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The effect of stable thermal stratification on shear flow stability

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We study the effect of buoyancy on shear flow stability. The vertical body force is induced by a vertical, constant, and positive thermal gradient. A linear stability analysis is carried out, in the spatial framework, focusing on both exponential and transient growth. In both cases, positive thermal stratification is found to stabilize the disturbances. © *2004 American Institute of Physics*. [DOI: 10.1063/1.1810751]

The disturbance growth mechanism in parallel and quasiparallel flows is of great interest to understand transition to turbulence and has been the object of several theoretical and experimental studies. In a low turbulence environment, the disturbance growth mechanism is linear and the main theoretical tool is the eigenvalue analysis of equations linearized around the laminar solution. Two kinds of transition processes can be distinguished in wall-bounded shear flows. For small turbulence levels ($<0.5\%$), the disturbances take the form of two-dimensional exponentially growing Tollmien–Schlichting waves. They are amplified up to the point where nonlinear breakdown occurs. In boundary layers this process is called natural transition. For high turbulence levels (above 1%), the classical (modal) path of laminar-turbulent transition is often unable to predict transitional Reynolds numbers and observed flow structures. The natural transition is bypassed and transition can be interpreted via the strong transient, algebraic growth of streamwise streaks, as observed by Klebanoff *et al.*[1]

Here, Poiseuille flow between two horizontal planes is considered; the buoyancy force is induced by upper wall heating T_h and/or lower wall cooling T_c. For weak thermal gradients all fluid properties are assumed to be constant. The mean velocity and temperature distributions, made dimensionless using the maximum velocity U_0, the half channel height h, and a characteristic temperature ($\Delta T = T_h - T_c$) as scales, have the simple forms

$$U(y) = 1 - y^2, \quad \Theta(y) = \frac{1+y}{2}.$$

The linear stability analysis to follow may be applicable to a more general class of nearly parallel shear flows and temperature or density distributions.

The behavior of infinitesimal three-dimensional disturbances is described by the linearized Navier–Stokes equations with the Boussinesq approximation. These can be written in Fourier-transformed form as a system of three ordinary differential equations: the Orr–Sommerfeld equation for the normal velocity $\tilde{v}(y)$, the Squire equation for the normal vorticity $\tilde{\eta}(y)$, and the energy equation for the thermal fluctuation $\tilde{\tau}(y)$. The dimensionless system reads

$$\begin{aligned} &[(-i\omega + i\alpha U - \mathrm{Re}^{-1}\Delta)\Delta - i\alpha U'']\tilde{v} = -(\alpha^2+\beta^2)\mathrm{Ri}\,\tilde{\tau}, \\ &[-i\omega + i\alpha U - \mathrm{Re}^{-1}\Delta]\tilde{\eta} = -i\beta U'\tilde{v}, \\ &[-i\omega + i\alpha U - (\mathrm{Pr}\,\mathrm{Re})^{-1}\Delta]\tilde{\tau} = -\Theta'\tilde{v}, \end{aligned} \tag{1}$$

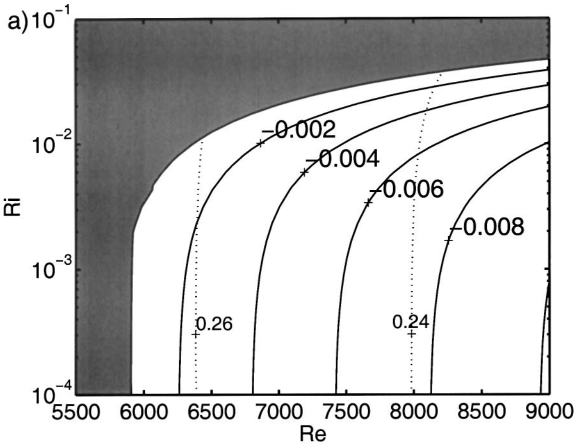
where $\Delta = \partial_{yy} - \alpha^2 - \beta^2$ is the Fourier-transformed Laplacian operator. ω, α, and β are, respectively, the frequency, streamwise, and spanwise wave numbers, and $\Theta' = d\Theta/dy = 1/2$. The boundary conditions are $\tilde{v} = \partial\tilde{v}/\partial y = \tilde{\eta} = \tilde{\tau} = 0$ at both walls. The independent parameters are the Reynolds, the Richardson, and the Prandtl numbers, defined as

$$\mathrm{Re} = \frac{U_0 h}{\nu}, \quad \mathrm{Ri} = \frac{\gamma g \Delta T\, h}{U_0^2}, \quad \mathrm{Pr} = \frac{\nu}{\kappa},$$

where γ is the coefficient of thermal expansion. The equations are discretized using a Chebyshev collocation method[2] with the software MATLAB.

The stability analysis adopts a spatial approach implying that the eigenvalue problem is solved for $\alpha \in \mathbb{C}$ with ω and β real. In compact form this system can be noted as $\mathcal{L}\tilde{q}_n = \alpha_n \tilde{q}_n$. The temporal modal stability was studied earlier by Gage and Reid.[3] Although the neutral curve is independent of a spatial or temporal viewpoint, these two problems are quite different. Because the problem is parabolic in time, a temporal study simply proceeds forward in time. On the other hand, the spatial problem is ill posed as an initial value problem. In fact, with the possible exception of the unstable mode, the upper half of the complex α plane contains downstream decaying modes while the lower half corresponds to upstream decaying modes. In the following, only downstream evolving modes are considered.

First, we study the evolution of disturbances with a modal analysis providing the asymptotic behavior for large values of x. This classical stability analysis is focused on the sign of the least stable mode, labeled as Tollmien–Schlichting mode. A generalization of Squire's theorem[3] states that "the three-dimensional problem is equivalent to a two-dimensional problem at smaller Reynolds number and larger Richardson number." Hence, for positive Richardson numbers, the modal analysis can be reduced to the Orr–Sommerfeld and energy equations with $\beta = 0$. The growth



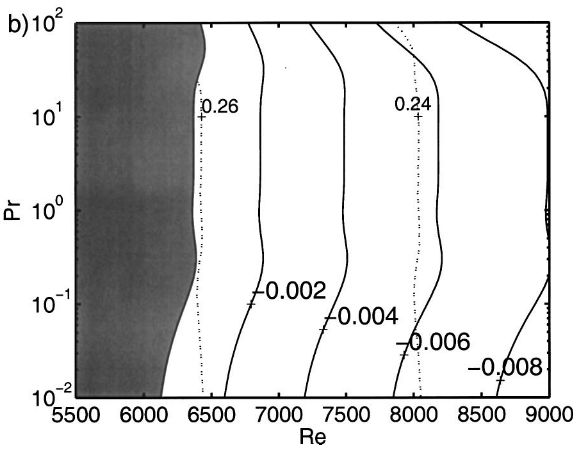


FIG. 1. Contours of growth rate α_i (continuous lines), and corresponding angular frequency of largest growth ω (dotted lines) in the (Re, Ri) plane for Pr=0.7 (a) and in the (Re, Pr) plane for $\mathrm{Ri}=10^{-2}$ (b). The shaded areas indicate the linearly stable regions.

rate and corresponding ω contours, in the (Re, Ri) and (Re, Pr) planes, are displayed in Fig. 1. First, we observe the decreasing value of the growth rate for increasing Richardson numbers; Gage and Reid[3] had already shown such a trend, and demonstrated that the flow becomes completely stabilized for Ri > 0.443. Some critical parameters are reported in Table I and compare acceptably well to some of the old results in Ref. 3. As can be seen, buoyancy strongly influences the critical value of the Reynolds number. At the same time, the Tollmien–Schlichting wave structure shows a weak sensitivity to heat transfer: the streamwise wave number α_r, wave speed c_r, and the eigenfunctions are but mildly affected. Second, we observe a weak influence of the Prandtl number on the results [cf. Fig. 1(b)].

As a next step, we investigate an amplification mechanism that cannot be attributed to single-mode exponential growth. Small perturbations in shear flow might experience transient growth before their eventual downstream asymptotic decay. Mathematically this is the consequence of the non-normal nature of the operator $\mathcal{L}$. Since eigenvectors are nonorthogonal, constructive or destructive interference amongst the various modes is possible before the asymptotic behavior sets in. The combination of this algebraic, basically inviscid, growth and the viscous damping effect lead to a phenomenon known as transient growth. If the transiently growing energy attains a sufficient amplitude, nonlinear interactions may ultimately give rise to transition. In this sense, algebraic growth could be a first stage of bypass transition. The search for transiently growing perturbations can be considered by determining the worst case scenario, i.e., searching for the inlet condition that provides the largest energy growth. Consequently, we define the maximum energy amplification $G_{\max}$ as

$$G_{\max} = \max_{\forall x \in [0;\infty[} \|q(x)\|_E,$$

with the normalization $\|q(0)\|_E = 1$. The subscript E denotes an energetic norm defined in the following. The disturbances corresponding to this maximum take the form of streamwise elongated and quasistationary structures, so that a new set of scales, analogous to Prandtl approximations for boundary layer, can be employed.[4] By scaling the cross-stream coordinates (y,z) with h, and the streamwise coordinate x with h/Re, it follows that U_0 should be used as the scale for the streamwise perturbation velocity u, together with U_0/Re for the cross-stream components (v,w). The pressure is normalized by $\rho(U_0/\mathrm{Re})^2$, with ρ the density of the fluid. If we apply these new scales to the linearized Navier–Stokes equations it follows that the system of equations at first order is independent of the Reynolds number and parabolic in the streamwise direction. It is simple to reduce the leading order equations to a set of three spatially parabolic disturbance equations:

$$[(-i\omega_L + i\alpha_L U - \Delta_n)\Delta_n - i\alpha_L U'']\tilde{v} = -\beta^2 \mathrm{Gr}\,\tilde{\tau},$$

$$[-i\omega_L + i\alpha_L U - \Delta_n]\tilde{u} = -U'\tilde{v},$$

$$[-i\omega_L + i\alpha_L U - \mathrm{Pr}^{-1}\Delta_n]\tilde{\tau} = -\Theta'\tilde{v}, \qquad (2)$$

with $\Delta_n = \partial_{yy} - \beta^2$ and Gr the Grashof number, $\mathrm{Gr} = \mathrm{Re}^2\mathrm{Ri}$. The subscript "$L$" has been employed to denote long scales, ω_L and α_L are related to the frequency and streamwise wave

TABLE I. Neutral stability results for various Ri values.

	Gage and Reid			Present results		
Ri	Re_c	α_c	$c_{r_{\mathrm{crit}}}$	Re_c	α_c	$c_{r_{\mathrm{crit}}}$
0	5396	1.022	0.2672	5772.2	1.020	0.2639
0.0304	7133	1.005	0.2482	7600.6	1.0031	0.2452
0.0616	9718	0.998	0.2285	10 246	0.9845	0.2261
0.0952	13 755	0.964	0.2075	14 451.5	0.9637	0.2057

number of the full model (1) by $\omega_L = \omega\,\mathrm{Re}$ and $\alpha_L = \alpha\,\mathrm{Re}$.

A first result obtained is that there are no differences between the steady results (i.e., $\omega = 0$) obtained using the parabolic (2) or the full model (1). In particular, the simplified model adequately describes the linear spatial evolution of instability in the Rayleigh–Bénard–Poiseuille problem ($\mathrm{Re} = \mathrm{Gr}\,\mathrm{Pr} < 0$) for which the first amplified mode takes the form of stationary longitudinal rolls oriented in the main flow direction. The corresponding critical values (Ra_c, β) are independent of the Prandtl number and (obviously) of the Reynolds number and correspond to those obtained for the natural Rayleigh–Bénard convection.

We now investigate the transient growth of streaks. A dimensionless disturbance energy is defined as

$$E(x) = \iiint |u|^2 + \frac{1}{\mathrm{Re}^2}\left(|v|^2 + |w|^2 + \frac{\mathrm{Gr}}{\Theta'}|\tau|^2\right) dy\ dz\ dt, \quad (3)$$

with perturbations expressed in the set of downstream developing eigenmodes,

$$q = \sum_{n=1}^{\infty} \kappa_n \tilde{q}_n(y) e^{i(\alpha_n x + \beta z - \omega t)},$$

where q is the generic disturbance variable. The coefficients κ_n are optimized to achieve the maximum energy amplification factor, and they are obtained using singular value decomposition.[5] The parameters ω_{opt}, β_{opt}, and x_{opt}, corresponding to the largest possible gain at fixed parameters values, are obtained using a shooting method. Notice that Squire theorem is only valid for computing the least stable mode, and earlier studies of transient growth have shown that disturbances which experience the maximum gain are three dimensional. Also in this case the optimal perturbations take the form of stationary ($\omega_{\mathrm{opt}} = 0$), counterrotating streamwise vortices and thermal streaks (v, w, τ) aligned with the mean flow at inception and evolve into streamwise velocity streaks (u) (cf. Fig. 2). The optimal spanwise distribution is nearly constant and close to $\beta_{\mathrm{opt}} = 2$, corresponding to quasicircular structures. The maximum value for the gain varies is proportion to Re^2 and this can be easily justified[6] since, for large Re, the optimal gain becomes

$$\frac{G_{\max}}{\mathrm{Re}^2} = \frac{\iiint |u|^2 dy\ dz\ dt\big|_{x_{\mathrm{opt}}}}{\iiint |v|^2 + |w|^2 + \frac{\mathrm{Gr}}{\Theta'}|\tau|^2 dy\ dz\ dt\big|_{x=0}}. \quad (4)$$

Figure 2 shows the stabilizing interaction of vortices and thermal streaks at the input. Such a stabilization is the consequence of vertical thermal gradients which induce restoring forces to counteract the vertical displacement of fluid particles within a streamwise vortex. This effect can also be interpreted by the streamwise oscillating behavior of the perturbations (cf. Fig. 3). As stated by Luchini,[6] oscillations of changing signs are unfavorable to streamwise streaks development.

The contours of the maximum gain $G_{\max}/\mathrm{Re}^2$ are traced in Fig. 4 as function of the Grashof and Prandtl numbers. We observe, in particular, the decreasing value of the gain for increasing Grashof numbers. Also, for $\mathrm{Pr} \ll 1$ or $\mathrm{Pr} \gg 1$, buoyancy is less effective as a consequence of the dissipation of, respectively, thermal streaks and vortices which hampers their interaction. The streamwise location for which the maximum gain is achieved $(x_{\mathrm{opt}}/\mathrm{Re})$ decreases with the gain

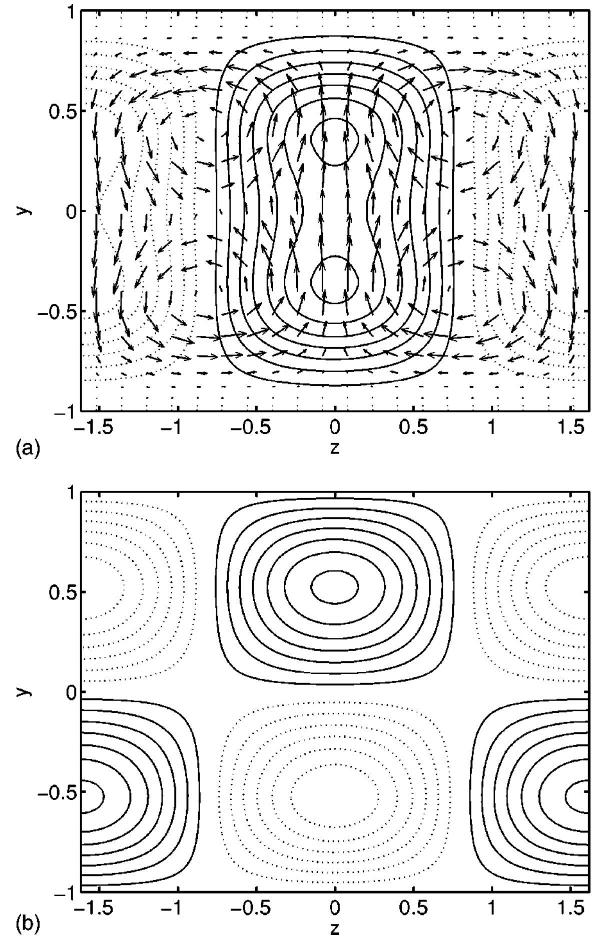


FIG. 2. Optimal inflow disturbances (a) at $x = 0$ and resulting streaks (b) at $x = x_{\mathrm{opt}}$ for $\mathrm{Pr} = 1$, $\mathrm{Gr} = 10^4$, $\beta = 1.95$, and $\omega = 0$. The inflow perturbation is represented through the cross-stream velocity (vectors) and thermal streaks (isolines). The outflow is displayed with isolines of the streamwise velocity. The continuous and dotted lines denote, respectively, positive and negative values of the isolines.

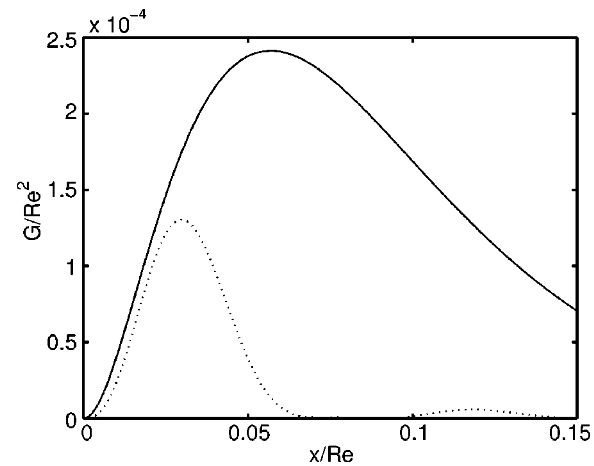


FIG. 3. Energy growth vs x. Continuous line, $\mathrm{Gr} = 0$ (with $\beta_{\mathrm{opt}} = 1.91, x_{\mathrm{opt}} = 0.057$); dotted line, $\mathrm{Gr} = 10^4$ and $\mathrm{Pr} = 1$ (with $\beta_{\mathrm{opt}} = 1.95, x_{\mathrm{opt}} = 0.030$).

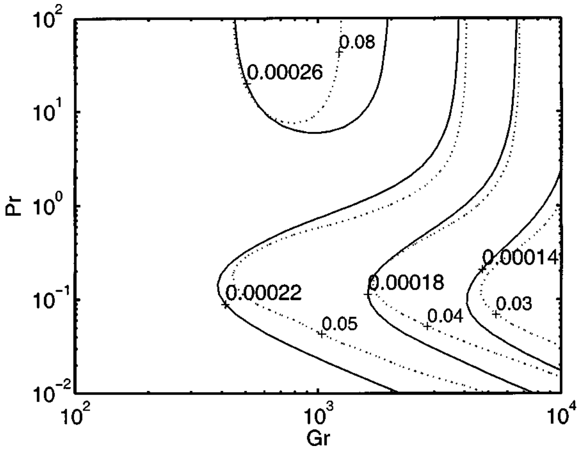


FIG. 4. Contours of maximal gain G_{max}/Re^2 in the (Gr, Pr) plane (continuous lines) together with the corresponding streamwise position x_{opt}/Re (dotted lines).

because the development of the velocity streaks is opposed by the rapid dissipation of the vortex. Finally, notice that for increasing values of Gr, the gain becomes less sensitive to spanwise wave number variations, in other words, for high Grashof numbers an increasingly large band of wave numbers can equally well excite transient amplification of disturbances. The optimal β value arises out of a balance between the energy transfer with the mean shear flow term and the viscous dissipation term. For high Grashof numbers, dissipation is ruled by buoyancy.

In this study, spatial linear stability theory is used to examine the stability of shear flows subject to positive thermal stratification. Stable stratification is an efficient mean to hamper the growth of modal and nonmodal disturbances; thus, the present results indicate a viable strategy to efficiently control transitional flows. Moreover, a simplified parabolic model was developed and validated in order to describe the evolution of streamwise elongated disturbances in shear flows with thermal effects.

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