Stability of the flow in a plane microchannel with one or two superhydrophobic walls

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Abstract

The modal and nonmodal linear stability of the flow in a microchannel with either one or both walls coated with a superhydrophobic material is studied. The topography of the bounding wall(s) has the shape of elongated micro-ridges with arbitrary alignment with respect to the direction of the mean pressure gradient. The superhydrophobic walls are modelled using the Navier slip condition through a slip-tensor, and the results depend parametrically on the slip-length and orientation angle of the ridges. The stability analysis is carried out in the temporal framework; the modal analysis is performed by solving a generalized eigenvalue problem, and the nonmodal, optimal perturbation analysis is done with an adjoint optimisation approach. We show theoretically and verify numerically that Squire's theorem does not apply in the present settings, despite the fact that Squire modes are found to be always damped. The most notable result is the appearance of a streamwise wall-vortex mode at very low Reynolds numbers when the ridges are sufficiently inclined with respect to the mean pressure gradient, in the case of a single superhydrophobic wall. When two walls are rendered water repellent, the exponential growth of the instability results from either a two-dimensional or a three-dimensional Orr-Sommerfeld mode, depending on the ridges orientation and amplitude. Nonmodal results for either one or two superhydrophobic wall(s) display but a mild modification with respect to the no-slip case.
I. INTRODUCTION

Superhydrophobic coatings represent an interesting technique for the possible reduction of drag in applications involving the flow of liquids over solid surfaces, for a wide range of Reynolds number, from laminar to turbulent conditions [1]. Such coatings work by the interposition of a gas layer between the liquid and the solid wall, trapped by distributed microscopic roughness elements present at the wall; over the gas layer the liquid can flow with negligible friction. Here, we are concerned with the initial development stages of the laminar-turbulent transition for the flow of a liquid in a micro-channel, with one or both walls characterized by a periodic, micro-patterned topography.

Min and Kim [2] addressed this same hydrodynamic stability problem, considering isotropic, superhydrophobic channel walls (characterized, in an averaged sense, by the same scalar slip length, $\lambda$), and studied the case of both exponentially growing two-dimensional modes and three-dimensional pseudo-modes excited algebraically over short time intervals. Whereas two dimensional Tollmien-Schlichting (TS) waves were stabilized by the use of a non-zero slip length, the effect of slip on the transient amplification of streak-like perturbations was found to be minor; Min and Kim performed also a few direct numerical simulations of transition to turbulence initiated by two-dimensional TS waves in different configurations (in the presence of only streamwise slip, only spanwise slip, or slip along both horizontal directions) finding that in some cases transition was advanced (with respect to the no-slip situation) and in others it was retarded. From the results it appears that it is the presence of spanwise slip (which is, in all practical cases, unavoidable when superhydrophobic surfaces are used) to favour the early triggering of transition.

Also Lauga and Cossu [3] considered isotropic, superhydrophobic surfaces using a scalar slip length to model the wall. Their modal stability results demonstrated a strong stabilizing effect for two-dimensional TS waves (particularly when both channel walls display slip), whereas only a minor influence was found on the maximum transient energy growth of streamwise streaks.

Both of the studies cited above considered, appropriately, only a two-dimensional modal analysis on account of Squire’s theorem; it will be demonstrated here that – under certain conditions not previously examined – three-dimensional modes can initiate the transition process when slip at the wall is present.
Very recently, Yu et al. [4] have re-considered the temporal, modal stability problem for the flow in a channel with longitudinal superhydrophobic grooves on one or both walls, without employing the (averaging) concept of a slip length. They resolved the two-dimensional problem for the base flow in the plane orthogonal to the mean flow direction, and the two-dimensional problem for the disturbance field, assuming the interface flat and pinned at the corners of the ribs. When both the spanwise periodicity of the grooves and the shear-free fraction are sufficiently small, compared to the channel thickness, the results of Yu et al.'s analysis reproduce those obtained by employing a slip length. As the periodicity and the shear-free fraction are increased, a new wall mode is found, apparently related to the presence of inflection points in the mean, streamwise velocity profile; it is such a new mode which can lead the flow to an early instability. A similar conclusion as to the effect of the periodicity of the grooves has been reached by Luchini [5] who conducted direct numerical simulations of turbulence in a channel with two superhydrophobic walls, by comparing slip-length boundary condition cases with simulations carried out on walls with alternating no-slip and no-shear conditions (with the shear-free interface of the same length as the no-slip portion above the ribs). The conclusion by Luchini is that the concept of a slip length can be employed as long as the periodicity $s^+$ of the longitudinal micro-ridges remains below about 30 (measured in wall units). To set ideas, let us see what the limit $s^+ = 30$ means for the practical case of a 20 m long, planar object moving in water at 20 knots ($U \approx 10 \text{ m/s}$). The momentum thickness $\theta$ varies from zero (at the leading edge of the body, where the boundary layer has not yet formed) to about 1.86 cm (near the trailing edge). The corresponding $Re_\theta$ thus reaches a value equal to about $1.86 \times 10^5$ at the end of the plate (where $Re_x = 2 \times 10^5$) which translates into a friction coefficient $c_f = 2\tau_{wall}/(\rho U^2)$ of approximately $1.5 \times 10^{-3}$. At the trailing edge of the plate ($x = 20 \text{ m}$) the wall shear is about $\tau_{wall} = 75 \text{ Pa}$ so that the friction velocity is $u_r = 0.274 \text{ ms}^{-1}$ and $s^+ = 30$ corresponds to a dimensional periodicity $s$ of about 100 $\mu m$. This latter value is close to being an upper bound of the characteristic length scales of superhydrophobic surfaces realized in practice.

Note, incidentally, that $s^+ = 15 \div 20$ provides optimal drag reducing properties for the case of longitudinal riblets [6–8], while for $s^+$ exceeding 30 skin friction drag is generally increased with respect to the base value. It is advantageous, for superhydrophobic drag reduction purposes, to have the periodicity $s$ of the micro-ridges as large as possible, with a large gas fraction exposed to water. The value $s^+ = 30$ yields dimensional values of
periodicity of the order of 100 µm (cf. arguments given above) which might be used in practice to prevent the escape, and limit the diffusion into water, of the gas bubbles. The limit imposed on $s^+$ justifies the use of simplified simulations with homogeneous Navier slip conditions. Furthermore, it is argued that longitudinal micro-ridges or micro-grooves of periodicity $s^+ = 20$ can be used advantageously also should the wetting transition [9] take place, by exploiting the "riblet effect".

The surface topography is rendered in the present work by a Navier slip condition [10, 11], which represents, in an homogenized sense, the alternation of no-slip and no-shear elongated regions which are found when micro-ridges cover the walls, under the assumption that the gas in the cavities exerts no shear stress on the liquid above it. For a comparison between different surface structures the reader is referred to Bottaro [1]; see in particular Fig. 10 there, showing that the effective slip length is of the order of the pattern periodicity, and decreasing with the solid area fraction. In the present paper linear stability results are obtained for both modal and nonmodal amplification of disturbances, for micro-ridges aligned, orthogonal, or at an angle, to the driving pressure gradient. In view of the recent paper by Yu et al. [4] it is clear that the results presented here apply only for $s$ sufficiently small.

II. PROBLEM FORMULATION

The effect of superhydrophobic (SH) surfaces on the instability onset, and consequently the initial stages of laminar-turbulent transition, is addressed in the framework of plane microchannels where the Reynolds number is typically small. We assume that the channel has thickness $2h^*$ and use $h^*$ to normalize distances, and the bulk speed $\bar{U}^*$ is employed to scale the velocity. Superscript $\star$ denotes dimensional quantities. The SH riblet-like wall considered here forms an anisotropic texture (Fig. 1) for which a slip tensor $\Lambda$ in the plane of the walls ($x, z$) can be defined [12–14] as

$$\Lambda = Q \begin{bmatrix} \lambda^\parallel & 0 \\ 0 & \lambda^\perp \end{bmatrix} Q^T,$$

with

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix},$$

where $\lambda^\parallel$ and $\lambda^\perp$ are the eigenvalues of the slip tensor $\Lambda$ for $\theta = 0^\circ$ and $90^\circ$, and the transformation (1) represents a rotation of the tensor by an angle $\theta$. For $\theta = 0^\circ$ the ridges
FIG. 1. (Color online) Sketch of the wall pattern with definition of axes, angle $\theta$ and ridges periodicity $s$. The gas-liquid interface is represented as a curved surface in light blue color for illustrative purposes; the way in which the Navier slip lengths are modified by the curvature of the interface has been addressed by Teo and Khoo [15]

are aligned with $x$, and for $\theta = 90^\circ$ they are aligned with $z$. In the special case of isotropic SH it is $\lambda^\parallel = \lambda^\perp$; for the case of microridges aligned along the mean pressure gradient [10, 11, 16] we have $\lambda^\parallel = 2\lambda^\perp$. This latter result will be used from now on, and the results will be expressed as a function only of $\lambda^\parallel$.

By denoting with $u, v$ and $w$ the streamwise, wall-normal and spanwise velocity components, respectively, the dimensionless boundary conditions for the horizontal velocity components at the two walls in $y = \pm 1$ read

\[
\begin{bmatrix}
  u(x, -1, z) \\
  w(x, -1, z)
\end{bmatrix}
= \Lambda \frac{\partial}{\partial y}
\begin{bmatrix}
  u(x, -1, z) \\
  w(x, -1, z)
\end{bmatrix},
\]

\[
\begin{bmatrix}
  u(x, 1, z) \\
  w(x, 1, z)
\end{bmatrix}
= -\Lambda \frac{\partial}{\partial y}
\begin{bmatrix}
  u(x, 1, z) \\
  w(x, 1, z)
\end{bmatrix},
\]

in the case of both walls being textured, plus vanishing conditions for the vertical velocity component $v$ at the two walls. If one of the two walls is not superhydrophobic, the condition there is simply $u = 0$. 

5
A. Base flow and linear stability equations

The velocity and pressure are decomposed into a steady base flow and an unsteady disturbance according to

\[ u(x, y, z, t) = U(x, y, z) + \epsilon u'(x, y, z, t), \quad p(x, y, z, t) = P(x, y, z) + \epsilon p'(x, y, z, t), \quad (4) \]

with \( \epsilon \ll 1 \). The governing equations for plane, incompressible and steady channel flow read

\[ \frac{dP}{dx} = \frac{1}{Re} \frac{d^2U}{dy^2}, \quad V = 0, \quad \frac{d^2W}{dy^2} = 0, \quad (5) \]

where the Reynolds number is defined as \( Re = \bar{U}^* h^*/\nu^* \). When the boundary conditions (2–3) are used, the analytical solution of the base flow, in the case of two superhydrophobic walls, reads

\[ U(y) = \frac{-3 y^2 - 1 - \lambda^\parallel (1 + \cos^2 \theta)}{2 + 3\lambda^\parallel (1 + \cos^2 \theta)}, \quad W(y) = \frac{3 \lambda^\parallel \sin \theta \cos \theta}{2 + 3\lambda^\parallel (1 + \cos^2 \theta)}. \quad (6) \]

When \( \theta \) differs from \( 0^\circ \) and \( 90^\circ \), a small component of the base flow orthogonal to the mean pressure gradient is created in the channel [17]. In the case in which only the bottom wall is superhydrophobic the basic flow is:

\[ U(y) = \frac{-3}{4} \frac{(y^2 - 1)(8 + 6\lambda^\parallel + \lambda^\parallel^2) + 2\lambda^\parallel(y - 1)(2 + 2 \cos^2 \theta + \lambda^\parallel)}{6\lambda^\parallel + 3\lambda^\parallel \cos^2 \theta + 4 + 2\lambda^\parallel^2}, \quad (7) \]

\[ W(y) = \frac{-3 \lambda^\parallel \sin \theta \cos \theta(y - 1)(4 - \lambda^\parallel \cos^2 \theta + 2\lambda^\parallel)}{[4 + \lambda^\parallel(1 + \sin^2 \theta)](6\lambda^\parallel + 3\lambda^\parallel \cos^2 \theta + 4 + 2\lambda^\parallel^2)}, \quad (8) \]

and this flow presents a streamwise component of the vorticity which is a maximized by \( \theta = \pm 45^\circ \) when \( \lambda^\parallel \) is smaller than about 0.1 (above this value of \( \lambda^\parallel \) the absolute value of the inclination angle of the grooves which displays the largest vorticity increases mildly). Examples of the base flow for \( \lambda^\parallel = 0.155 \), in the case of one and two superhydrophobic walls, are displayed in Fig. 2 for two values of \( \theta \).

The linear stability equations are obtained by introducing (4) into the Navier-Stokes equations and collecting terms of order \( \epsilon \). In primitive variable form they read:

\[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0, \quad (9) \]

\[ \frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v \frac{dU}{dy} + W \frac{\partial u'}{\partial z} = -\frac{\partial p'}{\partial x} + \frac{1}{Re} \nabla^2 u', \quad (10) \]
FIG. 2. Streamwise $U$ and spanwise $W$ velocity components of the base flow when $\lambda^\parallel = 0.155$ for the cases $\theta = 0^\circ$ (dashed) and $\theta = 45^\circ$ (solid). Left: one superhydrophobic wall. Right: two superhydrophobic walls. The symbols show the experimental micro-PIV data of Ou & Rothstein [18] for the case $\theta = 0^\circ$; the filled circles show measurements above the ribs, whereas the empty symbols are taken above the gas-water interface.

The disturbance field (denoted by primes) is expressed in terms of Fourier modes along the wall-parallel directions, i.e.

$$q'(x,y,z,t) = \tilde{q}(y,t) \exp[i(\alpha x + \beta z)] + c.c.,$$

for the generic variable $q'$, where $\alpha$ and $\beta$ are the streamwise and spanwise wavenumbers, respectively, and $c.c.$ denotes complex conjugate. The governing linear equations (9–12) are supplemented by the boundary conditions (2-3) for the disturbance variables in the case of two superhydrophobic walls. In the case of one SH wall, the other wall is given a no-slip condition, $\mathbf{u}' = \mathbf{0}$. The theory developed is applicable as long as the disturbance wavelength $2\pi/k$, with $k = \sqrt{\alpha^2 + \beta^2}$, is sufficiently longer than the spatial periodicity $s$ of the micro-ridges.
Modal analysis

The modal analysis is performed by assuming a temporal behaviour of the form

\[ \tilde{q}(y,t) = \tilde{q}(y) \exp(-i \omega t), \]  

where \( \omega \) is the complex angular frequency and \( \omega_i > 0 \) denotes unstable solutions. Substituting the asymptotic temporal behaviour (14) into the linearized equations (9–12) yields a generalized eigenvalue problem. In discrete form the resulting system of equations can be written as

\[ i\omega B \tilde{q} = A \tilde{q}, \]  

where \( \tilde{q} = (\hat{u}, \hat{v}, \hat{w}, \hat{p}) \); \( A \) and \( B \) are complex-valued \( 4n \times 4n \) matrices and \( n \) is the number of discrete points taken in the \( y \)-direction. In a channel with no-slip walls, Squire’s theorem states that the instability of the coupled system stems from the amplification of a two-dimensional Orr-Sommerfeld mode [19]. This is proven by applying Squire’s transformation \( (ku_2 = \alpha \hat{u} + \beta \hat{w}; k Re_2 = \alpha Re; p_2/\kappa = 1/\alpha; v_2 = \hat{v}; \omega_2/\kappa = \omega/\alpha) \) to the linearized system. The result is that, if a three-dimensional mode is unstable, a two-dimensional mode will be unstable at a lower value of the Reynolds number, \( Re_2 = \alpha Re/\kappa \leq Re \).

Furthermore, it can be shown, always in the no-slip case, that Squire modes (eigensolutions of the unforced Squire’s equation for the vertical vorticity component) are always damped. In the present wall-slip case, however, the statements above do not necessarily apply.

A digression on Squire’s theorem and Squire modes

By decomposing the velocity vector \( \hat{u} = (\hat{u}, \hat{v}, \hat{w}) \) into components parallel and perpendicular to the wavenumber vector, i.e.

\[ u_\parallel = \frac{\alpha \hat{u} + \beta \hat{w}}{k}, \quad u_\perp = \frac{\beta \hat{u} - \alpha \hat{w}}{k}, \]

the governing equations satisfied by \( (u_\parallel, v_2, p_2) \) are independent of \( u_\perp \):

\[ iku_\parallel + \frac{dv_2}{dy} = 0, \]

\[ -i\omega_2 u_\parallel + ikU u_\parallel + \frac{dU}{dy} v_2 = -ikp_2 + \frac{1}{Re_2} \left( \frac{d^2}{dy^2} - k^2 \right) u_\parallel, \]
\[-i\omega_{2D}v_{2D} + ikU v_{2D} = -\frac{dp_{2D}}{dy} + \frac{1}{Re_{2D}}\left(\frac{d^2}{dy^2} - k^2\right)v_{2D}.\]  

(18)

Whereas at first sight this appears to imply that Squire’s theorem is satisfied, it is not the case, since (i) the base flow of this new two-dimensional problem is different from \(U\) (it is \(\overline{U} = U + \frac{k}{\alpha} W\)) and (ii) the boundary conditions for the parallel component of the velocity do not decouple, i.e. at \(y = \pm 1\) the boundary conditions are expressed in terms of both \(u_\parallel\) and \(u_\perp\). The decoupling of the problem into two separate problems (a homogeneous problem for \((u_\parallel, v_{2D}, p_{2D})\) and a second problem for \(u_\perp\), forced by \(v_{2D}\)) is possible only in the case of isotropic SH walls (i.e. \(\lambda_\parallel = \lambda_\perp\) and \(W = 0\)), and it is only in this case that Squire’s theorem holds.

Furthermore, the fact that Squire modes are not necessarily damped can be seen by considering the equation for the velocity component perpendicular to the wavevector, i.e.

\[
\left[-i\omega + i\alpha U + i\beta W - \frac{1}{Re}\left(\frac{d^2}{dy^2} - k^2\right)\right] u_\perp = \left(\beta \frac{dU}{dy} - \alpha \frac{dW}{dy}\right) v_{2D},
\]

(19)

known as Squire’s equation. By multiplying the unforced equation (19) by \(u_\perp^*\), with the * superscript denoting complex conjugate, and integrating in \(y\) across the fluid domain, we find

\[
\omega \int_{-1}^{1} u_\perp^* u_\perp \ dy = \int_{-1}^{1} \left[(\alpha U + \beta W)u_\perp^* u_\perp + \frac{i}{Re} u_\perp^* \left(\frac{d^2}{dy^2} - k^2\right) u_\perp\right] \ dy.
\]

(20)

Integrating by parts once and taking the imaginary part (subscript \(i\) denotes imaginary part, subscript \(r\) denotes real part) we are left with

\[
\omega_i \int_{-1}^{1} |u_\perp|^2 \ dy = -\frac{1}{Re} \int_{-1}^{1} \left(\left|\frac{du_\perp}{dy}\right|^2 + k^2|u_\perp|^2\right) \ dy + \frac{A}{Re},
\]

(21)

where

\[
A = \frac{1}{k^2} \left[\alpha^2 \left(\frac{\hat{w}_r}{dy} - \frac{\hat{u}_i}{dy}\right) - \beta^2 \left(\frac{\hat{u}_i}{dy} - \frac{\hat{w}_r}{dy}\right) + \frac{\alpha \beta}{k^2} \left(\frac{\hat{u}_i}{dy} + \frac{\hat{w}_r}{dy} - \frac{\hat{u}_i}{dy} - \frac{\hat{w}_r}{dy}\right)\right]_{-1}^{1}.
\]

(22)

There is no evident reason why \(A\), which contains boundary terms arising from integration by parts, should be negative (or positive and small, so as not to render positive the right-hand-side of equation (21)); thus, Squire modes (characterized by \(\hat{v} \equiv 0\) throughout \(y\)) can, in principle, be amplified (since \(\omega_i\) in equation (21) is not necessarily negative).
In our experience, however, Squire modes remain damped (cf. Section III), both those in
the so-called \(A\) branch (also known as \textit{wall modes}) and those in the \(P\) branch (\textit{center modes}).
Conversely, recent results by Szumbarski [20] and Mohammadi et al. [21] for the flow in
channels with streamwise-invariant and spanwise-periodic corrugations demonstrate that it
is precisely the least stable Squire mode (in the \(P\) branch) which can become unstable for
a sufficiently large corrugation amplitude. When the amplitude of the corrugation exceeds
a value of \(O(10^{-2})\) an inviscid mechanism – driven by the spanwise gradient of the main
velocity component – forces the destabilisation of the Squire \textit{center} mode. These findings are
related to those by Yu et al. [4], who focussed however on \textit{wall} modes. We re-emphasize here
that the rough walls considered have spatial scales sufficiently small for an homogenisation
procedure – leading to the Navier-slip concept – to be tenable.

\textbf{Nonmodal analysis}

The nonmodal behaviour is studied by computing the maximum finite-time amplification;
the initial disturbance velocity field, \(\tilde{u}_0\), is \textit{optimal} when the gain

\[
G(Re, \alpha, \beta, T, \lambda, \theta) = \frac{e(T)}{e(0)},
\]

is maximized, where

\[
e(t) = \frac{1}{2} \int_{-1}^{1} (\tilde{u}\tilde{u}^* + \tilde{v}\tilde{v}^* + \tilde{w}\tilde{w}^*)dy,
\]

and \(T\) is the target time of the optimization. This is conducted by introducing Lagrange
multipliers enforcing the constraints given by the governing linear equations and the boundary
conditions. The corresponding adjoint equations are derived using a discrete approach
[22]. We further define

\[
G_M(Re, \lambda, \theta) = \max_{\forall \alpha, \beta, T} G,
\]

when \(G\) is maximized with respect to the wavenumbers \((\alpha, \beta)\) and the final time \(T\). The final
time and spanwise wavenumber corresponding to \(G_M\) are denoted \(T_M\) and \(\beta_M\), respectively.

\textbf{Numerical procedure}

The equations are written in primitive variable form and discretized on a staggered grid. The spatial derivatives are treated with second order finite differences and a semi-implicit
second-order scheme is used to advance in time. A uniform grid is adopted along the \( y \)-direction and 300 discrete points are sufficient to obtain converged eigenvalues, with errors with respect to reference solutions lower than 0.1%. The code used to compute the nonmodal growth has been tested on several cases found in the literature (using no-slip boundary conditions); in particular, the value of the optimal gain \( G_M = 2 \times 10^{-4} \text{Re}^2 \) and the corresponding time at which it is achieved, \( T_M = 0.076 \text{Re} \), with \( \alpha = 0 \) and \( \beta = 2.04 \), are recovered within less than 0.1% [19]. Results are obtained imposing that convergence is reached when the relative difference in gain between two consecutive iterations is below \( 10^{-8} \).

III. RESULTS

The instability onset, in the case of modal growth, and the largest temporal amplification, in the case of nonmodal growth, are studied parametrically for both one and two SH channel walls.

Modal analysis

We initiate the discussion of the modal results by showing some representative behaviors for the case of a single superhydrophobic wall. Fig. 3 (top, left) illustrates the variation of the growth rate \( \omega_i \) of the most unstable (or least stable) Orr-Sommerfeld (OS) mode as a function of the slip length, for the parameters indicated in the figure’s caption. The wave angle considered is \( \Phi = \tan^{-1} \beta/\alpha = 20^\circ \); this three-dimensional mode is initially damped at low \( \lambda^\parallel \). However, past a threshold value of the slip length, the mode becomes unstable with a maximum growth rate which is achieved at \( \lambda^\parallel = 0.25 \). The disturbance mode shapes in correspondence to this point are plotted in the left frame, center row, of Fig. 3; they correspond to classical OS eigenfunctions, asymmetric about \( y = 0 \) because of the slip condition at \( y = -1 \). On the right side of Fig. 3 the behavior of a different mode is represented, at a much smaller value of the Reynolds number than the one considered so far. This instability mode displays a comparable behavior of the growth rate as a function of \( \lambda^\parallel \) (an initial decrease of \( \omega_i \), followed by an increase, with a maximum amplification for \( \lambda^\parallel = 0.15 \)), but radically different eigenfunctions, displayed in the center row, right frame. This mode, which is found to be dominating when the ridges are at an angle around 45°
FIG. 3. (Color online) Growth rate $\omega_i$ as a function of $\lambda^\parallel$ (top frame), and absolute value of the disturbance velocity components $(u, v, w)$ and disturbance pressure (middle frames), using one SH wall. With reference to the four top frames, in the left column the parameters are $Re = 10000$, $\theta = 80^\circ$, $\Phi = 20^\circ$ and $\alpha = 0.65$, whereas in the right column we have $Re = 2000$, $\theta = 45^\circ$, $\Phi \approx 90^\circ$ and $\alpha \approx 0$. The values of $\lambda^\parallel$ in the middle row, where eigenfunctions are plotted, correspond to the maximum growth rate for the respective case, i.e. $0.25$ and $0.145$. In the bottom row vectors and contours, in the $(y, z)$ plane, of the velocity components relative to the case in the right column are shown over three spanwise periods ($\beta = 2.5$). The shaded contours represent the positive and negative streamwise disturbance velocity component, whereas the vectors represent wall-normal and spanwise components.
to the mean pressure gradient, takes the form of near-wall vortices, as exemplified on the bottom frame of Fig. 3. Alternating high and low speed streaks, elongated in the streamwise direction \( x \) (\( \alpha = 10^{-3} \) in all the calculations for which we state \( \alpha \approx 0 \), the case \( \alpha \) exactly equal to zero being ill-posed numerically), are present near the SH wall, with corresponding low amplitude secondary vortices. While it is not a surprise that inclined ridges at the wall yield low frequency streamwise or quasi-streamwise vortices, it is remarkable that this behavior is rendered so clearly by the homogenized Navier-slip boundary condition. This new instability mode depends crucially on the wall ridges’ amplitude (a threshold value \( \lambda^\parallel = 0.038 \) is found with the present settings) and orientation with respect to the mean pressure gradient (i.e. \( \theta \)), and displays a temporal amplification factor typically larger than the most unstable three-dimensional OS wave (cf. the top two frames of the figure).

It is now instructive to examine the spectra, in terms of either the complex phase speed \( c \) or the complex frequency \( \omega \), depending on the value of the streamwise wavenumber, for the two cases discussed so far; such spectra are plotted in Fig. 4. The figure on the top is the classical spectrum which can be observed when \( \alpha \) is not close to zero, with the three branches, classically denoted as \( A \), \( P \) and \( S \) branch; this figure displays, in fact, all of the eigenvalues which exist when \( \lambda^\parallel \) varies in the range \( [0, 0.4] \). It is interesting to observe that the degenerate Squire modes of branch \( A \) (shown with red/grey bullets) split: such degenerate modes correspond, in the no-slip case, to a symmetric/antisymmetric couple of \( \hat{u} \) eigenfunctions. When slip occurs on one wall, one of the two wall modes of the initially degenerate pair in branch \( A \) moves rapidly away from the \( \lambda^\parallel = 0 \) value, thus displaying a very strong sensitivity (in fact, also OS wall modes are highly sensitive). Despite this, the Squire eigenvalues, both the wall modes and the center modes, never cross the real axis in all cases considered here, and the mode which becomes unstable is the three-dimensional OS mode with \( c_r \) close to 0.2. The picture is radically different for the case of ridges at 45° to the mean pressure gradient (bottom frame); as \( \lambda^\parallel \) increases, the modes which are initially degenerate, all damped and concentrated along a single vertical line with \( \omega_r \approx 0 \) (for \( \lambda^\parallel = 0 \)), separate and diverge from one another. The continuous line in the bottom frame joins all the least stable modes found for \( \lambda^\parallel < 0.038 \) and the unstable modes which emerge when \( \lambda^\parallel \) exceeds 0.038.

The results obtained so far indicate that a new wall-vortex mode, driven by the presence of inclined wall ridges of sufficiently large amplitude, exists when \( Re \) is rather small, to
FIG. 4. (Color online) (Top) spectra of the complex phase velocity $c$ in the case of one SH wall when $Re = 10000$, $\theta = 80^\circ$, $\Phi = 20^\circ$ and $\alpha = 0.65$ and different values of $\lambda^\parallel$. The filled circles correspond to $\lambda^\parallel = 0$; in particular, the red/grey bullets show the Squire modes on branch $A$. The open squares represent the spectrum for $\lambda^\parallel = 0.4$ and the dots show the trajectory of each eigenmode when $\lambda^\parallel$ varies from 0 to 0.4. (Bottom) spectra of the complex frequency $\omega$ in the case of one SH wall when $Re = 2000$, $\theta = 45^\circ$, $\Phi \approx 90^\circ$ and $\alpha \approx 0$. The open squares, diamonds and filled circles show the spectra for $\lambda^\parallel = 0, 0.1, 0.2$, respectively. The continuous line traces the least stable (or the most unstable) mode for $\lambda^\parallel$ varying in the range $[0, 0.2]$. 
FIG. 5. Critical Reynolds number $Re_c$ (left), the corresponding wave angle (middle) and streamwise wavenumber (right) as a function of $\theta$ for the case of $\lambda^\parallel = 0.07$ (dashed line) and $\lambda^\parallel = 0.155$ (solid line) for one SH wall.

presumably dominate the early stages of the transition process.

A parametric study, with $\theta$ and $\lambda^\parallel$ varied systematically to infer trends is reported in Figs. 5 and 6. The first of these figures show that the OS mode identifies the critical conditions only when $\theta$ is close to $0^\circ$ and $90^\circ$; for $\theta$ in a range around $45^\circ$ (range which is wider with the increase of $\lambda^\parallel$) the wall-vortex mode is the dominating instability. The smallest critical Reynolds numbers, $Re_c$, are found at $45^\circ$ for both cases examined in Fig. 5 and are around a value of 1000, much smaller than the values of the corresponding neutral OS modes. The critical wave angle is $90^\circ$ in the range of $\theta$’s where this new instability dominates. Fig. 6 shows the behavior of the most unstable, two-dimensional OS mode (solid lines) which leads the instability when $\theta = 0^\circ$, and the switch between the OS wave
FIG. 7. Critical Reynolds number $Re_c$ (left) and corresponding wave angle (middle) and streamwise wavenumber (right) as a function of $\theta$ for the case of $\lambda^\parallel = 0.02$ (solid line) and $\lambda^\parallel = 0.05$ (dashed line) in the presence of two SH walls.

FIG. 8. Critical Reynolds number $Re_c$ (left) and corresponding wave angle (middle) and streamwise wavenumber (right) as a function of $\lambda^\parallel$ for the case of $\theta = 0^\circ$ (solid line) and $\theta = 45^\circ$ (dashed line) in the presence of two SH walls.

and the wall-vortex mode, when $\theta = 45^\circ$, taking place at $\lambda^\parallel = 0.033$. As expected from previous studies, a stabilization effect (i.e. an increase of $Re_c$) is found for the OS mode as $\lambda^\parallel$ grows from zero ($Re_c = 3848$ for $\lambda^\parallel = 0$). However, when the ridges are at an angle of 45° the OS mode is eventually overruled by the streamwise wall-vortex mode, which becomes unstable at progressively smaller values of the Reynolds number; for $\lambda^\parallel$ above around 0.15 an asymptotic value of the critical $Re$ close to 600 is reached for the onset of the wall-vortex mode.

The case of two superhydrophobic walls is considered next, focussing on lower values of $\lambda^\parallel$ since it is known [1] that, when the walls are isotropic, a comparable stabilizing effect is achieved in the case of two SH walls for a value of the slip length ten times smaller than
FIG. 9. (Color online) Spectrum of temporal eigenvalues $\omega$, with the unstable mode marked with a red/grey bullet (left), and absolute value of the disturbance velocity components ($u, v, w$) and disturbance pressure of the unstable mode (right), for $Re = 10000$, $\lambda^\parallel = 0.05$, $\theta = 45^\circ$, $\alpha = 0.1$, $\Phi = 86^\circ$ ($\beta = 1.4$). Both walls are superhydrophobic.

for a single SH wall. The results are summarized by Figs. 7 and 8. The notable effect in this case is that the streamwise wall-vortex mode does not emerge, with a competition which is now instaured between two-dimensional and three-dimensional OS modes; Fig. 7 shows that the onset of an exponential instability is delayed when $\lambda^\parallel$ is increased and that the two-dimensional OS wave (with $\Phi_c = 0$) dominates the transition process only for $\theta$ sufficiently large (the switch-over value increasing with $\lambda^\parallel$). The stabilizing effect of $\lambda^\parallel$ is confirmed by Fig. 8; for $\lambda^\parallel$ below 0.01 the stability characteristics are similar to those of the no-slip case, and two-dimensional OS modes prevail (for any value of $\theta$). In the case of ridges inclined at an angle of $45^\circ$ to the mean streamwise velocity component, the mode which takes the lead past $\lambda^\parallel = 0.033$ is quasi-streamwise ($\alpha$ is small and decreasing). The spectrum of eigenvalues for a representative case is presented in the left part of Fig. 9 for $\lambda^\parallel = 0.05$, $\theta = 45^\circ$ and $Re = 10000$. The classical branches, $A$, $P$ and $S$, are present, with the unstable mode on the $A$ branch. The shape of the unstable mode is found in the right frame of Fig. 9, where the shape of a (distorted) three-dimensional OS wave can be seen.
FIG. 10. (Color online) Vectors and contours, in the \((y,z)\) plane, of the optimal disturbance at \(t = 0\) (left column) and the ensuing solution at the target time \(T = 105\) (right column), shown over two spanwise periods, for \(\lambda^\parallel = 0\) (top row), \(\lambda^\parallel = 0.05\) and one SH wall (middle row), \(\lambda^\parallel = 0.05\) and two SH walls (bottom row). The shaded contours represent the positive and negative streamwise disturbance velocity component, whereas the vectors represent wall-normal and spanwise components. The parameters are \(Re = 1333\), \(\beta = 2\), \(\alpha \approx 0\), \(\theta = 30^\circ\).

**Nonmodal analysis**

Fig. 10 displays representative optimal perturbations (left column) for a given target time, for both no-slip and SH cases, together with their output fields (right column). The intermediate row (one slip wall at \(y = -1\)) is interesting since the initial disturbance field is more intense near the bottom wall than near the top one, and is oblique in the \((y,z)\) plane.

The maximum gain \(G_M\) of a disturbance over a given time, maximized with respect to the wavevector, depends parametrically on \(Re\), \(\lambda^\parallel\) and \(\theta\). The results shown in Figs. 11 through
FIG. 11. Gain $G_M$ (left), corresponding time $T_M$ (middle) and spanwise wavenumber $\beta_M$ (right) as a function of $\lambda^\|$ in the case of $\theta = 0^\circ$ (−), $\theta = 15^\circ$ (∗), $\theta = 30^\circ$ (−−), $\theta = 60^\circ$ (⋄), for $Re = 1333$ and two SH walls. In all cases the corresponding optimal streamwise wavenumber is $\alpha_M \approx 0$.

FIG. 12. Gain $G_M$ (left), corresponding time $T_M$ (middle) and spanwise wavenumber $\beta_M$ (right) as a function of $\lambda^\|$ in the case of $\theta = 0^\circ$ (−), $\theta = 15^\circ$ (∗), $\theta = 30^\circ$ (−−), $\theta = 60^\circ$ (⋄). In all cases $Re = 1333$, $\alpha_M \approx 0$ and only one wall is superhydrophobic.

13 are computed for a fixed value of $Re = 1333$, which is the same used by Min & Kim [2] (they scaled $Re$ with the centerline velocity which is why they quote a value of 2000). This Reynolds number is subcritical from a modal analysis point of view in the no-slip case.

In Fig. 11 $G_M$ is given as a function of $\lambda^\|$ for different values of $\theta$, in the case of two SH walls. For $\lambda^\| = 0$ we recover the no-slip case and for $\lambda^\| > 0$ there is a monotonic decrease of the gain for all $\theta$’s. In all cases the corresponding $\alpha_M \approx 0$ and the variation of both $\beta_M$ and $T_M$ with the slip length is weak. In the case of a single SH wall the results show a different trend, as demonstrated in Fig. 12. Again, the gain $G_M$ is presented as a function of $\lambda^\|$ for different values of $\theta$. For values of the ridge angle larger than zero the gain always increases as the Navier slip length is increased. Moreover, for some values of the ridge angle $\theta$, and
FIG. 13. Gain $G$ (left) and corresponding optimal spanwise wavenumber $\beta$ (right) as a function of the final time $T$, for the case of $\lambda^\parallel = 0$ (○), $\lambda^\parallel = 0.03$ (△), $\lambda^\parallel = 0.06$ (○), $\theta = 30^\circ$, $\alpha \approx 0$ and $Re = 1333$.

above a threshold $\lambda^\parallel$, the flow becomes unstable from a modal point of view: in these cases no finite value of $T_M$ is found, since the gain increases monotonically with the increase of the final target time. An example is presented in Fig. 13 where the gain $G$ is plotted as a function of the final time $T$ of the optimization and three different values of $\lambda^\parallel$ for the case in which $\theta = 30^\circ$. For $\lambda^\parallel = 0$ and 0.03 the gain decreases for large enough values of $T$; conversely, when $\lambda^\parallel = 0.06$ the gain increases, albeit slowly, with $T$, with the spanwise wavenumber $\beta$ reaching an asymptotic value equal to 1.78. The unbounded increase of $G$ with $T$ is the indication of the occurrence of the streamwise wall-vortex exponential instability.

IV. SUMMARIZING REMARKS

A thorough modal and nonmodal linear stability analysis of the flow in a channel with the walls coated with a SH material has been conducted, for the case of a surface topography constituted by micro-ridges with arbitrary alignment. The main motivation is to understand under which conditions and parameters the most stabilizing or destabilizing effects are obtained. The results obtained give indications on transition delay or enhancement from laminar to turbulent flow and consequently on the possibility of drag reduction.

The modal behavior has yielded surprising results in two senses: on the one hand, a
new streamwise wall-vortex mode has been found in the case of a single SH wall, driven by
the wall boundary condition, and capable to reduce significantly the value of the Reynolds
number for the onset of the instability. This new mode is enhanced by the increase of $\lambda^\parallel$ and is found to be most effective when the ridges are inclined by an angle of about 45° to the mean pressure gradient. On the other hand, when two walls are superhydrophobic, the instability is ruled by either a two-dimensional or a three-dimensional Orr-Sommerfeld mode, as function of $\theta$ and $\lambda^\parallel$, demonstrating a posteriori the inapplicability of Squires theorem for this flow.

The nonmodal analysis shows that while the presence of two SH walls yields a slight reduction in energy growth over time, the case of only one SH wall produces an increase of the disturbance kinetic energy for a large range of values of $\lambda^\parallel$ when $\theta$ is sufficiently greater than zero. It is further shown that, for a single SH wall, beyond a threshold slip length, for values of the inclination angle of the micro-ridges around 45° the gain becomes unbounded with the final target time, a sign of the onset of the wall-vortex instability.

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