Modelling of poroelastic carpets

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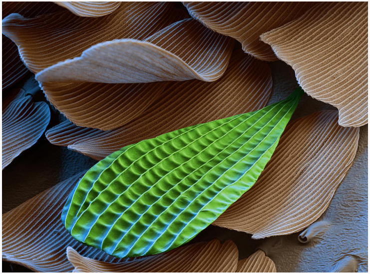
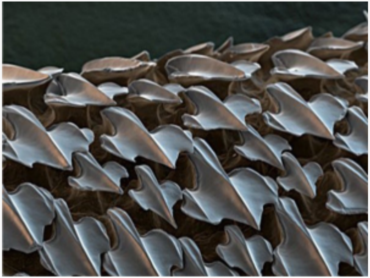
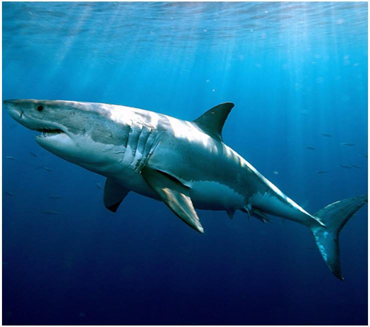
London, 9th May 2016

Man often tries to achieve technical surfaces which are rigid and smooth...

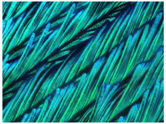
... but in Nature, porous, anisotropic, irregular, elastic, rough is the norm!

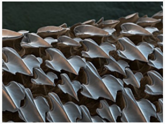
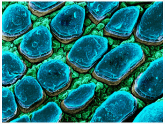





Motivation

In biomimetics we deal with several separation of scales phenomena



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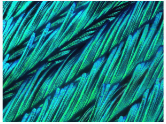

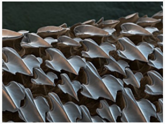
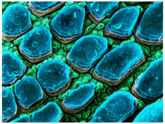




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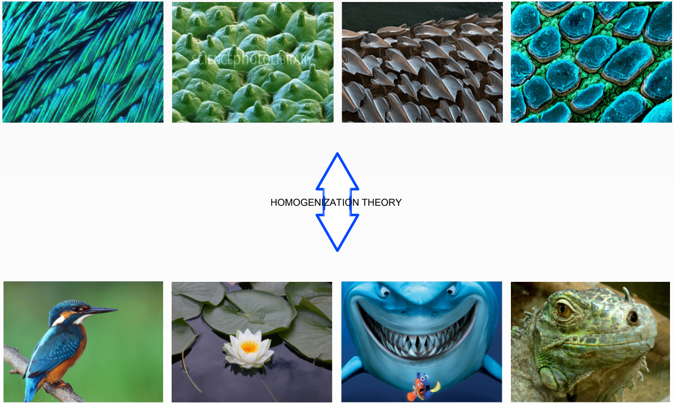
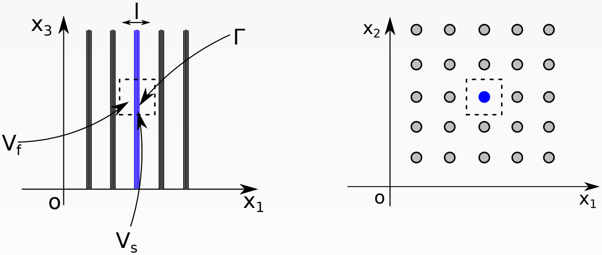


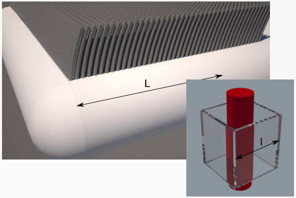
- Theory of homogenization applied to poroelastic media
- Resolution of the microscopic equations
  - Permeability tensor
  - Elasticity tensor
- Resolution of the macroscopic equations
  - Oscillating channel flow
- Left to do ...

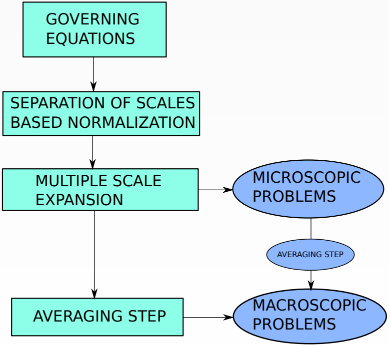
Introduction: carpet of elastic fibres



Transversely isotropic porous medium, made by fibers shown in the (x_1, x_3) and (x_1, x_2) plane, respectively. The dotted rectangle in the two frames represents the elementary cell V. V_f is the volume occupied by the fluid and V_s is that occupied by the solid, so that $V = V_f + V_s$. Γ is the fluid-solid microscopic interface. The porosity ϑ is defined as V_f/V. All the unknowns are periodic over V.

Introduction: carpet of elastic fibres





$$\epsilon = \frac{l}{L} \ll 1$$

We introduce $x, x' = \epsilon x$

$$\mathbf{u}(x, x', t) = \sum_{n=0}^{N} \epsilon^n \mathbf{u}^{(n)}(x, x', t)$$

$$\mathbf{v}(x, x', t) = \sum_{n=0}^{N} \epsilon^n \mathbf{v}^{(n)}(x, x', t)$$

$$p(x, x', t) = \sum_{n=0}^{N} \epsilon^n p^{(n)}(x, x', t)$$

$$\langle f \rangle := \frac{1}{V} \int_{V_f | V_s} f \, dV .$$

The scales considered

$$U = \frac{V}{T_S} \qquad \text{No slip on } \Gamma$$

$$E\frac{Pl^2}{\mu L^2}T_S = P, \qquad \text{macroscopic solid stresses balanced by pressure on } \Gamma$$

$$\frac{P}{L} = \frac{\mu U}{l^2} \qquad \text{macroscopic press forces balanced by viscous dissipation}$$

$$\Rightarrow T_S = \frac{\mu L^2}{El^2} = \frac{\mu}{\epsilon^2 E} \qquad \text{solid time scale}$$

$$\frac{\rho_s}{T_S^2} = \frac{E}{L^2}, \qquad \text{inertia of the solid of the same order of the solid stress}$$

(Fluid and solid variables)

$$\hat{\mathbf{x}} = l\mathbf{x}, \quad \hat{p} = Pp, \quad \hat{t}_f = \frac{lt_f}{U}, \quad \hat{\mathbf{u}} = \epsilon\frac{Pl}{\mu}\mathbf{u}$$

$$\hat{\mathbf{v}} = \frac{PL}{E}\mathbf{v}, \quad \hat{t}_s = \frac{\mu t_s}{E\epsilon^2}$$

The homogenized model

$$\frac{\partial u_i}{\partial x_i} = 0 \text{ on } V_f$$

$$\epsilon \mathrm{Re}_l \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \epsilon \nabla^2 u_i \text{ on } V_f$$

$$\epsilon^2 \frac{\partial^2 v_i}{\partial t_s^2} = \frac{\partial}{\partial x_j} C_{ijkl} \varepsilon_{kl}(\mathbf{v}) \text{ on } V_s$$

linked by

$$u_i = \frac{\partial v_i}{\partial t} \quad \text{and} \quad -p\, n_i + 2\epsilon \varepsilon_{ij}(\mathbf{u}) n_j = \frac{1}{\epsilon} \left[C_{ijkl} \varepsilon_{kl}(\mathbf{v}) \right] n_j \quad \text{on } \Gamma$$

$$\mathrm{Re}_l = \frac{\rho_f U l}{\mu} = \epsilon \frac{\rho_f U L}{\mu} = \epsilon \mathrm{Re}_L$$

DEVELOPED MODELS

- $\mathrm{Re}_l = \mathcal{O}(\epsilon)$ & $\mathcal{O}(1)$ for poroelastic media, isotropic and anisotropic

After homogenization for the macroscopic fields $\mathbf{u}^{(0)},\ \mathbf{v}^{(0)},\ p^{(0)}$ we have

$$\begin{cases} (1-\vartheta)\frac{\partial^2 v_i^{(0)}}{\partial t^2} = \frac{\partial}{\partial x_j'}\left[\mathcal{C}_{ijpq}\varepsilon'_{pq}(\mathbf{v}^{(0)}) - \alpha'_{ij}p^{(0)}\right] \\ \frac{\partial \langle u_i^{(0)}\rangle}{\partial x_i'} = \langle \frac{\partial \chi_i^{pq}}{\partial x_i}\rangle \varepsilon'_{pq}(\dot{\mathbf{v}}^{(0)}) - \langle \frac{\partial \eta_i}{\partial x_i}\rangle \dot{p}^{(0)} \\ \langle u_i^{(0)}\rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij}\frac{\partial p^{(0)}}{\partial x_j'} \end{cases}$$

valid in the homogenized macroscopic domain, and the equations for the microscopic fields $\boldsymbol{\chi},\ \boldsymbol{\eta}$, $\mathbf{K}$ and $\mathbf{A}$ valid in the microcell

$$\begin{cases} \frac{\partial}{\partial x_j}\left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} = 0, \\ \left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} n_j = 0 \quad \text{on } \Gamma \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial x_j}\left[C_{ijkl}\ \varepsilon_{kl}(\eta)\right] = 0, \\ \left[C_{ijkl}\ \varepsilon_{kl}(\eta)\right] n_j = -n_i \quad \text{on } \Gamma \end{cases} \qquad \begin{cases} \frac{\partial A_j}{\partial x_i} - \frac{\partial^2 K_{ij}}{\partial x_k^2} = \delta_{ij}, \\ \frac{\partial K_{ij}}{\partial x_i} = 0, \\ K_{ij}(\mathbf{x},t) = 0 \quad \text{on } \Gamma \end{cases}$$

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$$
\begin{cases}
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After homogenization for the macroscopic fields $\mathbf{u}^{(0)}, \ \mathbf{v}^{(0)}, \ p^{(0)}$ we have

$$\begin{cases} (1-\vartheta)\frac{\partial^2 v_i^{(0)}}{\partial t^2} + \boxed{\mathrm{Re}_l U_j \langle \frac{\partial u_i^{(0)}}{\partial x_j} \rangle} = \frac{\partial}{\partial x_j'} \left[\mathcal{C}_{ijpq} \, \varepsilon'_{pq}(\mathbf{v}^{(0)}) - \alpha'_{ij} \, p^{(0)} \right] \\ \frac{\partial \langle u_i^{(0)} \rangle}{\partial x_i'} = \langle \frac{\partial \chi_i^{pq}}{\partial x_i} \rangle \, \varepsilon'_{pq}(\dot{\mathbf{v}}^{(0)}) - \langle \frac{\partial \eta_i}{\partial x_i} \rangle \, \dot{p}^{(0)} \\ \langle u_i^{(0)} \rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij} \frac{\partial p^{(0)}}{\partial x_j'} \end{cases}$$

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$$\begin{cases} \boxed{\mathrm{Re}_l U_k \frac{\partial K_{ij}}{\partial x_k}} = -\frac{\partial A_j}{\partial x_i} + \frac{\partial^2 K_{ij}}{\partial x_k^2} + \delta_{ij}, \\ \frac{\partial K_{ij}}{\partial x_i} = 0, \\ K_{ij}(\mathbf{x}, t) = 0 \quad \text{on } \Gamma \end{cases}$$

$Re_l = \mathcal{O}(\epsilon)$: packed rigid spheres

$$\mathcal{K}_{ij} = \mathcal{K}\delta_{ij}$$

Macroscopic level

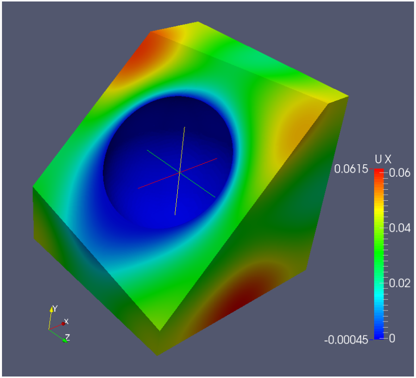
$$\langle u_i^{(0)} \rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij} \frac{\partial p^{(0)}}{\partial x_j'}$$

Microscopic level

$$\begin{cases} -\frac{\partial A_j}{\partial x_i} + \nabla^2 K_{ij} = -\delta_{ij} \\ \frac{\partial K_{ij}}{\partial x_i} = 0 \end{cases}$$

$$K_{ij} = 0 \quad on\ \Gamma$$

K_{ij}, A_j (x_1, x_2)-*periodic*



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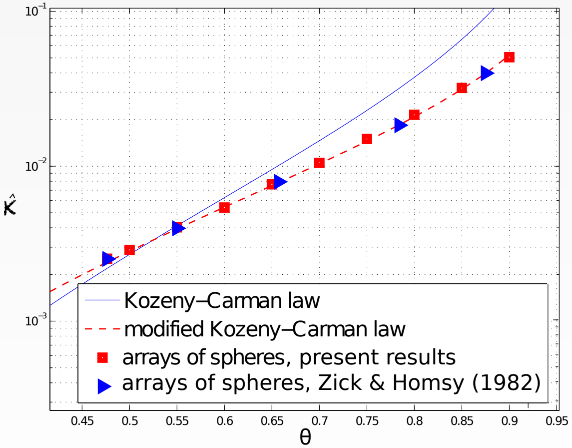


$$\hat{\mathcal{K}} = \frac{1}{5}\left(\frac{V_s}{|\Gamma|}\right)^2 \frac{\vartheta^3}{(1-\vartheta)^2}, \qquad \hat{\mathcal{K}} = \frac{1}{5}\left(\frac{V_s}{|\Gamma|}\right)^2 \frac{\vartheta^{\frac{5}{2}}}{(1-\vartheta)^{\frac{47}{30}}}$$

$Re_l = \mathcal{O}(\epsilon)$: arrays of rigid cylinders

Macroscopic level

$$\langle u_i^{(0)} \rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij} \frac{\partial p^{(0)}}{\partial x'_j}$$

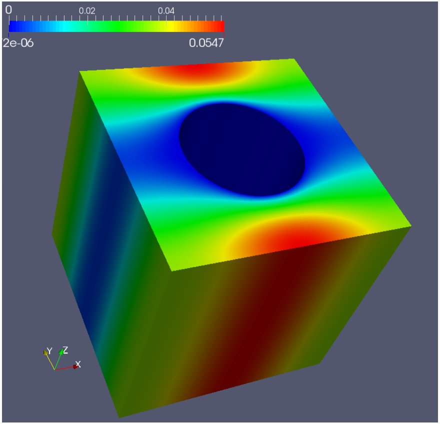
Microscopic level

$$\begin{cases} -\frac{\partial A_j}{\partial x_i} + \nabla^2 K_{ij} = -\delta_{ij} \\ \frac{\partial K_{ij}}{\partial x_i} = 0 \end{cases}$$

$$K_{ij} = 0 \quad on\ \Gamma$$

K_{ij}, A_j (x_1, x_2)*-periodic*

$\mathcal{K}_{11} = \mathcal{K}_{22}$



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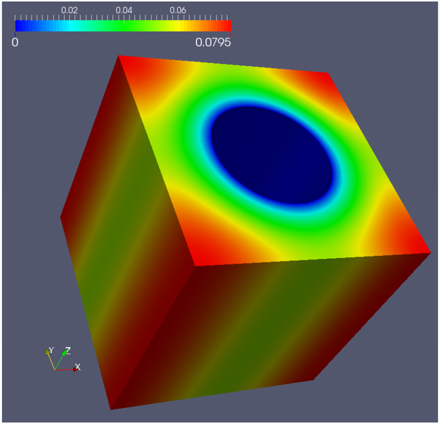
K_{ij}, A_j (x_1, x_2)*-periodic*

$\mathcal{K}_{33}$



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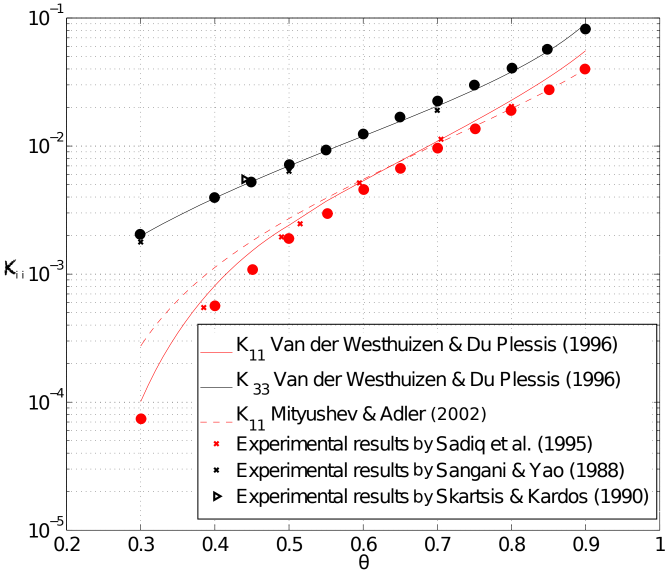


$Re_l = \mathcal{O}(1)$

Macroscopic level

$$\begin{cases} (1-\vartheta)\frac{\partial^2 v_i^{(0)}}{\partial t^2} + Re_l U_j \langle \frac{\partial u_i^{(0)}}{\partial x_j} \rangle = \frac{\partial}{\partial x_j'} \left[\mathcal{C}_{ijpq} \varepsilon'_{pq}(\mathbf{v}^{(0)}) - \alpha'_{ij} p^{(0)} \right] \\ \frac{\partial \langle u_i^{(0)} \rangle}{\partial x_i'} = \langle \frac{\partial \chi_i^{pq}}{\partial x_i} \rangle \varepsilon'_{pq}(\dot{\mathbf{v}}^{(0)}) - \langle \frac{\partial \eta_i}{\partial x_i} \rangle \dot{p}^{(0)} \\ \langle u_i^{(0)} \rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij} \frac{\partial p^{(0)}}{\partial x_j'} \end{cases}$$

Microscopic level

$$Re_l U_k \frac{\partial K_{ij}}{\partial x_k} \simeq -\frac{\partial A_j}{\partial x_i} + \frac{\partial^2 K_{ij}}{\partial x_k^2} + \delta_{ij}, \qquad \frac{\partial K_{ij}}{\partial x_i} = 0,$$

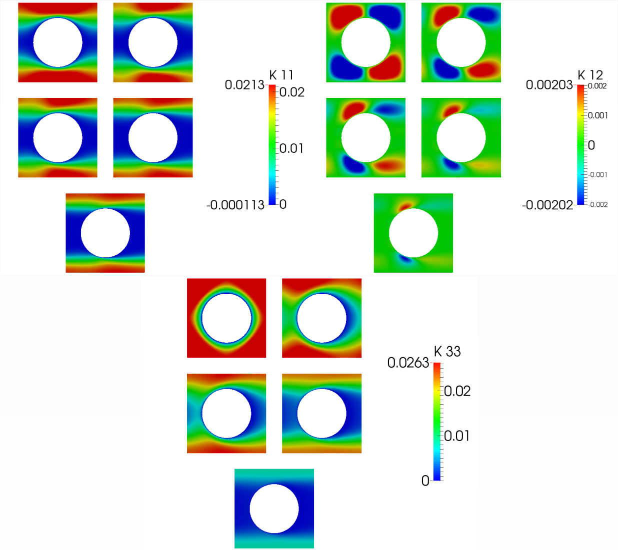
$$K_{ij}(\mathbf{x}, t) = 0 \quad on\ \Gamma, \quad plus\ periodicity\ over\ V_f$$

$$Re_l = \frac{Ul}{\nu}, \quad U_k := \frac{1}{V_{Tot}} \int_{V_{Tot}} \langle u_k^{(0)} \rangle \, dV$$

MICRO and MACRO level linked by iterations over U_k.

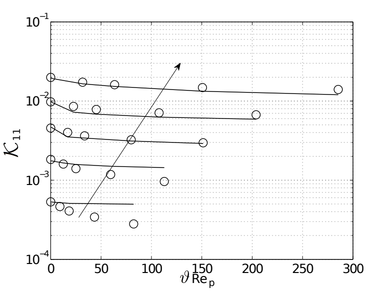
cf. Gustaffson & Protas (2013) on the use of Oseen's closure for high Re

$Re_l U_k \in [0, 150]\delta_{1k},\ \vartheta = 0.7$



$\mathrm{Re}_l U_k = (c, 0, 0)$

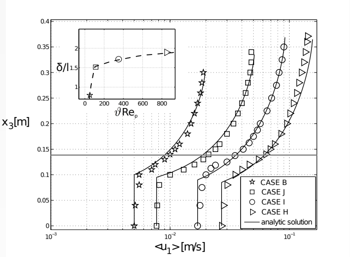
Edwards et al. (1990)

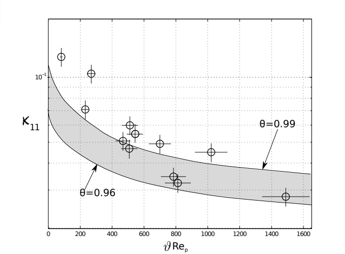


$$\mathrm{Re}_l = \mathcal{O}(1) \rightarrow \mathcal{K}_{ij} = \begin{pmatrix} ☞ & \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & ☞ & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) & ☞ \end{pmatrix}$$

$\mathrm{Re}_l U_k = (c, 0, 0)$

Ghisalberti & Nepf (2004,2006,2009)





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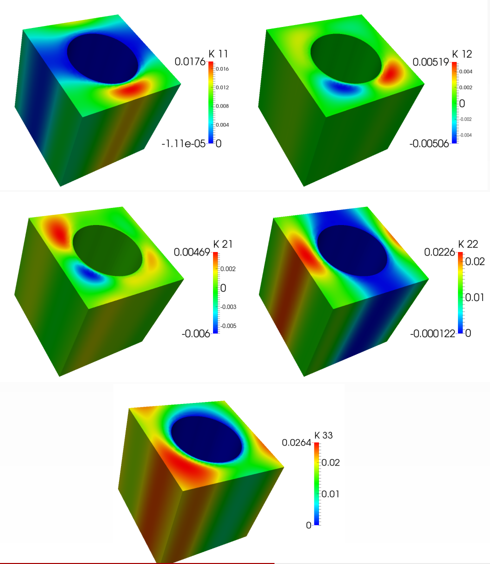
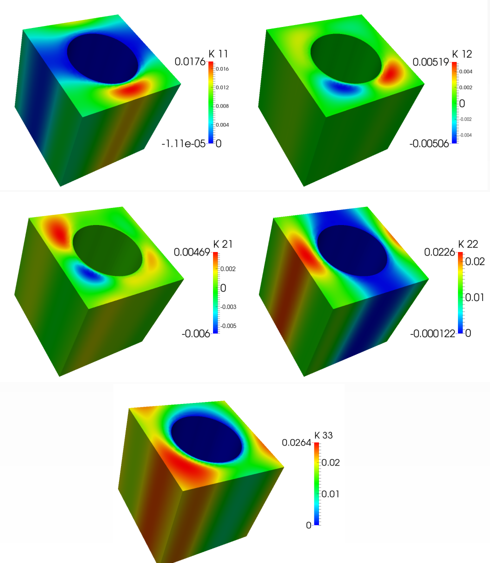
$\mathrm{Re}_l U_k = (10, 20, 15)$, $\vartheta = 0.7$



$$\mathrm{Re}_l = \mathcal{O}(\epsilon) \rightarrow \mathcal{K}_{ij} = \begin{pmatrix} 0.009 & \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & 0.009 & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) & 0.0190 \end{pmatrix}$$

$$\mathrm{Re}_l = \mathcal{O}(1) \rightarrow \mathcal{K}_{ij} = \begin{pmatrix} 0.0046 & 0.0003 & \mathcal{O}(10^{-9}) \\ 0.0003 & 0.0057 & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) & 0.0112 \end{pmatrix}$$

$Re_l U_k = (10, 20, 15)$, $\vartheta = 0.7$



$$Re_l = \mathcal{O}(\epsilon) \rightarrow \mathcal{K}_{ij} =$$

$$\begin{pmatrix} 0.009 & \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & 0.009 & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) & 0.0190 \end{pmatrix}$$

$$Re_l = \mathcal{O}(1) \rightarrow \mathcal{K}_{ij} =$$

$$\begin{pmatrix} \text{☞} & \text{☞} & \mathcal{O}(10^{-9}) \\ \text{☞} & \text{☞} & \mathcal{O}(10^{-9}) \\ \mathcal{O}(10^{-9}) & \mathcal{O}(10^{-9}) & \text{☞} \end{pmatrix}$$

After homogenization for the macroscopic fields $\mathbf{u}^{(0)}, \ \mathbf{v}^{(0)}, \ p^{(0)}$ we have

$$\begin{cases} (1-\vartheta)\frac{\partial^2 v_i^{(0)}}{\partial t^2} = \frac{\partial}{\partial x_j'}\left[\mathcal{C}_{ijpq}\,\varepsilon'_{pq}(\mathbf{v}^{(0)}) - \alpha'_{ij}\,p^{(0)}\right] \\ \frac{\partial \langle u_i^{(0)}\rangle}{\partial x_i'} = \langle\frac{\partial \chi_i^{pq}}{\partial x_i}\rangle\,\varepsilon'_{pq}(\dot{\mathbf{v}}^{(0)}) - \langle\frac{\partial \eta_i}{\partial x_i}\rangle\,\dot{p}^{(0)} \\ \langle u_i^{(0)}\rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij}\frac{\partial p^{(0)}}{\partial x_j'} \end{cases}$$

valid in the homogenized macroscopic domain, and the equations for the microscopic fields $\boldsymbol{\chi}, \ \boldsymbol{\eta}$, $\mathbf{K}$ and $\mathbf{A}$ valid in the microcell

$$\begin{cases} \frac{\partial}{\partial x_j}\left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} = 0, \\ \left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} n_j = 0 \quad \text{on } \Gamma \end{cases}$$

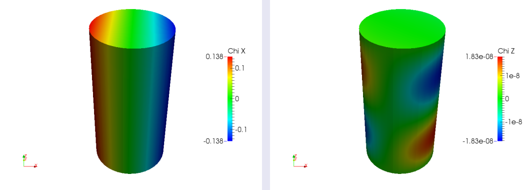
$$\begin{cases} \frac{\partial}{\partial x_j}\left[C_{ijkl}\ \varepsilon_{kl}(\eta)\right] = 0, \\ \left[C_{ijkl}\ \varepsilon_{kl}(\eta)\right] n_j = -n_i \quad \text{on } \Gamma \end{cases}$$

$$\begin{cases} \frac{\partial A_j}{\partial x_i} - \frac{\partial^2 K_{ij}}{\partial x_k^2} = \delta_{ij}, \\ \frac{\partial K_{ij}}{\partial x_i} = 0, \\ K_{ij}(\mathbf{x}, t) = 0 \quad \text{on } \Gamma \end{cases}$$

$\mathrm{Re}_l = O(\epsilon)$ & $\mathrm{Re}_l = O(1)$ effective tensors

(Cylinders, $\vartheta = 0.3 - 0.99$)

$$\begin{cases} \frac{\partial}{\partial x_j} \left\{ C_{ijkl} \left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq} \right] \right\} = 0, \\ \left\{ C_{ijkl} \left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq} \right] \right\} n_j = 0 \quad \text{on } \Gamma, \end{cases}$$



$$\mathcal{C}_{ijpq} = \langle C_{ijkl}\ \varepsilon_{kl}(\chi^{pq}) \rangle + \langle C_{ijpq} \rangle = \begin{pmatrix} \bigcirc & \blacksquare & \otimes & 0 & 0 & 0 \\ \blacksquare & \bigcirc & \otimes & 0 & 0 & 0 \\ \otimes & \otimes & \bigstar & 0 & 0 & 0 \\ 0 & 0 & 0 & \clubsuit & 0 & 0 \\ 0 & 0 & 0 & 0 & \spadesuit & 0 \\ 0 & 0 & 0 & 0 & 0 & \spadesuit \end{pmatrix}$$

$\text{Re}_l = O(\epsilon)$ & $\text{Re}_l = O(1)$ effective tensors

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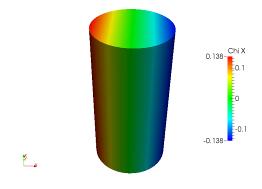
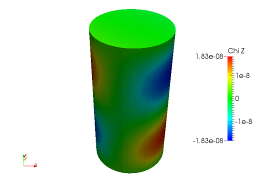


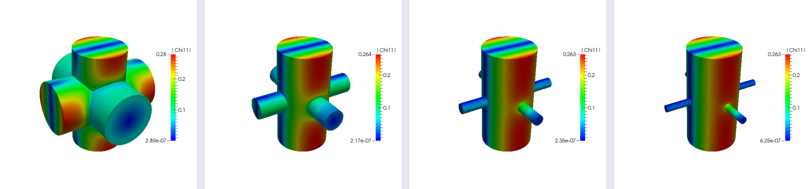


$$\mathcal{C}_{ijkl} = \begin{pmatrix} \bigcirc & \blacksquare & \otimes & 0 & 0 & 0 \\ \blacksquare & \bigcirc & \otimes & 0 & 0 & 0 \\ \otimes & \otimes & \bigstar & 0 & 0 & 0 \\ 0 & 0 & 0 & \clubsuit & 0 & 0 \\ 0 & 0 & 0 & 0 & \spadesuit & 0 \\ 0 & 0 & 0 & 0 & 0 & \spadesuit \end{pmatrix} \qquad \mathcal{C}_{ijkl} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bigstar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\mathrm{Re}_l = O(\epsilon)$ & $\mathrm{Re}_l = O(1)$ effective tensors

(Linked cylinders, $\vartheta \approx 0.8$)

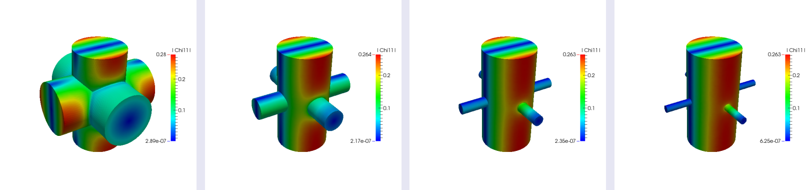
$$\begin{cases} \dfrac{\partial}{\partial x_j} \left\{ C_{ijkl} \left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq} \right] \right\} = 0, \\ \left\{ C_{ijkl} \left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq} \right] \right\} n_j = 0 \quad \text{on } \Gamma, \end{cases}$$



$$\mathcal{C}_{ijpq} = \langle C_{ijkl}\ \varepsilon_{kl}(\chi^{pq}) \rangle + \langle C_{ijpq} \rangle = \begin{pmatrix} \bigcirc & \blacksquare & \otimes & 0 & 0 & 0 \\ \blacksquare & \bigcirc & \otimes & 0 & 0 & 0 \\ \otimes & \otimes & \bigstar & 0 & 0 & 0 \\ 0 & 0 & 0 & \clubsuit & 0 & 0 \\ 0 & 0 & 0 & 0 & \spadesuit & 0 \\ 0 & 0 & 0 & 0 & 0 & \spadesuit \end{pmatrix}$$

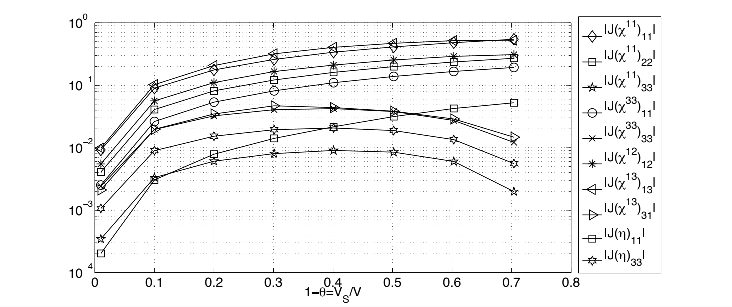
$\mathrm{Re}_l = O(\epsilon)$ & $\mathrm{Re}_l = O(1)$ effective tensors

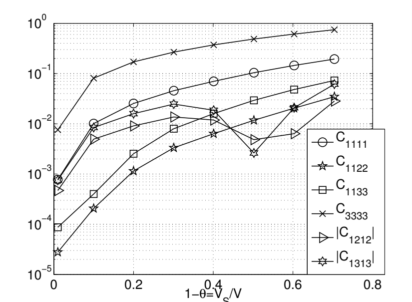
$$
\begin{cases}
\frac{\partial}{\partial x_j}\left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} = 0, \\
\left\{C_{ijkl}\left[\varepsilon_{kl}(\chi^{pq}) + \delta_{kp}\delta_{lq}\right]\right\} n_j = 0 \quad \text{on } \Gamma,
\end{cases}
$$

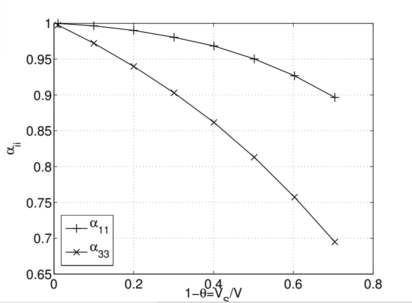


$$
\mathcal{C}_{ijpq} = \langle C_{ijkl}\ \varepsilon_{kl}(\chi^{pq})\rangle + \langle C_{ijpq}\rangle =
\begin{pmatrix}
\text{☞} & \text{☞} & \text{☞} & 0 & 0 & 0 \\
\text{☞} & \text{☞} & \text{☞} & 0 & 0 & 0 \\
\text{☞} & \text{☞} & \bigstar & 0 & 0 & 0 \\
0 & 0 & 0 & \text{☞} & 0 & 0 \\
0 & 0 & 0 & 0 & \text{☞} & 0 \\
0 & 0 & 0 & 0 & 0 & \text{☞}
\end{pmatrix}
$$

Averaged components of the effective elasticity tensors

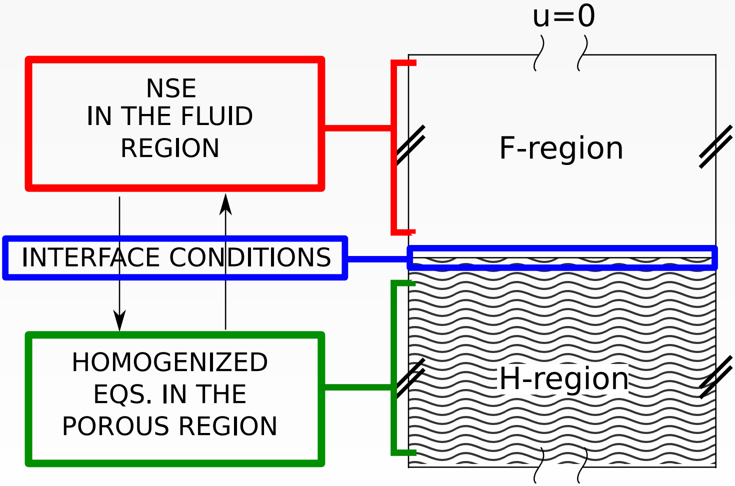






Macroscopic simulations: oscillating channel flow

A domain–decomposition–based solver



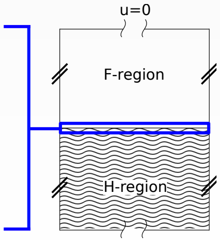
Macroscopic results: linked cylinders

$$\begin{cases} (1-\vartheta)\dfrac{\partial^2 v_i^{(0)}}{\partial t^2} = \dfrac{\partial}{\partial x_j'}\left[\mathcal{C}_{ijpq}\varepsilon_{pq}'(\mathbf{v}^{(0)}) - \alpha_{ij}' p^{(0)}\right] \\[2ex] \dfrac{\partial \langle u_i^{(0)}\rangle}{\partial x_i'} = \langle \dfrac{\partial \chi_i^{pq}}{\partial x_i}\rangle \varepsilon_{pq}'(\dot{\mathbf{v}}^{(0)}) - \langle \dfrac{\partial \eta_i}{\partial x_i}\rangle \dot{p}^{(0)} \\[2ex] \langle u_i^{(0)}\rangle - \vartheta \dot{v}_i^{(0)} = -\mathcal{K}_{ij}\dfrac{\partial p^{(0)}}{\partial x_j'} \end{cases}$$

$$\Sigma_{ij}\Big|_H n_j = \Sigma_{ij}\Big|_F n_j \qquad \text{(Gopinath \& Mahadevan, 2011)}$$

$$\left[u_i - \bar{\mathcal{L}}_{ijk}\varepsilon_{jk}(\mathbf{u})\right]\Big|_F = -\bar{\mathcal{K}}_{ij}\frac{\partial p}{\partial x_j'}\Big|_H \qquad \text{(Lācis \& Bagheri, 2016)}$$

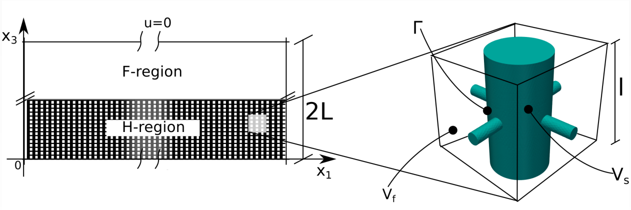
$$\Delta P\Big|_{F-H} = \bar{\mathcal{Q}}_{ij}\frac{\partial u_i}{\partial x_j'}\Big|_F \qquad \text{(Carraro et al., 2013)}$$



Macroscopic results: oscillating channel

NSE are forced by an oscillating pressure gradient of the form $\Re(Ae^{i\omega t})$. Solution shown for

- $\rho_f = 1.22 \ kg/m^3$, air,
- $\text{Re}_L = 100$,
- $\text{Ca} = 9.15 \times 10^{-8}$ (polyurethane foam, $E = 3 \times 10^5$ Pa, $\nu_P = 0.39$),
- $A = \omega = 1$.



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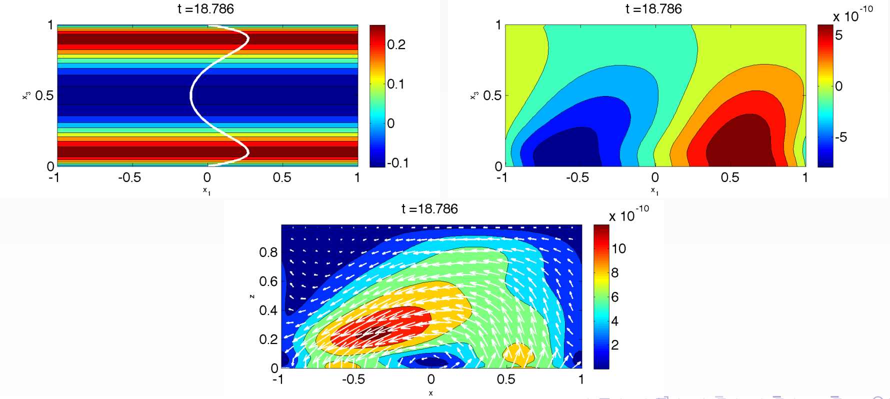

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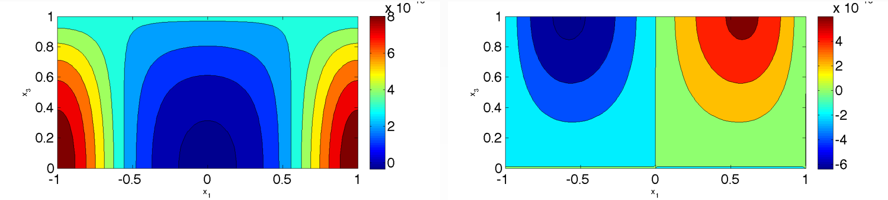

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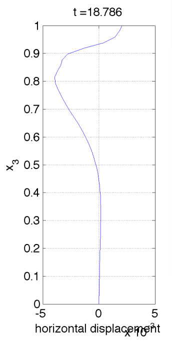
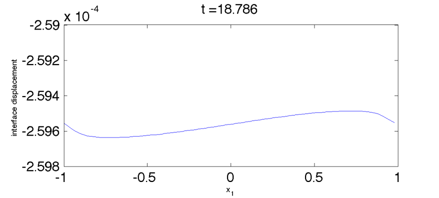
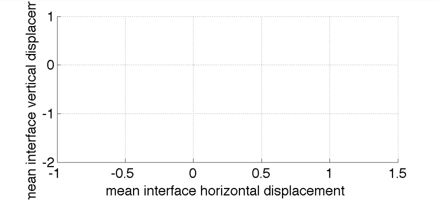






Left to do ...

$\epsilon < \mathrm{Re} < 1$

Considering higher order approximation (in $\epsilon = \frac{l}{L}$), $\langle u_i \rangle = \langle u_i^{(0)} \rangle + \epsilon \langle u_i^{(1)} \rangle$ and $\langle p \rangle = \langle p^{(0)} \rangle + \epsilon \langle p^{(1)} \rangle$:

Macroscopic level

$$\langle u_i^{(1)} \rangle = -\mathcal{L}_{ijk} \frac{\partial p^{(0)}}{\partial x'_j} \frac{\partial p^{(0)}}{\partial x'_k} - \mathcal{M}_{ijk} \frac{\partial^2 p^{(0)}}{\partial x'_j \partial x'_k} - \mathcal{K}_{ij} \frac{\partial p_0^{(1)}}{\partial x'_j}$$

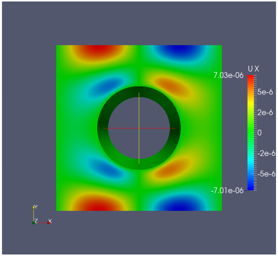
$\epsilon < \text{Re} < 1$

Microscopic level

$$\begin{cases} \frac{\partial L_{ijk}}{\partial x_i} = 0 \\ \frac{\partial B_{jk}}{\partial x_i} - \frac{\partial L_{ijk}}{\partial x_g \partial x_g} = K_{lj} \frac{\partial K_{ik}}{\partial x_l} \end{cases} \begin{cases} \frac{\partial M_{ijk}}{\partial x_i} = -K_{kj} \\ \frac{\partial C_{jk}}{\partial x_i} - \frac{\partial M_{ijk}}{\partial x_g \partial x_g} = -A_j \delta_{ik} + 2 \frac{\partial K_{ij}}{\partial x_k} \end{cases}$$

$$L_{ijk} = S_{ij} = T_j = 0, \; M_{ijk} = -\frac{V}{|\Gamma|} \langle K_{kj} \rangle n_i \quad on \; \Gamma,$$

$$L_{ijk}, \; M_{ijk}, \; B_{jk}, \; C_{jk}, \; S_{ij}, T_j \; V\text{-}periodic$$



$$\langle L_{ijk} \rangle = 0$$

Nield and Bejan (2006),
Skjetne and Auriault (1999)

Macroscopic simulations: the homogenized model

$$(\sigma_{ij}n_j)n_i\big|_{NS} = (\sigma_{ij}n_j)n_i\big|_{BR} \qquad (\sigma_{ij}n_j)t_i\big|_{NS} = (\sigma_{ij}n_j)t_i\big|_{BR}$$

can be written as

$$\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\Bigg|_{NSE} = \frac{\mu_e}{\mu}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)\Bigg|_{BRINK}$$

and

$$\left(-p + \frac{1}{2\mathrm{Re}}\frac{\partial u_3}{\partial x_3}\right)\Bigg|_{NSE} = \left(-p + \frac{\mu_e}{2\mu\mathrm{Re}}\frac{\partial u_3}{\partial x_3}\right)\Bigg|_{BRINK}$$

Macroscopic simulations: the homogenized model

Case 1

- $$\langle u_i^{(0)} \rangle = -\mathcal{K}_{ij}\epsilon^2 \mathrm{Re}_L \frac{\partial p^{(0)}}{\partial x'_j}$$

- $$p\big|_{NS} = p^{(0)}\Big|_{DARCY}$$

  $$u_i\big|_{NS} = u_i^{(0)}\Big|_{DARCY}$$

  imposed at $y_{ITF} - \delta$

  $$\delta = c\sqrt{\frac{\mathcal{K}}{\theta}}$$

  Le Bars & Worster (2006).

Case 2

- $$\langle u_i^{(0)} \rangle = -\mathcal{K}_{ij}\epsilon^2 \mathrm{Re}_L \frac{\partial p^{(0)}}{\partial x'_j} + \\ +\mathcal{K}_{ij}\epsilon^2 \frac{\mu_e}{\mu} \nabla^2 \langle u_j^{(0)} \rangle$$

- $$(\sigma_{ij} n_j) n_i\big|_{NS} = (\sigma_{ij} n_j) n_i\big|_{BR}$$
  $$(\sigma_{ij} n_j) t_i\big|_{NS} = (\sigma_{ij} n_j) t_i\big|_{BR}$$

  $$u_i\big|_{NS} = u_i^{(0)}\Big|_{BR}$$

  imposed at y_{ITF}.

Macroscopic simulations: Re_L=100