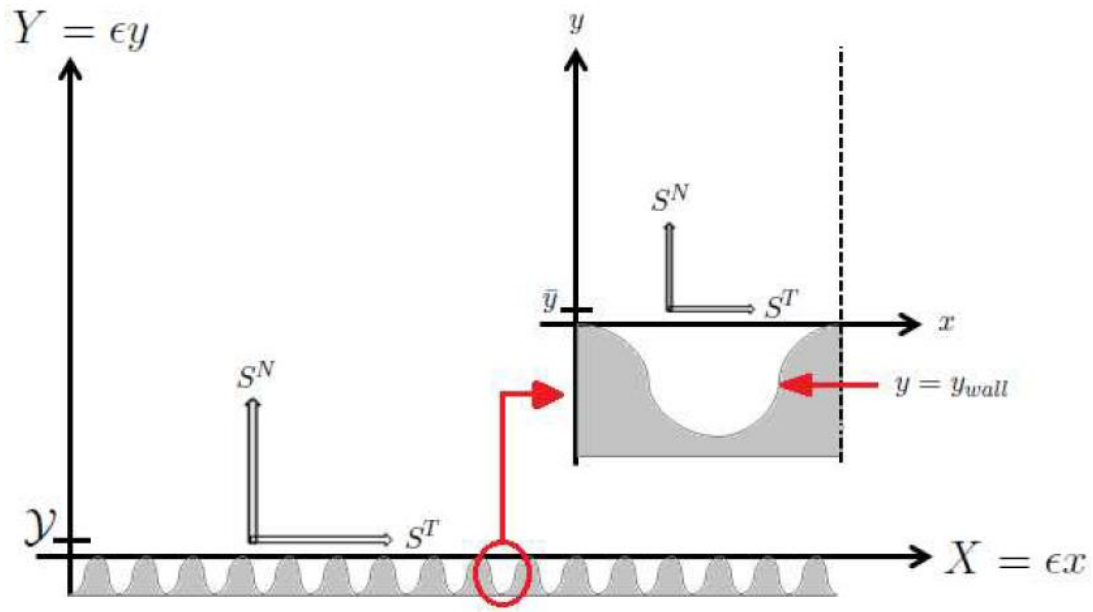
Università di Genova
DICCA DIPARTIMENTO DI INGEGNERIA CIVILE, CHIMICA E AMBIENTALE

Effective boundary conditions at a regularly micro-structured wall: a second-order homogenization approach



Based on joint work with Sahrish B. Naqvi.

(the smooth wall)

Henri Navier, 1823

Si elle était perpendiculaire à l'axe des y,

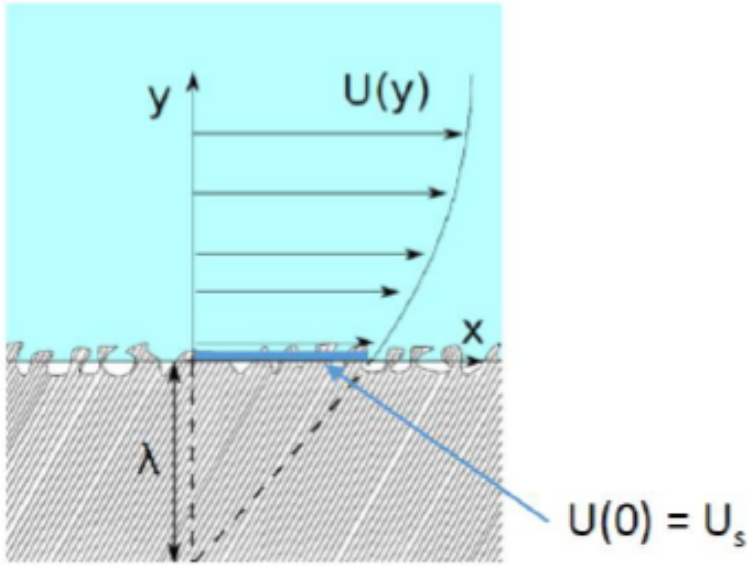
$$Eu + \varepsilon \frac{du}{dy} = 0, \quad Ew + \varepsilon \frac{dw}{dy} = 0;$$

La valeur de la constante E doit varier suivant la nature des corps avec lesquels le fluide est en contact, et (ce qui est physiquement impossible) s'il y avait un espace vide au-dessus de la portion libre de la surface du fluide, ces équations devraient encore être satisfaites pour les points appartenant à cette portion, en y supposant $E = 0$.

.... la constante ε représente en unités de poids la résistance provenant du glissement de deux couches quelconques l'une sur l'autre, pour une étendue égale à l'unité superficielle.

In today's words and terminology, Navier's argument was that there is *partial slip*, and the resistance of the wall is proportional to the slip velocity U_s:

$$E\,U_s = -\varepsilon\, dU/dy\big|_{y=0} \rightarrow \quad U(0) = (-\varepsilon/E)\, dU/dy\big|_{y=0} = \lambda\, dU/dy\big|_{y=0}$$



λ = slip length

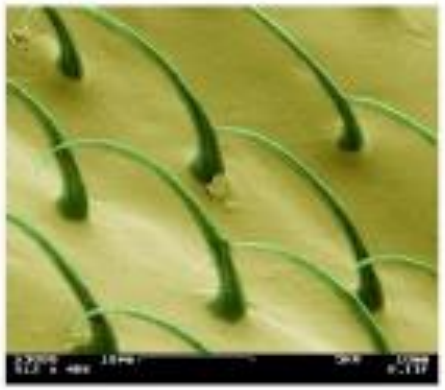
- Navier's slip condition is a first-order development around a fictitious wall (the position y = 0 is arbitrary) applicable when either
  - *(i)* the surface geometry is microstructured or
  - *(ii)* the continuum approximation breaks down.

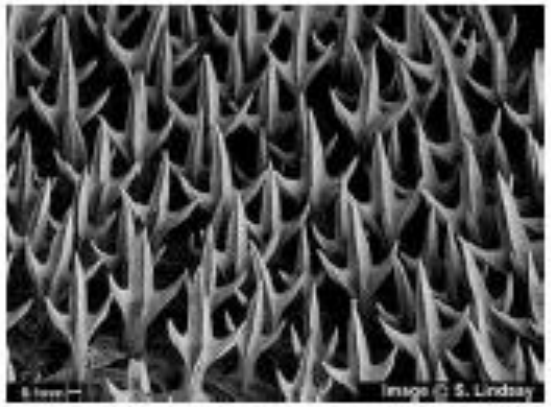
- There is a unique slip length λ for U and W only for isotropic (in x, z) walls. The general (anisotropic) case requires (to first order) that:

$$\begin{bmatrix} U(x,0,z) \\ W(x,0,z) \end{bmatrix} = \Lambda \frac{\partial}{\partial y} \begin{bmatrix} U(x,0,z) \\ W(x,0,z) \end{bmatrix},$$

with Λ a slip tensor (plus a *non-penetration* condition for V).

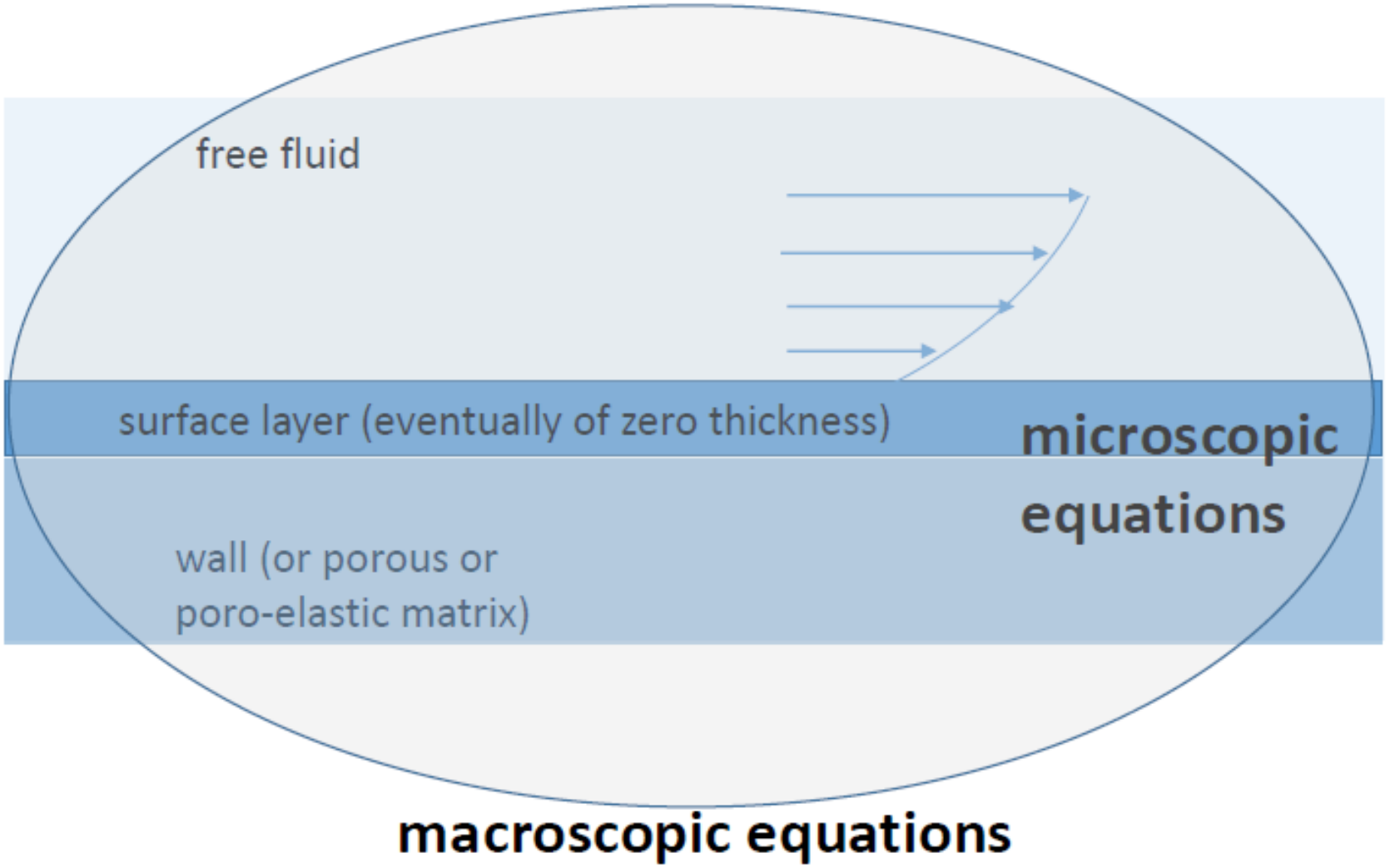
Irregular surfaces are the norm, not the exception

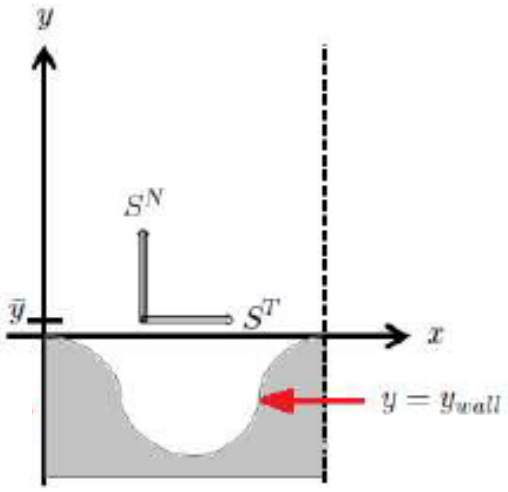




Literature has always turned around the (first order) Navier slip condition, for both regularly microstructured walls and random roughness (with few exceptions):

Achdou, Pironneau & Valentin,	*J. Comp. Phys*. 1996
Jäger & Mikelić,	*J. Diff. Equations* 2001
Basson & Gérard-Varet,	*Comm. Pure Appl. Math*. 2008
Kamrin, Bazant & Stone,	*J. Fluid Mech*. 2010
Luchini,	*J. Fluid Mech*. 2013
Introïni, Quintard & Duval,	*Int. J. Heat Mass Transf.* 2011
Guo, Vera-Tissoires & Quintard,	*J. Comp. Phys.* 2016
Jimenez-Bolanos & Vernescu,	*Phys. Fluids* 2017
Zampogna, Magnaudet & Bottaro,	*J. Fluid Mech*., 2019
Lacis et al.,	*J. Fluid Mech*., in press


free fluid
surface layer (eventually of zero thickness)
microscopic equations
wall (or porous or poro-elastic matrix)
macroscopic equations


y
S^N
$\bar{y}$
S^T
x
$y = y_{wall}$

unit cell

Microscopic equations:

$$u_x + v_y = 0,$$

$$\mathcal{R}(u_t + \mathbf{u} \cdot \nabla u) = -p_x + \nabla^2 u + f^x,$$

$$\mathcal{R}(v_t + \mathbf{u} \cdot \nabla v) = -p_y + \nabla^2 v + f^y,$$

$$\mathcal{R} = \rho \mathcal{U} l / \mu$$

Macroscopic equations:

$$U_X + V_Y = 0,$$

$$U_T + \mathbf{U} \cdot \nabla' U = -P_X + Re^{-1}(U_{XX} + U_{YY}),$$

$$V_T + \mathbf{U} \cdot \nabla' V = -P_Y + Re^{-1}(V_{XX} + V_{YY}),$$

$$Re = \rho U_{out} L / \mu$$

$$\mathcal{U} = \epsilon U_{out} \longrightarrow \mathcal{R} = \epsilon^2 Re \qquad \epsilon = l/L$$

Power series expansion for microscopic variables:

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots,$$

$$\nabla \rightarrow \nabla + \epsilon \nabla'$$

Microscopic system at leading order:

Homogeneous Stokes equations, plus periodicity in x, plus no-slip at the wall, plus forcing from the outer flow

$$u_y^{(0)} + v_x^{(0)} = \mathbb{S}^T, \quad -p^{(0)} + 2\, v_y^{(0)} = \mathbb{S}^N$$

at the edge of the unit cell ($y = y_\infty$).

Microscopic system at next order:

$$
\begin{aligned}
u_x^{(1)} + v_y^{(1)} &= -u_X^{(0)} - v_Y^{(0)}, \\
-p_x^{(1)} + u_{xx}^{(1)} + u_{yy}^{(1)} &= p_X^{(0)} - 2u_{Xx}^{(0)} - 2u_{Yy}^{(0)}, \\
-p_y^{(1)} + v_{xx}^{(1)} + v_{yy}^{(1)} &= p_Y^{(0)} - 2v_{Xx}^{(0)} - 2v_{Yy}^{(0)}.
\end{aligned}
$$

plus boundary conditions.

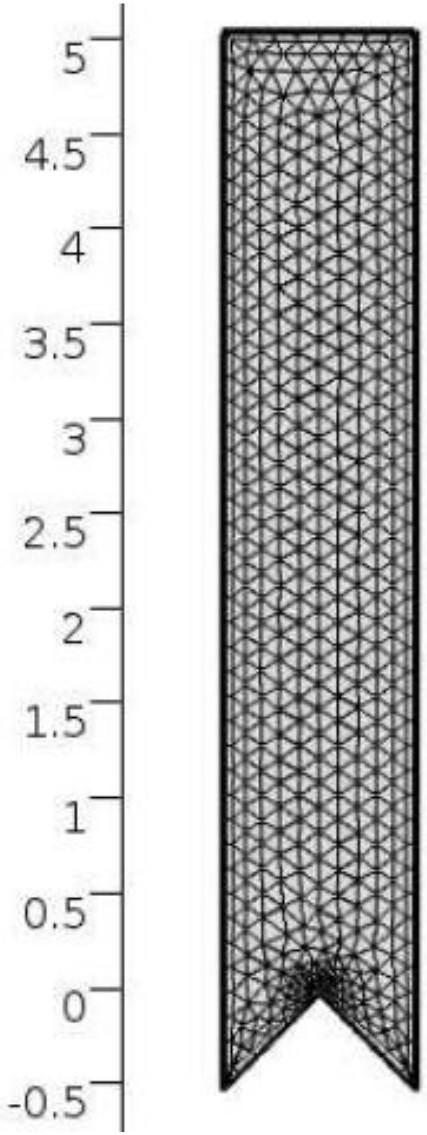
Linearity permits to express the order zero solution as

$$u^{(0)} = u^{\dagger}(x,y)\,\mathbb{S}^{T},$$
$$v^{(0)} = v^{\dagger}(x,y)\,\mathbb{S}^{T},$$
$$p^{(0)} = p^{\dagger}(x,y)\,\mathbb{S}^{T} - \mathbb{S}^{N}.$$

so that a Stokes system for the ‘dagger’ variables ensues with

$$u^{\dagger}_{y} + v^{\dagger}_{x} = 1, \quad -p^{\dagger} + 2\,v^{\dagger}_{y} = 0$$

at y_{∞}



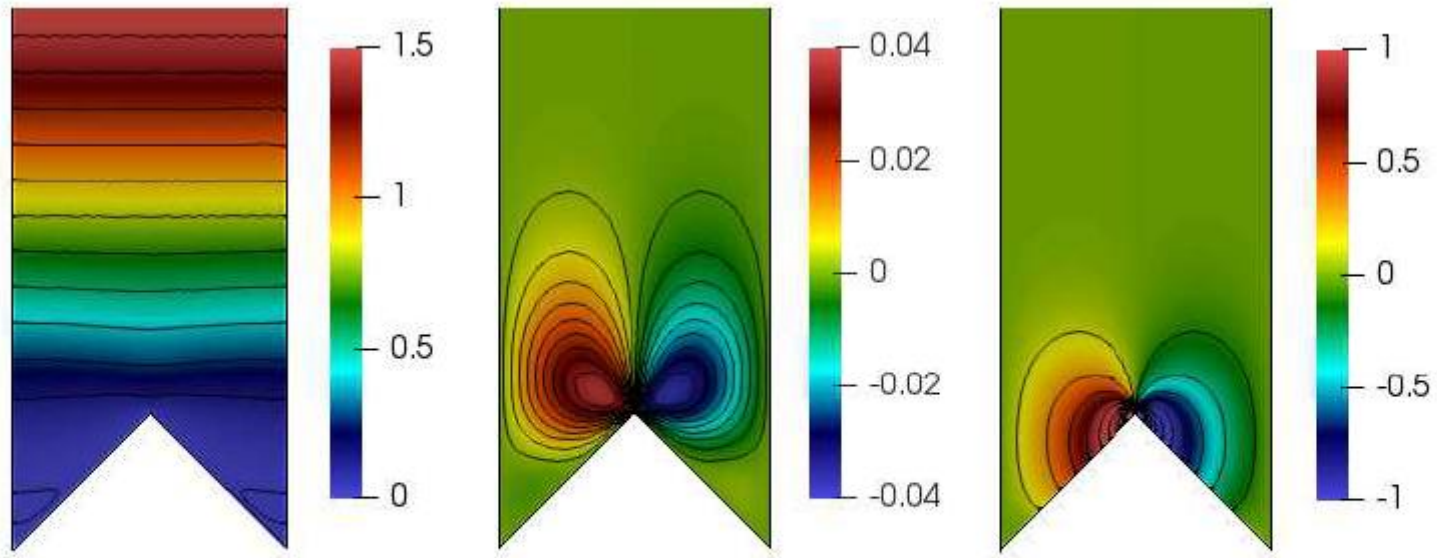


FIG. 5. Isolines of $u^\dagger$ (left), $v^\dagger$ (center) and $p^\dagger$.

- At order one the system becomes

$$u_x^{(1)} + v_y^{(1)} = -u^\dagger \, \mathbb{S}_X^T,$$
$$-p_x^{(1)} + u_{xx}^{(1)} + u_{yy}^{(1)} = p^\dagger \, \mathbb{S}_X^T - \mathbb{S}_X^N - 2u_x^\dagger \, \mathbb{S}_X^T,$$
$$-p_y^{(1)} + v_{xx}^{(1)} + v_{yy}^{(1)} = -2v_x^\dagger \, \mathbb{S}_X^T,$$

and the generic solution reads:

$$f^{(1)} = \hat{f}(x,y) \, \mathbb{S}_X^T + \breve{f}(x,y) \, \mathbb{S}_X^N$$

Forcing by X-gradient of shear stress

$$\hat{u}_x + \hat{v}_y = -u^\dagger,$$

$$-\hat{p}_x + \hat{u}_{xx} + \hat{u}_{yy} = p^\dagger - 2u^\dagger_x,$$

$$-\hat{p}_y + \hat{v}_{xx} + \hat{v}_{yy} = -2v^\dagger_x.$$

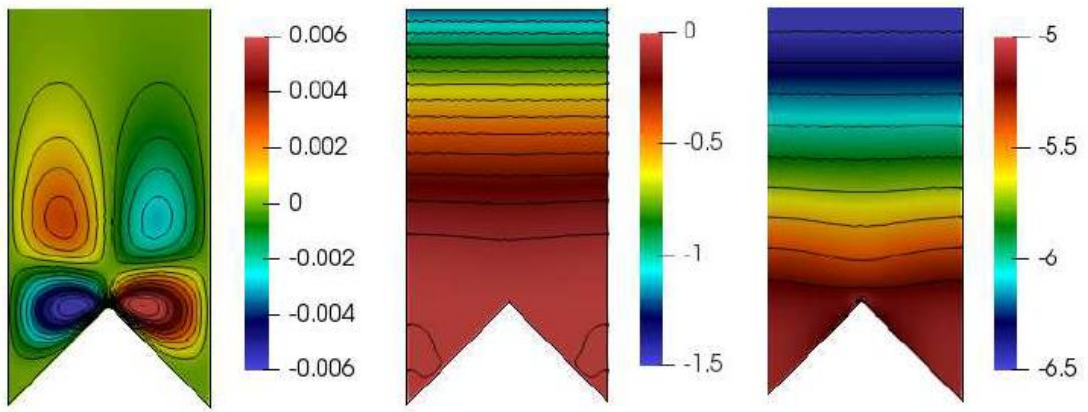


FIG. 6. Isolines of $\hat{u}$ (left), $\hat{v}$ (center) and $\hat{p}$.

Forcing by X-gradient of normal stress

$$\breve{u}_x + \breve{v}_y = 0,$$

$$-\breve{p}_x + \breve{u}_{xx} + \breve{u}_{yy} = -1,$$

$$-\breve{p}_y + \breve{v}_{xx} + \breve{v}_{yy} = 0,$$

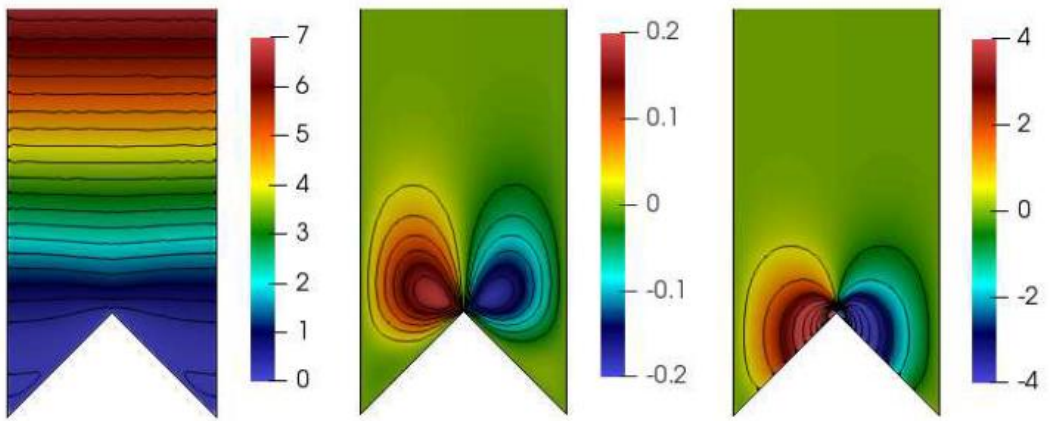


FIG. 7. Isolines of $\breve{u}$ (left), $\breve{v}$ (center) and $\breve{p}$.

The **macro**scopic effective conditions at $Y_\infty = \epsilon\, y_\infty$ are:

$$U(X, \epsilon y_\infty, t) = \epsilon \left[\int_0^1 \left(u^0 + \epsilon u^{(1)} \right) \Big|_{y=y_\infty} \mathrm{d}x \right] + \mathcal{O}(\epsilon^3),$$

$$V(X, \epsilon y_\infty, t) = \epsilon \left[\int_0^1 \left(v^{(0)} + \epsilon v^{(1)} \right) \Big|_{y=y_\infty} \mathrm{d}x \right] + \mathcal{O}(\epsilon^3).$$

i.e.

$$U(X, \epsilon y_\infty, t) \approx \epsilon\, \gamma_x\, \mathbb{S}^T + \epsilon^2\, n_{12}\, \mathbb{S}^N_X,$$
$$V(X, \epsilon y_\infty, t) \approx \epsilon^2\, n_{21}\, \mathbb{S}^T_X.$$

and, for $y_\infty = 5$, we find

$$\gamma_x := \int_0^1 u^\dagger(x, y_\infty)\, \mathrm{d}x = 5.07778$$

$$n_{12} := \int_0^1 \breve{u}(x, y_\infty)\, \mathrm{d}x = 12.89469$$

$$n_{21} := \int_0^1 \hat{v}(x, y_\infty)\, \mathrm{d}x = -12.89469$$

y_∞	4	5	6	7	8	9
γ_x	4.07778	5.07778	6.07778	7.07778	8.07778	9.07778
$n_{12} = -n_{21}$	8.31693	12.89469	18.47249	25.05028	32.62806	41.20585

TABLE I. Variation of coefficients with the position of the inner domain's edge.

$$n_{12} = -n_{21} = \frac{y_\infty^2}{2} + \lambda_x\, y_\infty + m_{12},$$

$$\gamma_x = \frac{\mathrm{d}n_{12}}{\mathrm{d}y_\infty} = y_\infty + \lambda_x,$$

with $\lambda_x = 0.07778$ and $m_{12} = -m_{21} = 0.00581$.

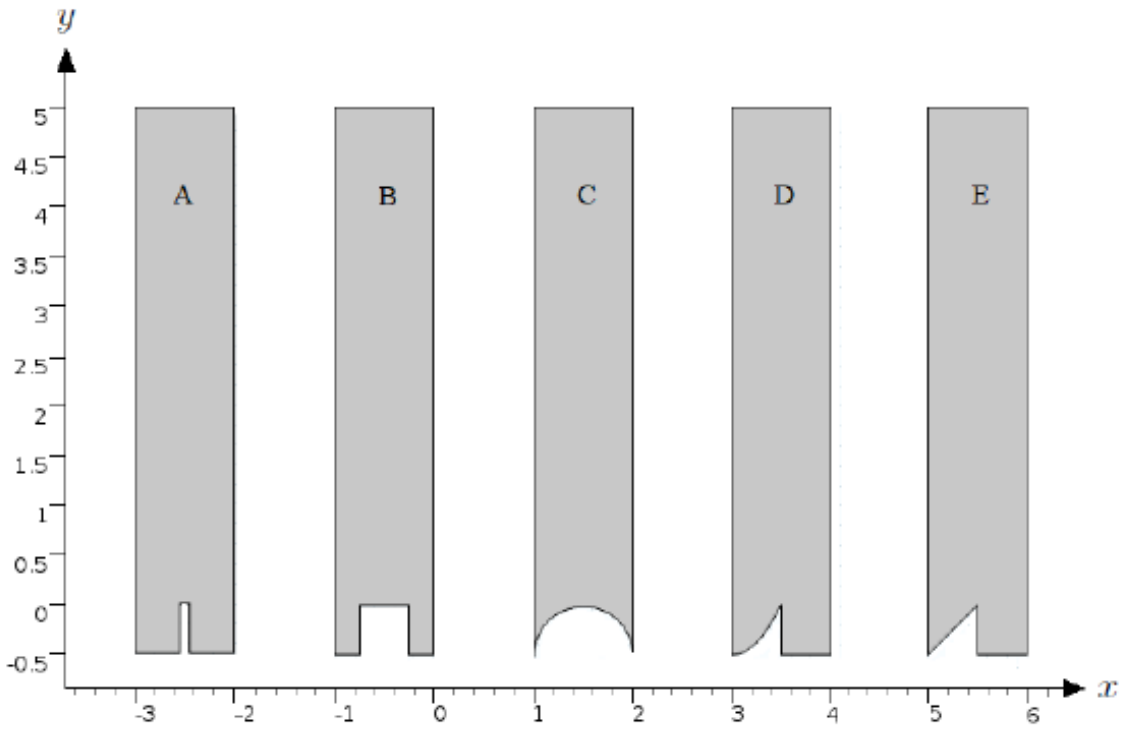
It is thus convenient to transfer the conditions to $Y = 0$, rim of the roughness elements, i.e.

$$U(X,0,t) = \epsilon\,\lambda_x\,S^T + \epsilon^2\,m_{12}\,S^N_X + \mathcal{O}(\epsilon^3)$$

$$V(X,0,t) = \epsilon^2\,m_{21}\,S^T_X + \mathcal{O}(\epsilon^3)$$

with $S^T = U_Y + V_X, \quad S^N = -ReP + 2V_Y,$ evaluated at $Y = 0$

Other roughness shapes:

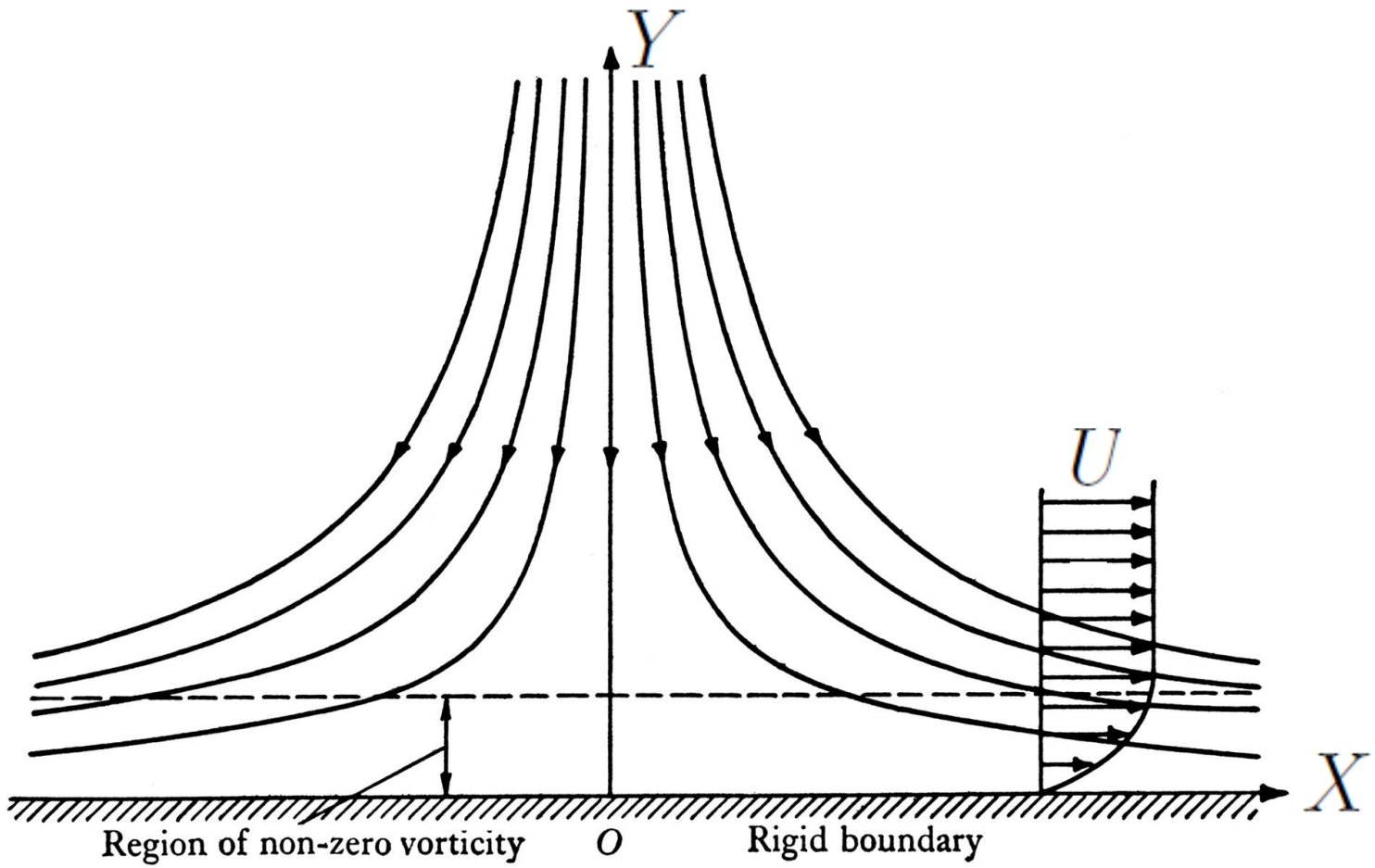


	A	B	C	D	E
λ_x	0.06293	0.01781	0.04087	0.08119	0.07992
$m_{12} = -m_{21}$	0.00324	0.00042	0.00183	0.00546	0.00550

TABLE II. Variation of slip and transpiration parameters for different roughness geometries.

Macroscopic test

Macroscopic test: the Hiemenz stagnation point flow



Rough Hiemenz flow: a *similarity solution* exists also when roughness is present

$$U = X\, f'(Y), \quad V = -f(Y), \quad P = P_0 - \frac{1}{2}\left[X^2 + g(Y)\right]$$

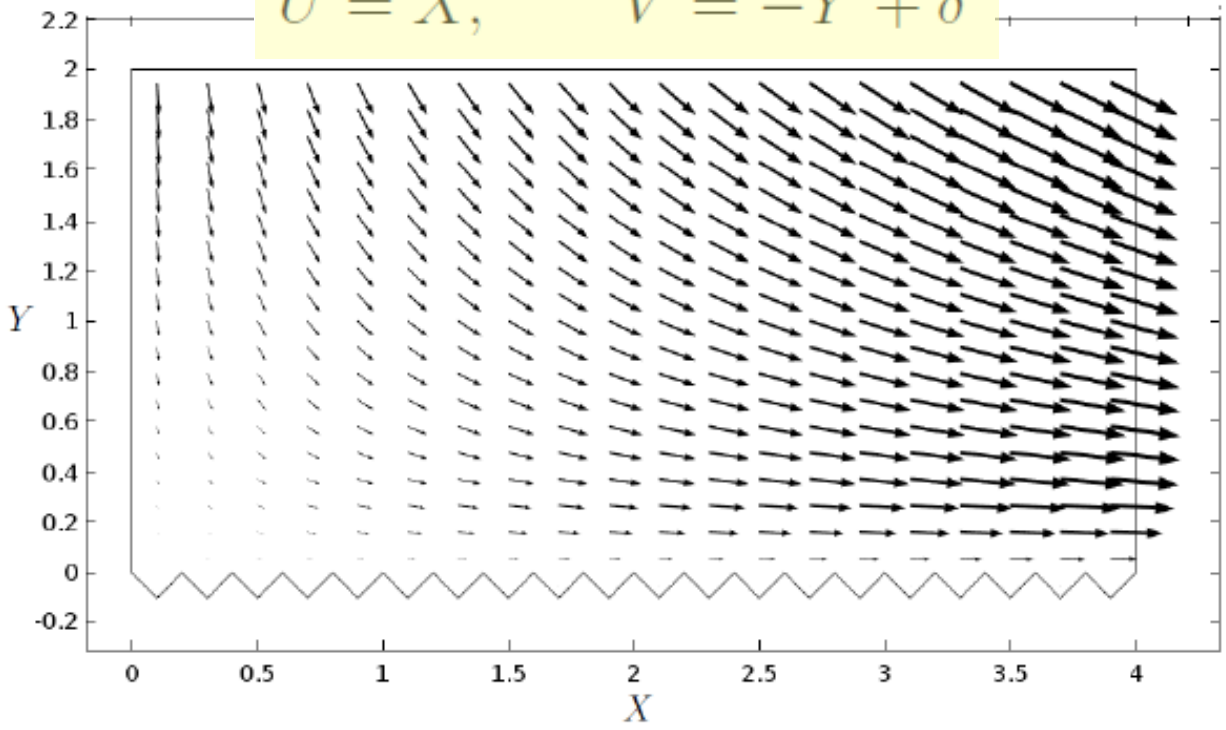
$$\frac{1}{Re} f''' + f\, f'' - f'^2 + 1 = 0,$$

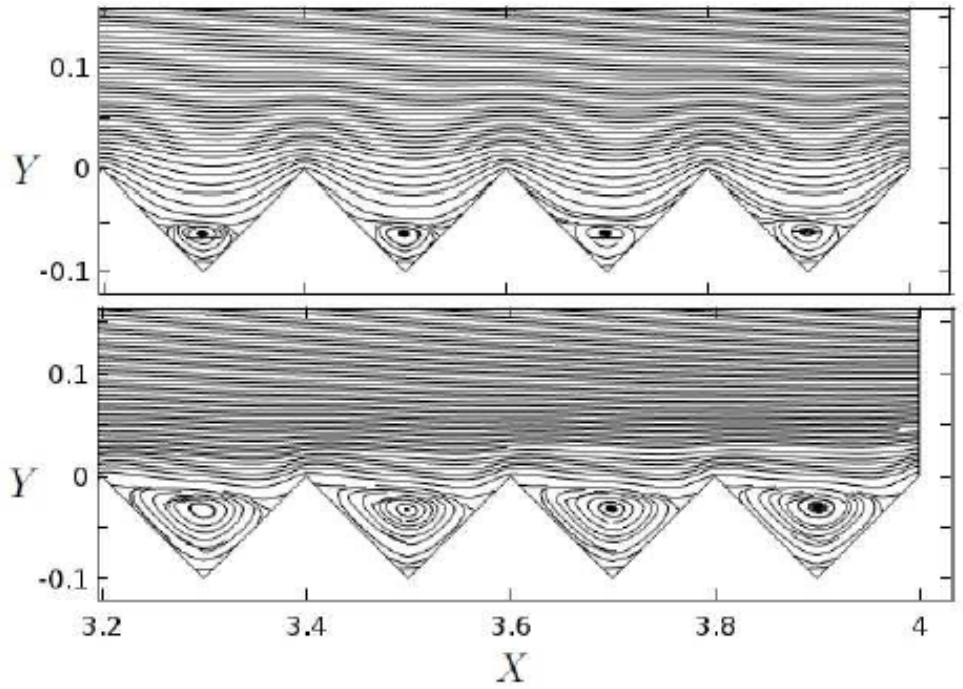
$$\epsilon\, \lambda_x\, f'' - f' + \epsilon^2\, m_{12}\, Re = 0, \quad \epsilon^2\, m_{21}\, f'' + f = 0 \quad \text{at} \quad Y = 0,$$

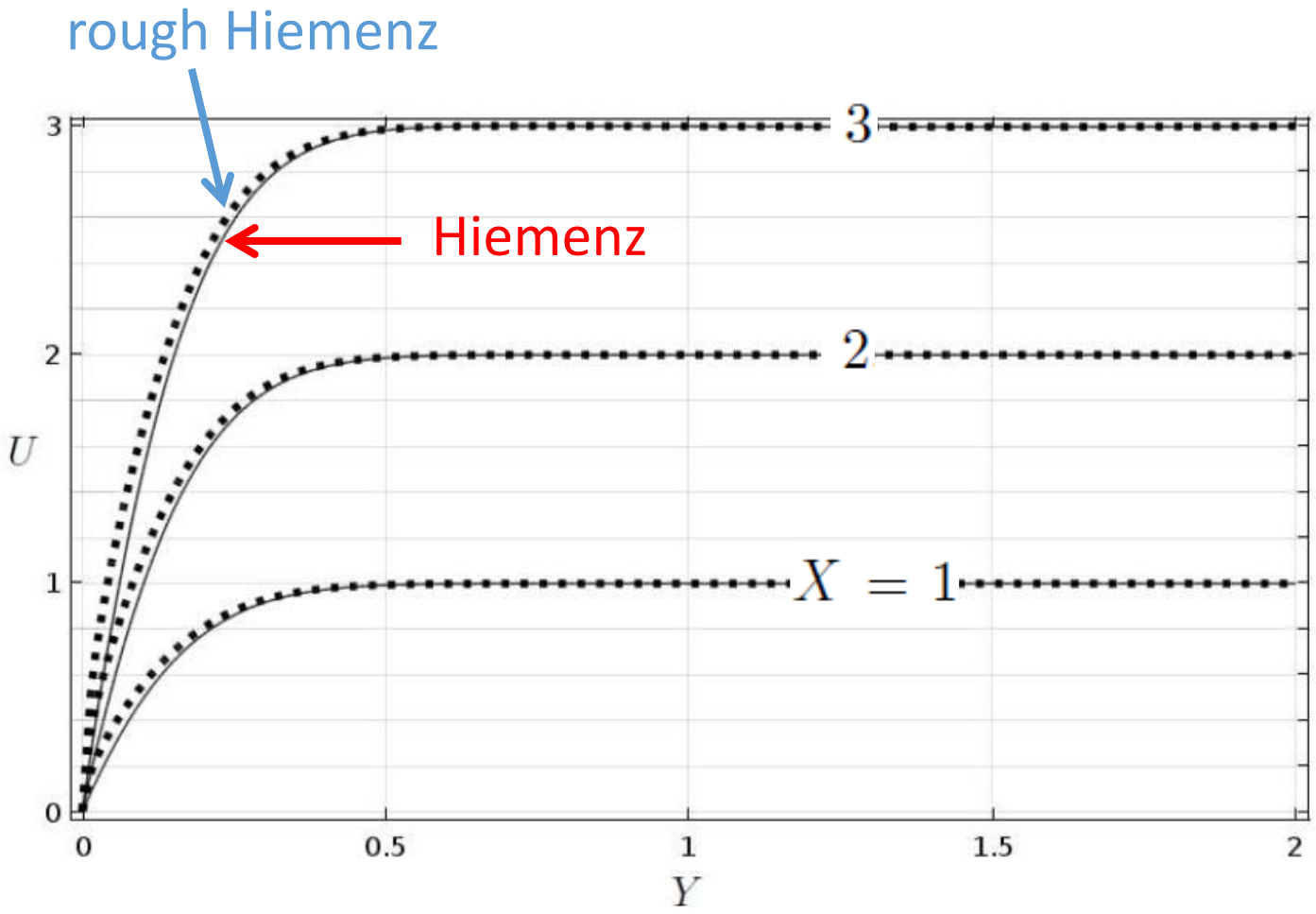
$$f' = 1 \quad \text{at} \quad Y \to \infty,$$

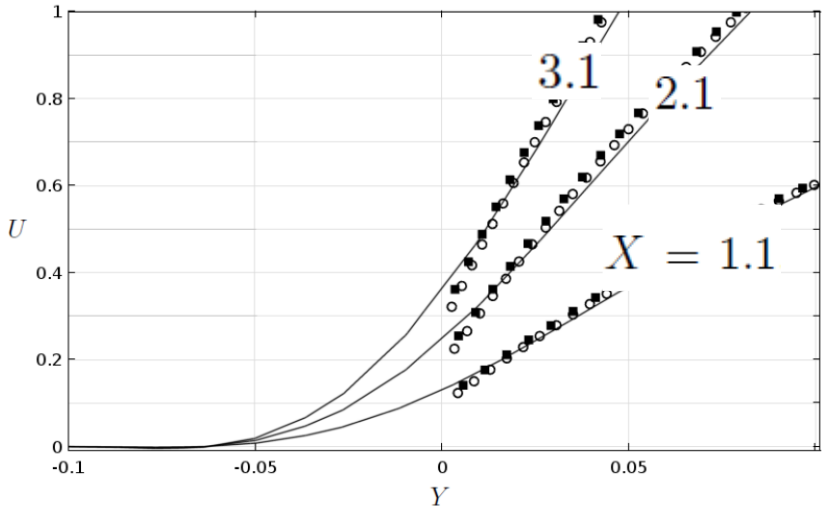
FreeFEM numerical resolution: $Re = 25, \epsilon = 0.2, \mathcal{R} = \epsilon^2\, Re = 1$

$$U = X, \qquad V = -Y + \delta^*$$

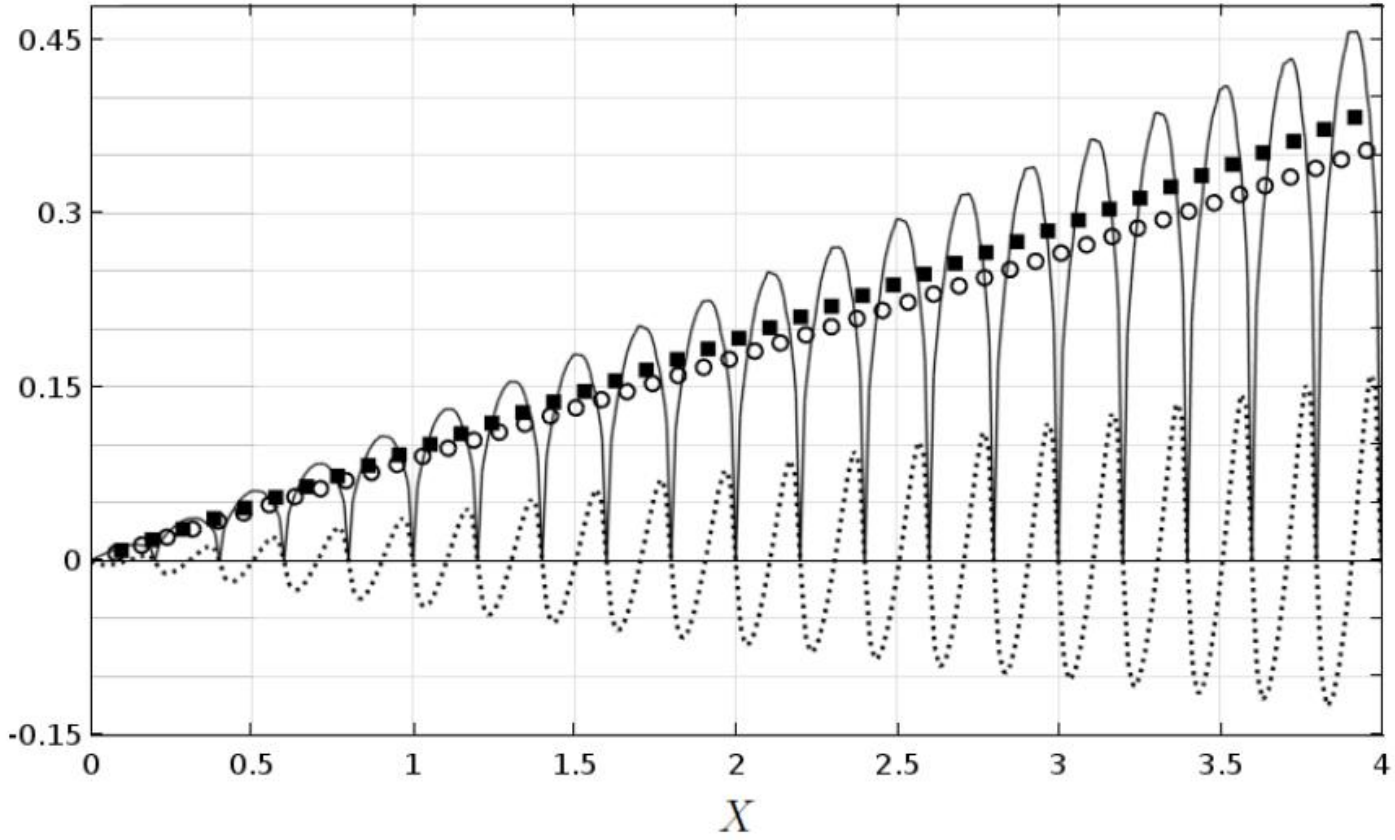





rough Hiemenz
Hiemenz
3
2
$X = 1$
U
Y
0
0.5
1
1.5
2
0
1
2
3

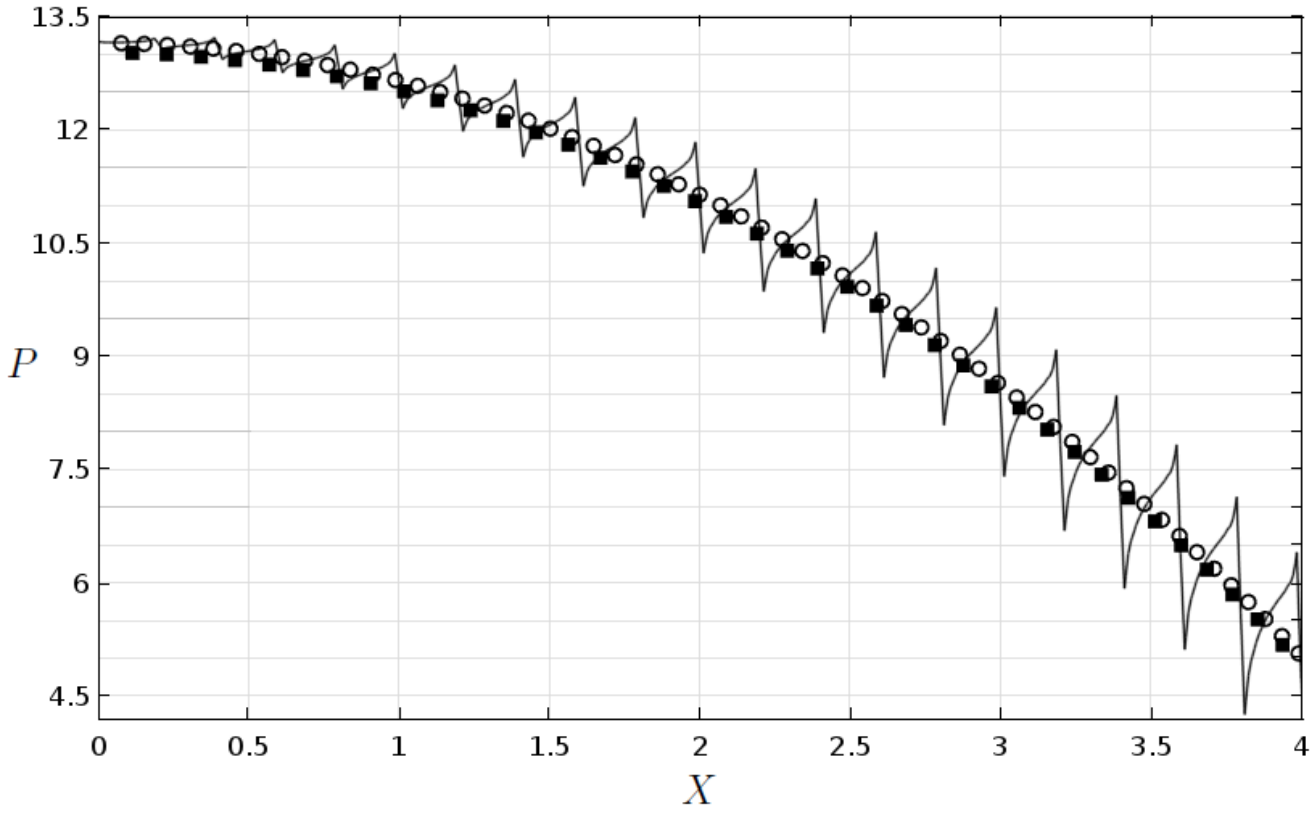

3.1
2.1
$X = 1.1$
U
Y
0
0.2
0.4
0.6
0.8
1
-0.1
-0.05
0
0.05

slip and transpiration velocity components



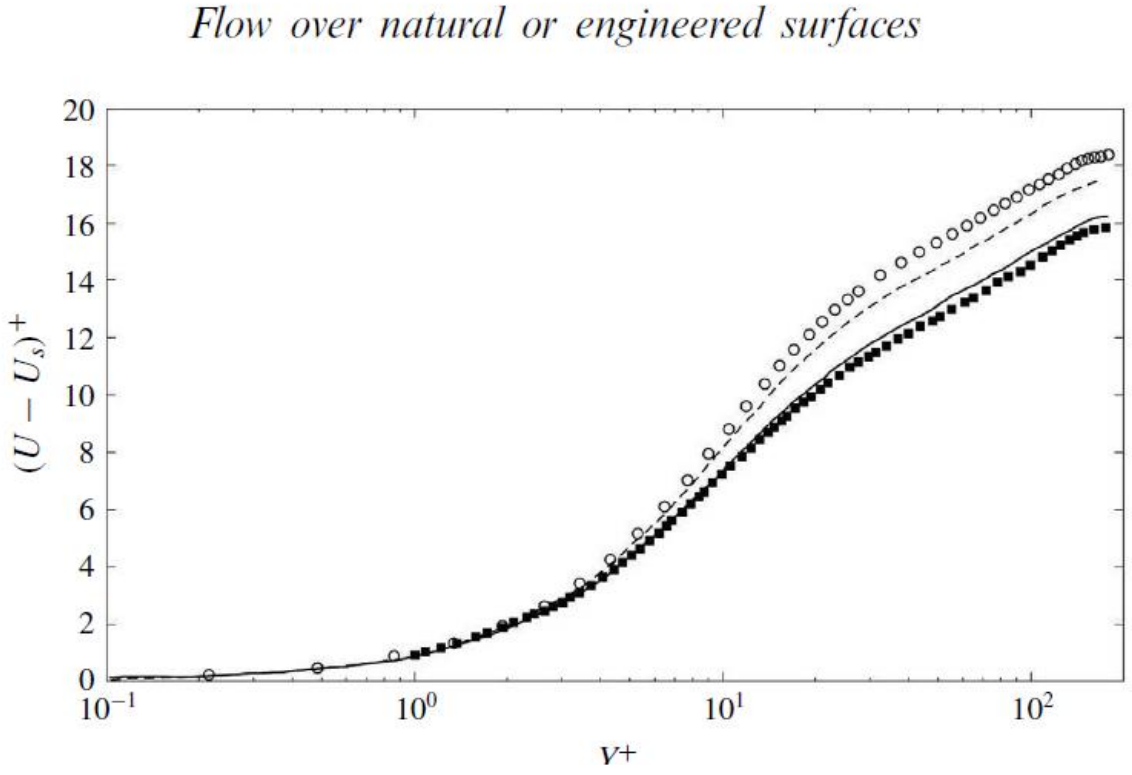
(appreciable difference between the solutions at order ε and ε^2)

pressure at $Y = 0$

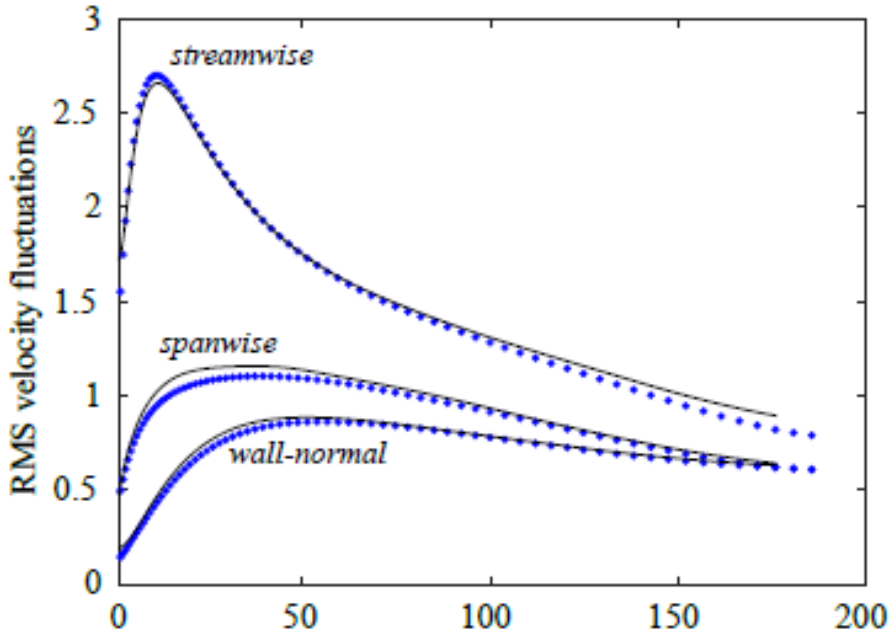


Larger differences between the order 1 and 2 conditions are expected for the case of 3D, laminar or turbulent flow cases

In a three-dimensional turbulent channel flow with a rough wall the transpiration condition seems to work well (cuboid roughness, $\varepsilon = 0.2$) …

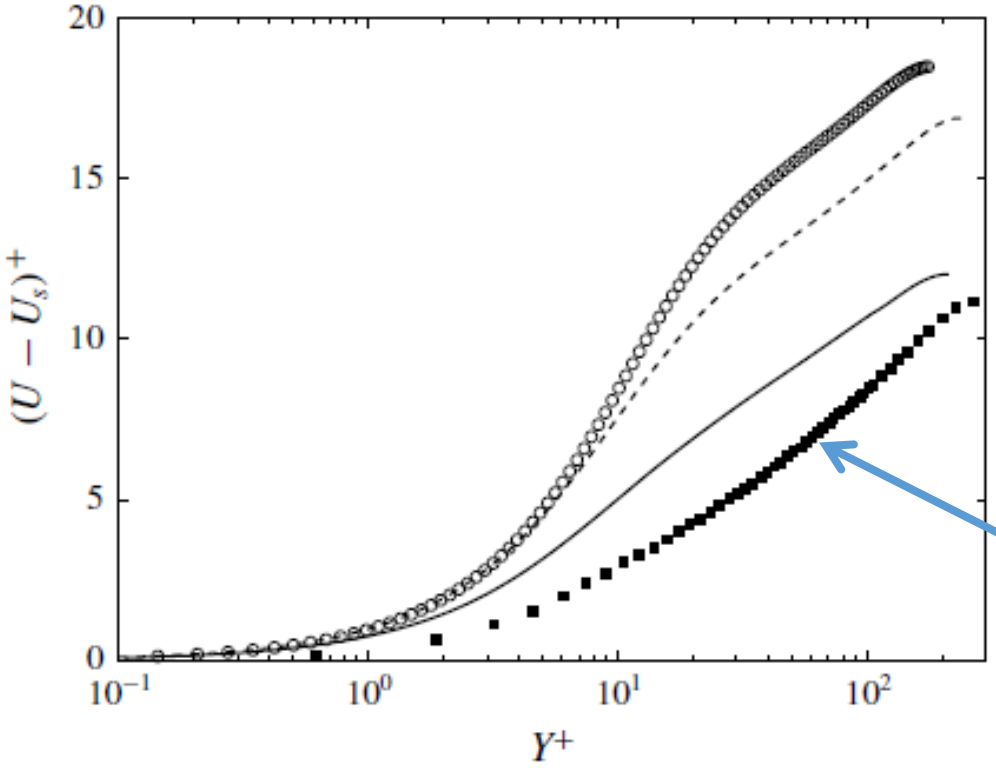


Bottaro, *J. Fluid Mech*. 2019



Lacis et al., *J. Fluid Mech*. in press

… although this is not always true (cubic roughness elements, $\varepsilon = 0.4$) …



feature-resolving simulations by Orlandi & Leonardi, *J. Turbulence* 2006

BUT …

BUT …

in the models used above (both Bottaro and Lacis et al.) the conditions used at the fictitious wall in $Y = 0$ are first order for U (i.e. simple Navier slip) and second order for V.

Need to do: DNS with second order conditions throughout

Summary

1. Developed second order conditions to be employed at a fictitious wall to model a micro-structured wall
2. Very simple to extract coefficients of the second order conditions by solving as few as two Stokes-like equations in unit cells of different heights and extrapolating results to $y_\infty = 0$
3. Conditions is formally correct for $\varepsilon \to 0$, but results seem to be acceptable for rather large values of ε
4. Trivial extension to 3D roughness

Future developments (SHS/LIS):

1. include a lubricant fluid within the roughness (VOF in unit cell)
2. derive slip/transpiration conditions to second order in this two-phase flow case (1st order problem already solved using BEM: Alinovi & Bottaro, *Phys. Rev. Fluids*, 2018)
3. perform DNS with slip/transpiration for turbulent channel flow over a LIS and compare to feature-resolving DNS
4. optimise morphology of rough surface impregnated with lubricant and optimise properties of lubricant fluid