{\rm mean}$. (a) Disturbance energy, normalized by E_0, as a function of x. (b) Optimal suction at the wall normalized with $\sqrt{E_{\rm w}}$.

Optimal laminar flow control of streaks

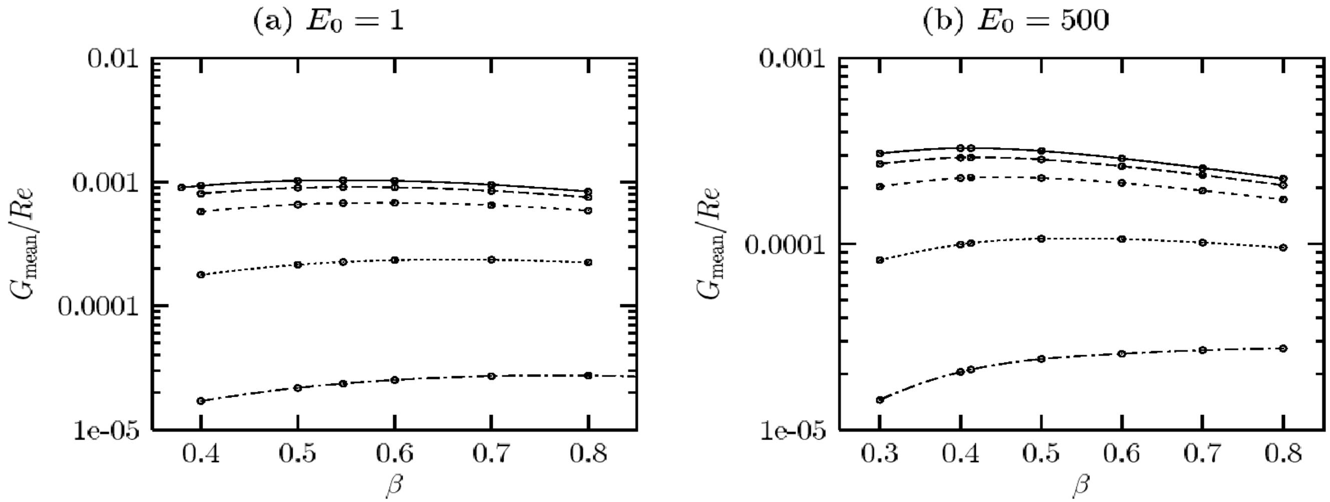


FIGURE 2. Curve of the gain as a function of the spanwise wavenumber β for different values of the control energy at the wall $E_{\rm w}$. (a) Initial energy $E_0 = 1$; (b) initial energy $E_0 = 500$. Uncontrolled: ———; $E_{\rm w} = 0.01$: -----; $E_{\rm w} = 0.1$: ------; $E_{\rm w} = 1.0$: ········; $E_{\rm w} = 5.0$: —·—·—.

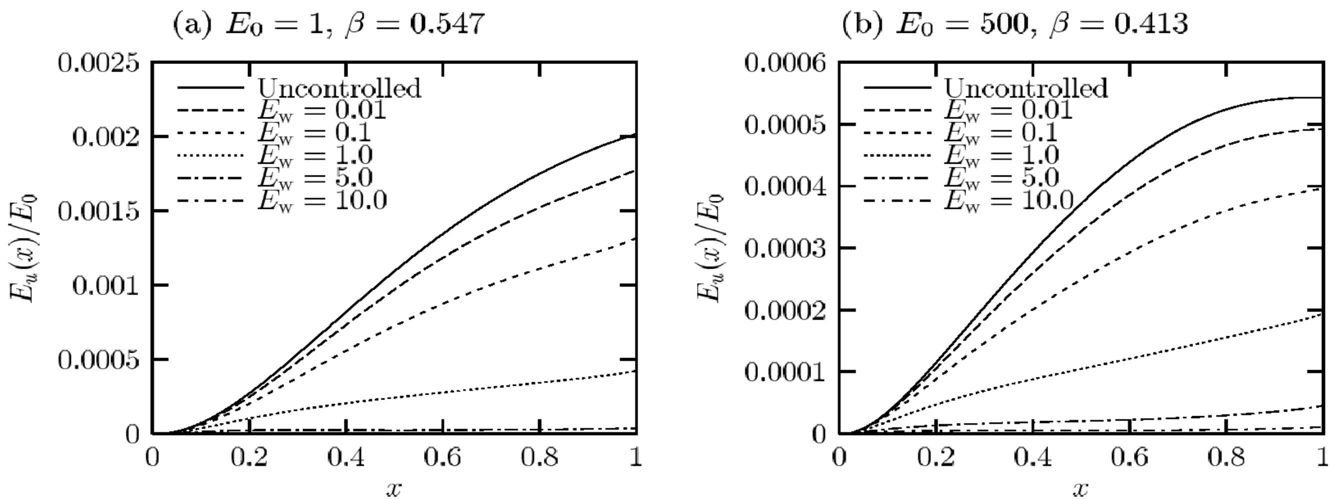


FIGURE 3. Comparison at fixed wavenumber β: perturbation energy $E_u(x)/E_0$ for increasing control energy $E_{\rm w}$ at the wall. (a) $E_0 = 1$ and $\beta = 0.547$; (b) $E_0 = 500$ and $\beta = 0.413$.

Optimal laminar flow control of streaks

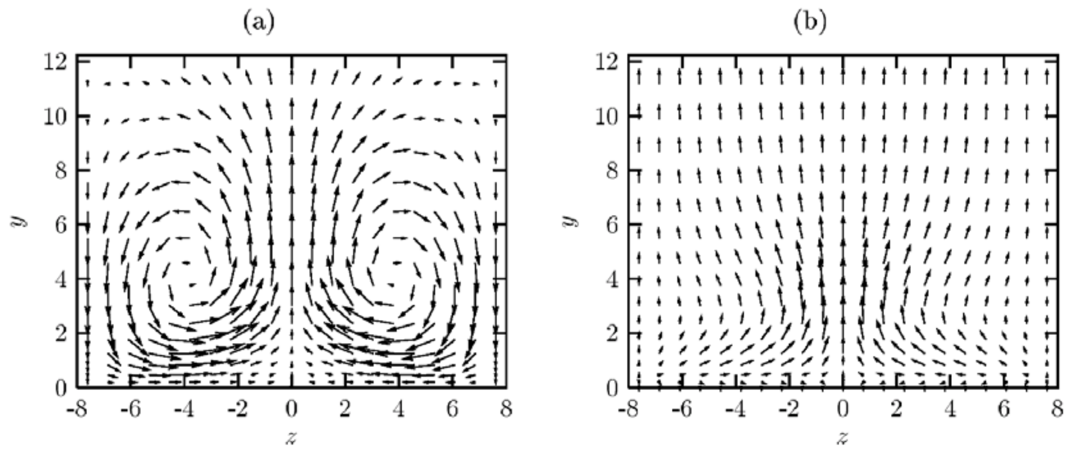


FIGURE 6. $E_0 = 500$, $\beta = 0.413$. Velocity vectors (v, w) in the (z, y) plane at $x = 1$. In both figures the vectors are scaled in the same manner. (a) Uncontrolled. (b) $E_w = 10$.

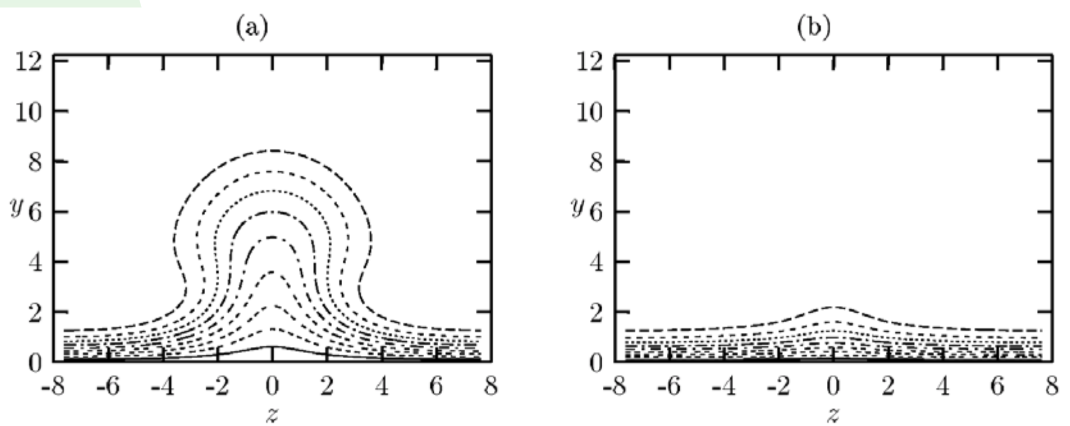


FIGURE 7. $E_0 = 500$, $\beta = 0.413$. Streamwise velocity u isolines in the (z, y) plane at $x = 1$. (a) Uncontrolled. (b) $E_w = 10$.

Laminar flow control creates a thin boundary layer …

… what about the skin friction then?

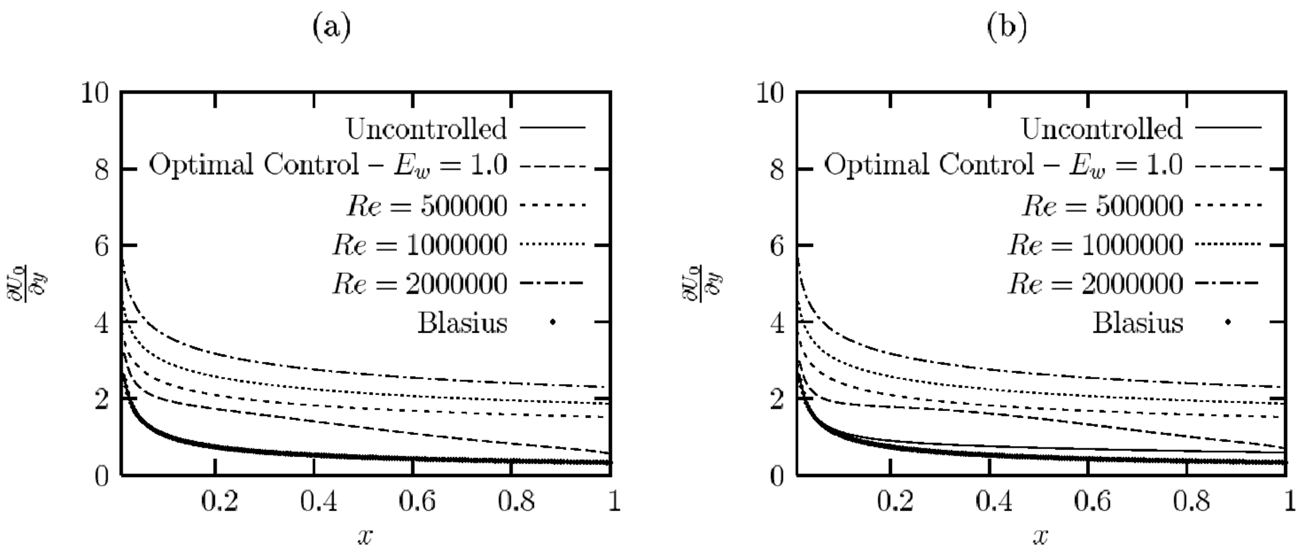


FIGURE 12. $\dfrac{\partial U_0}{\partial y}$ at the wall, in the uncontrolled, controlled ($E_w = 1$) and turbulent boundary layer: (a) $E_0 = 1$ and $\beta = 0.547$; (b) $E_0 = 500$ and $\beta = 0.413$.

Prandtl low-Re turbulent correlation:

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y=0} = 0.0296\frac{U_\infty^2}{\nu}\left(\frac{xU_\infty}{\nu}\right)^{-1/5}$$

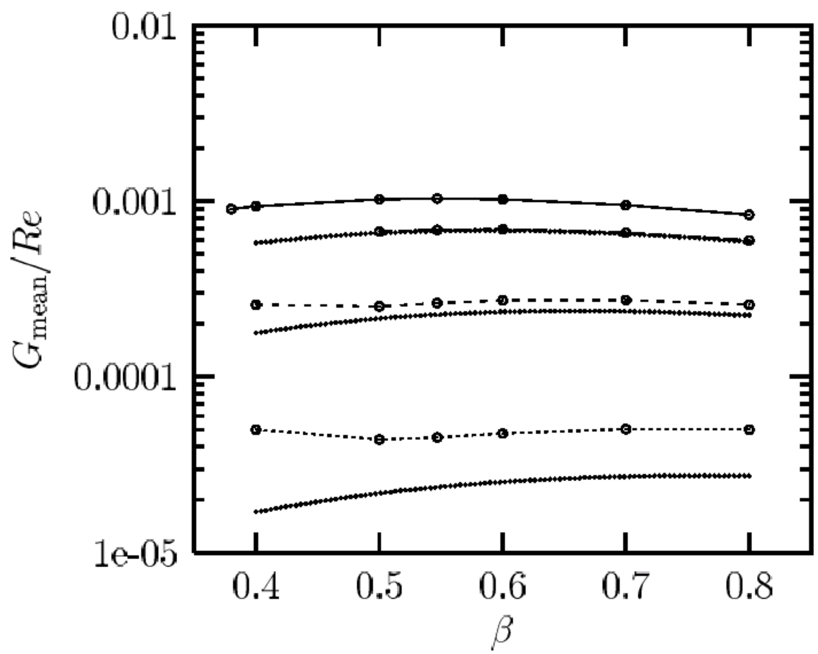


FIGURE 14. Comparison between optimal control and robust control: the mean gain is shown as a function of the spanwise wavenumber β for different values of the control energy E_{w} at $E_0 = 1$. Uncontrolled: ———; $E_{\mathrm{w}} = 0.1$: —·—·—; $E_{\mathrm{w}} = 1.0$: - - - - -; $E_{\mathrm{w}} = 5.0$: ------; Optimal control (see figure 2): ············.

Robust laminar flow control of streaks

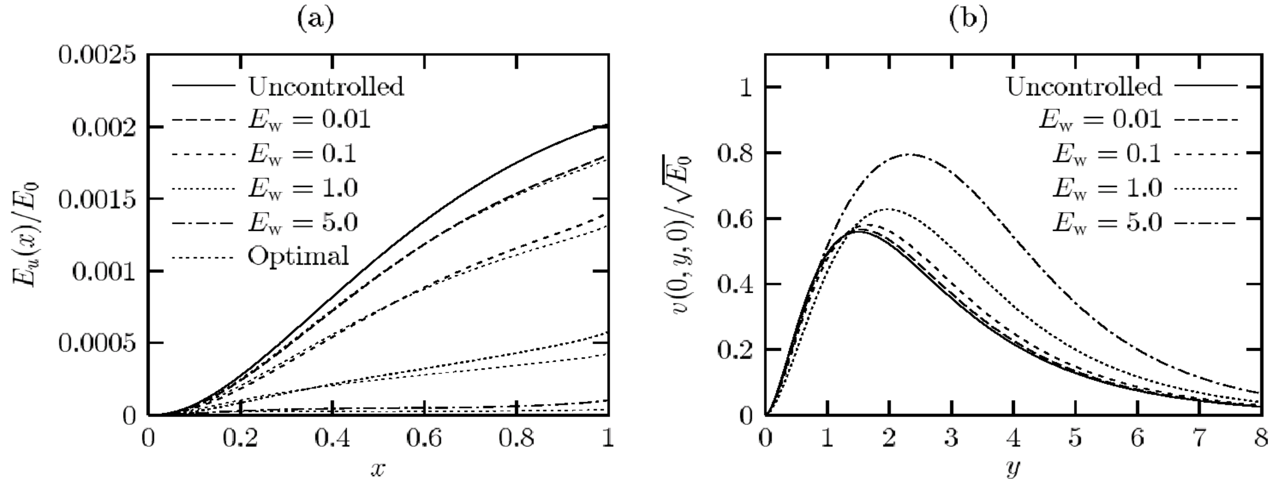


FIGURE 15. Comparison at constant wavenumber $\beta = 0.547$ for increasing control energy E_w at $E_0 = 1$: (a) perturbation energy $E_u(x)/E_0$; (b) optimal perturbation $v_0(y,z)/\sqrt{E_0}$ in the plane $z = 0$.

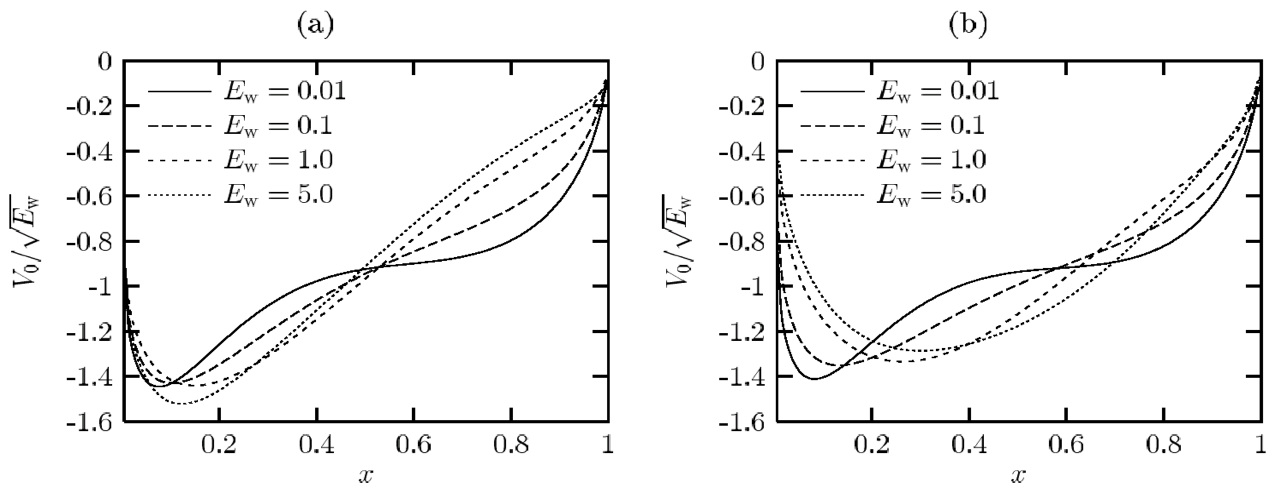


FIGURE 16. Comparison at constant wavenumber $\beta = 0.547$ for increasing control energy E_w. $E_0 = 1$, optimal suction profile at the wall $V_0(x,0)/\sqrt{E_w}$: (a) optimal control; (b) robust control.

Optimal and robust laminar flow control of streaks

- Mean flow suction can be found to optimally damp the growth of streaks in the linear and non-linear regimes
- Both in the optimal and robust control case the control laws are remarkably self-similar → Bonus for applications
- No need for feedback
- Technically feasible (*cf.* ALTTA EU project)

Conclusions

- Optimal control theory is a powerful tool
- Optimal feedback strategies underway
- Optimal control via tailored magnetic fields should be possible