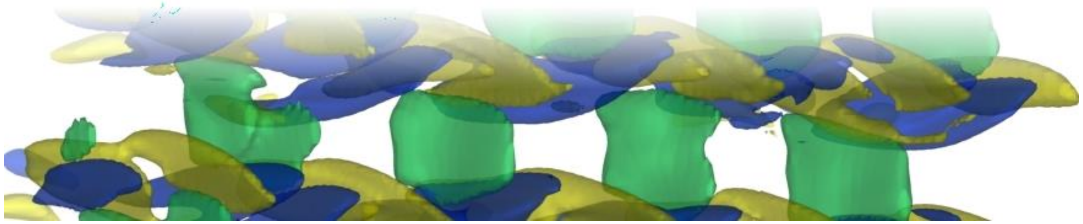

Non-normality and non-linearity in fluid systems

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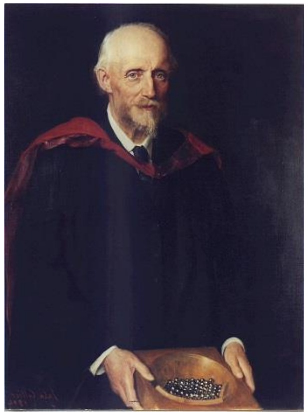
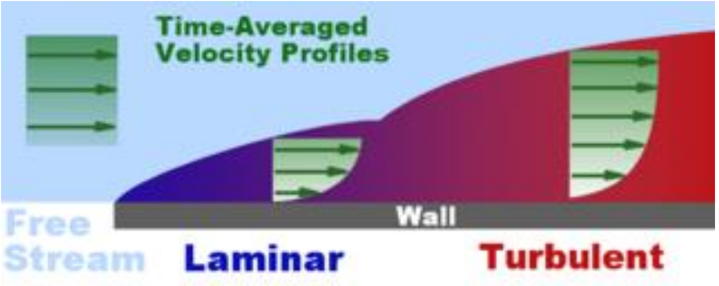
UNIVERSITÀ DEGLI STUDI DI GENOVA

Transition to turbulence, a burning question for 100+ years …



The simplest problem: incompressible boundary layer

What happens/why?



http://en.wikipedia.org/wiki/Boundary_layer_transition

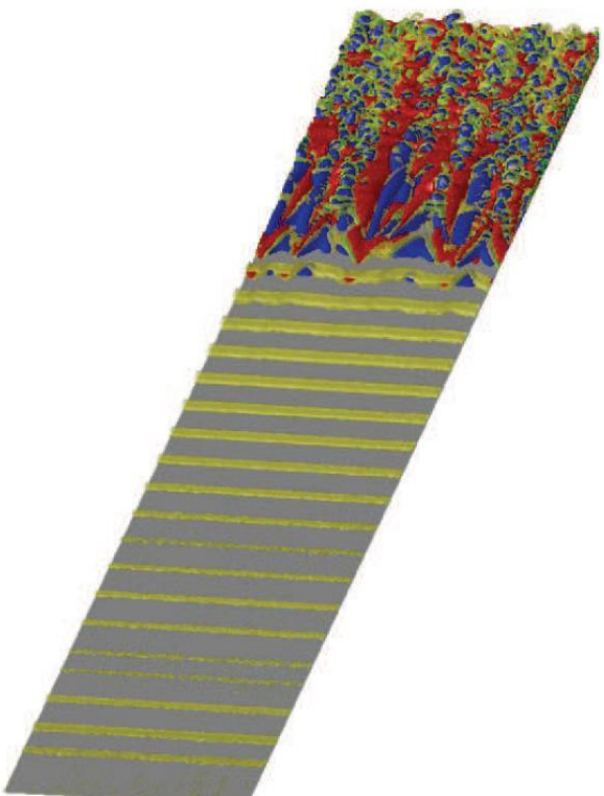
"… the concept of boundary layer transition is a complex one and still lacks a complete theoretical exposition."

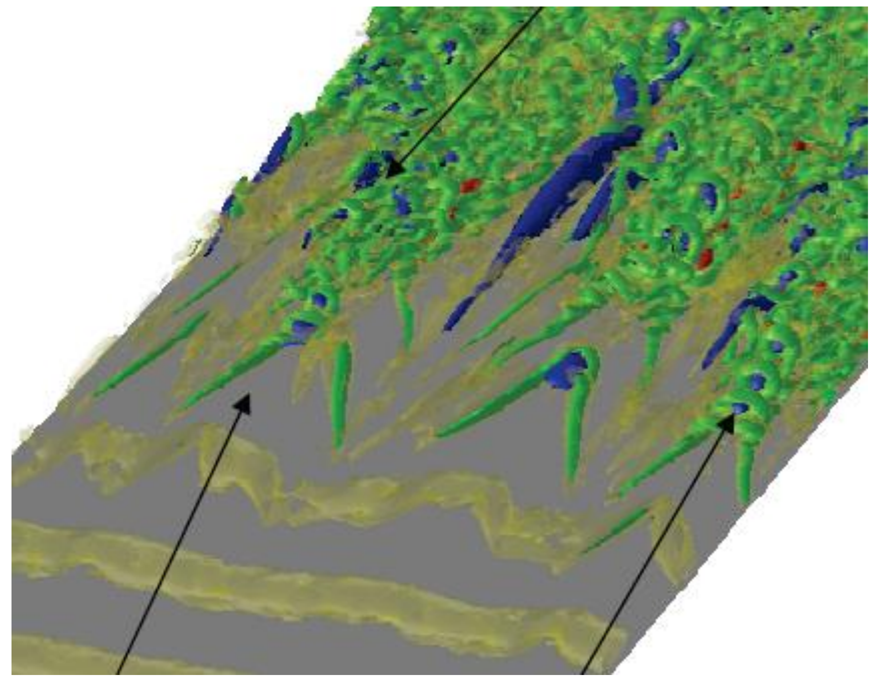
What we know already

- 2D TS waves

<u>**SUPERCRITICAL TRANSITION**</u>

(for ‘small’ disturbance levels)



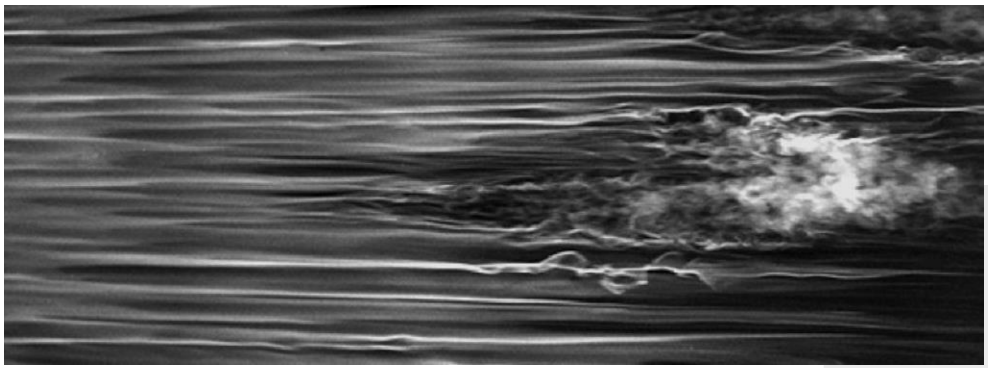


Λ-vortices hairpin vortices

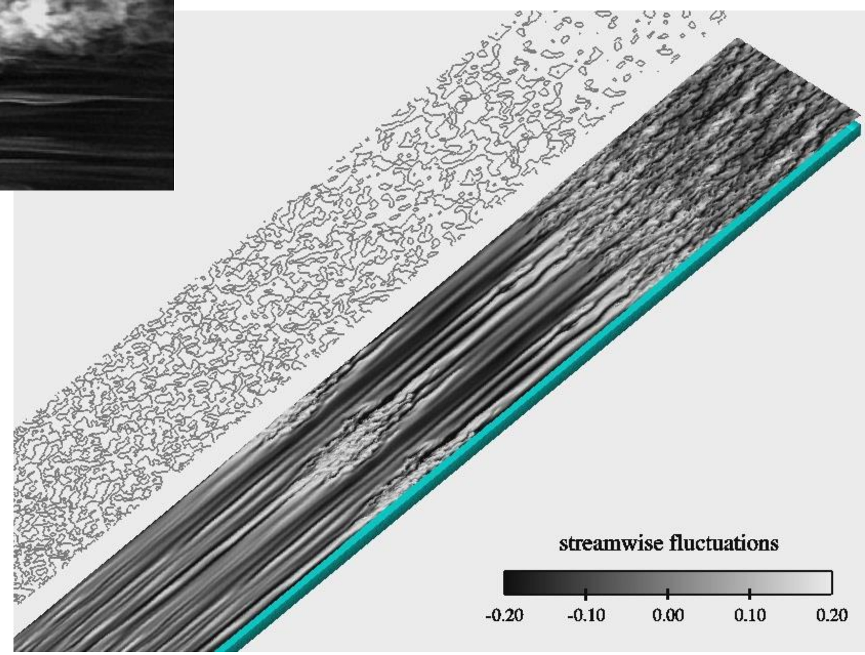
Schlatter, 2009

What we know already

- Emmons (1951) spots, induced by free-stream turbulence



Matsubara & Alfredsson, 2005



SUBCRITICAL (BYPASS) TRANSITION
(for 'large' Tu disturbance levels)

Zaki & Durbin, 2005

What we know already

Classical linear stability theory provides $\mathrm{Re}_{\mathrm{crit}}$ above which one eigenmode is unstable, but it seems to be of little use ...

	Poiseuille	Couette	Hagen-Poiseuille	Square duct
$\mathrm{Re}_{\mathrm{crit}}$	5772	∞	∞	∞
$\mathrm{Re}_{\mathrm{trans}}$	~ 2000	~ 400	~ 2000	~ 2000

What we know already

Can ‘optimal perturbations ’ (because of non-normal evolution operator) explain bypass transition?

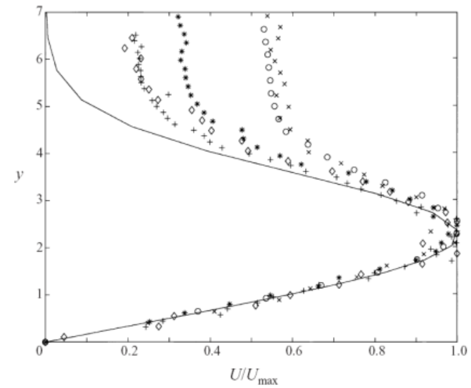
- Linear (based on B/L scalings):

    Andersson, Bergreen, Henningson, 1999

    Luchini, 2000

- Nonlinear (based or not on B/L scalings):

    Zuccher, Luchini, Bottaro, 2004



… but **α = 0** streaks are not good at kicking transition

Waleffe, 1995

Andersson et al., 2001

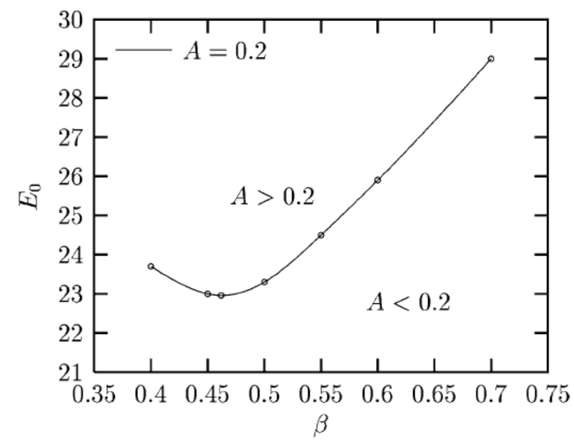
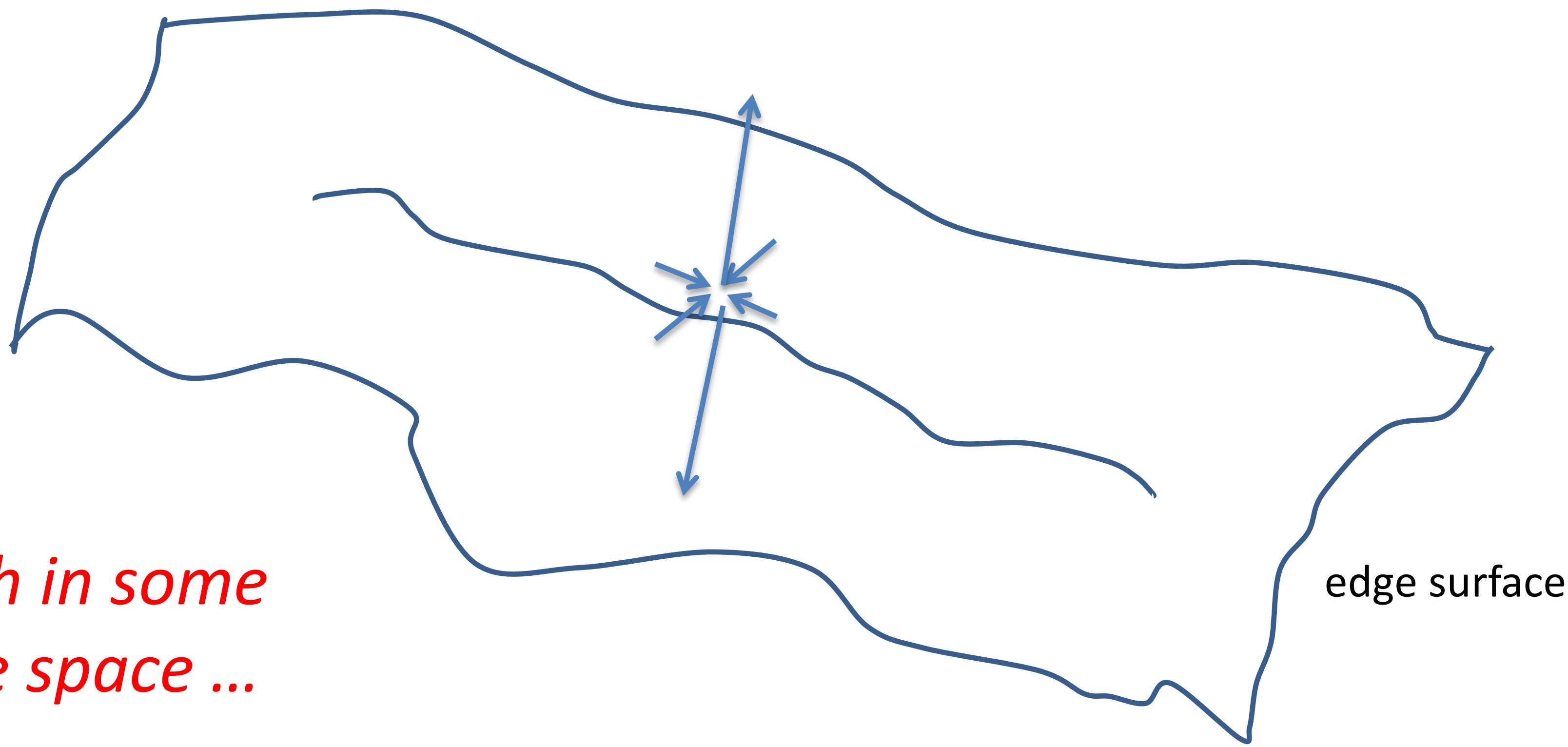


Fig. 12. Curve of initial perturbation energy E_0 as a function of β for which $A = 0.2$ somewhere in the domain.

What we know already

What about ECS, saddles, edge states, etc.?

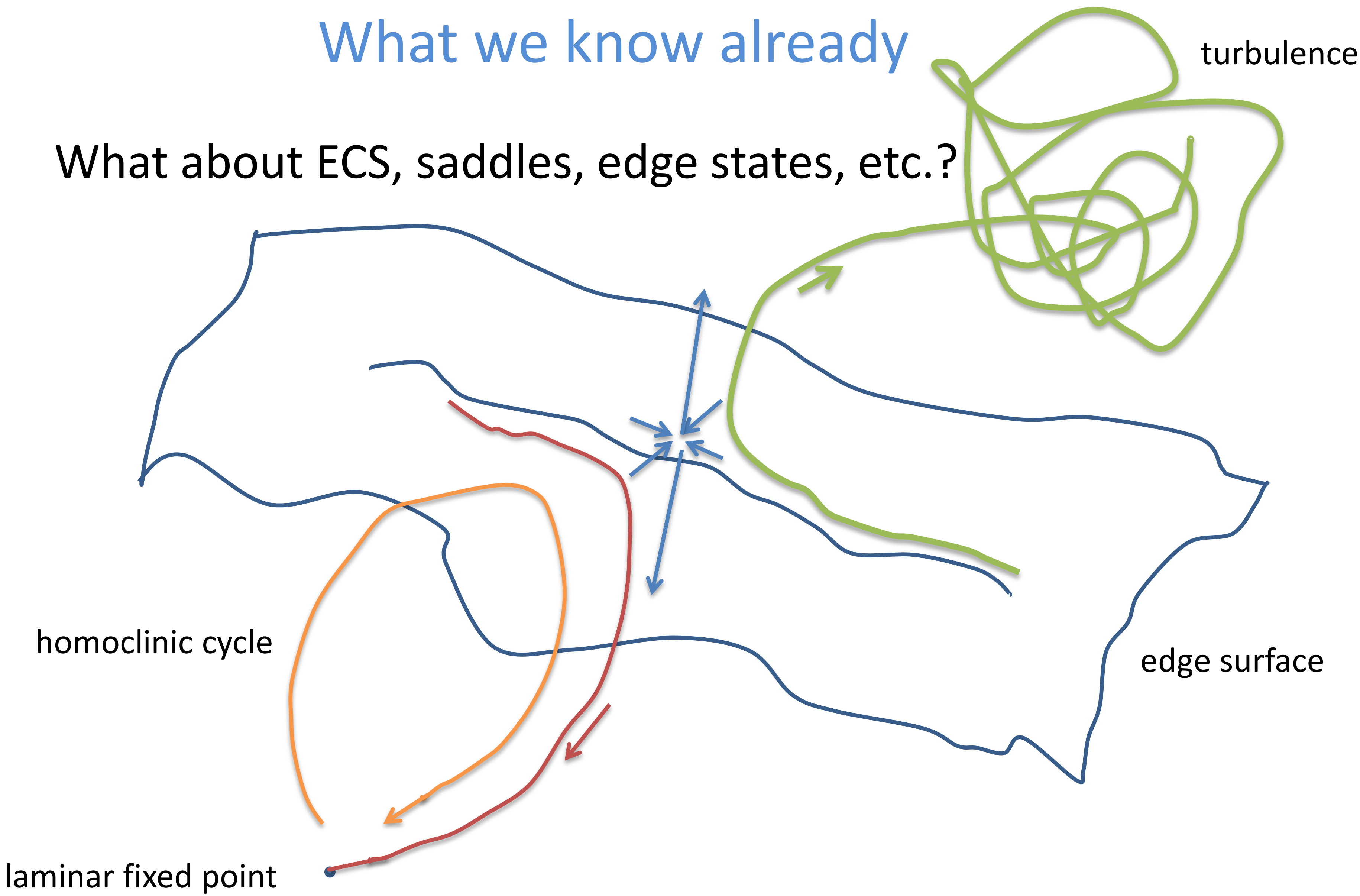


Sketch in some phase space …

laminar fixed point •

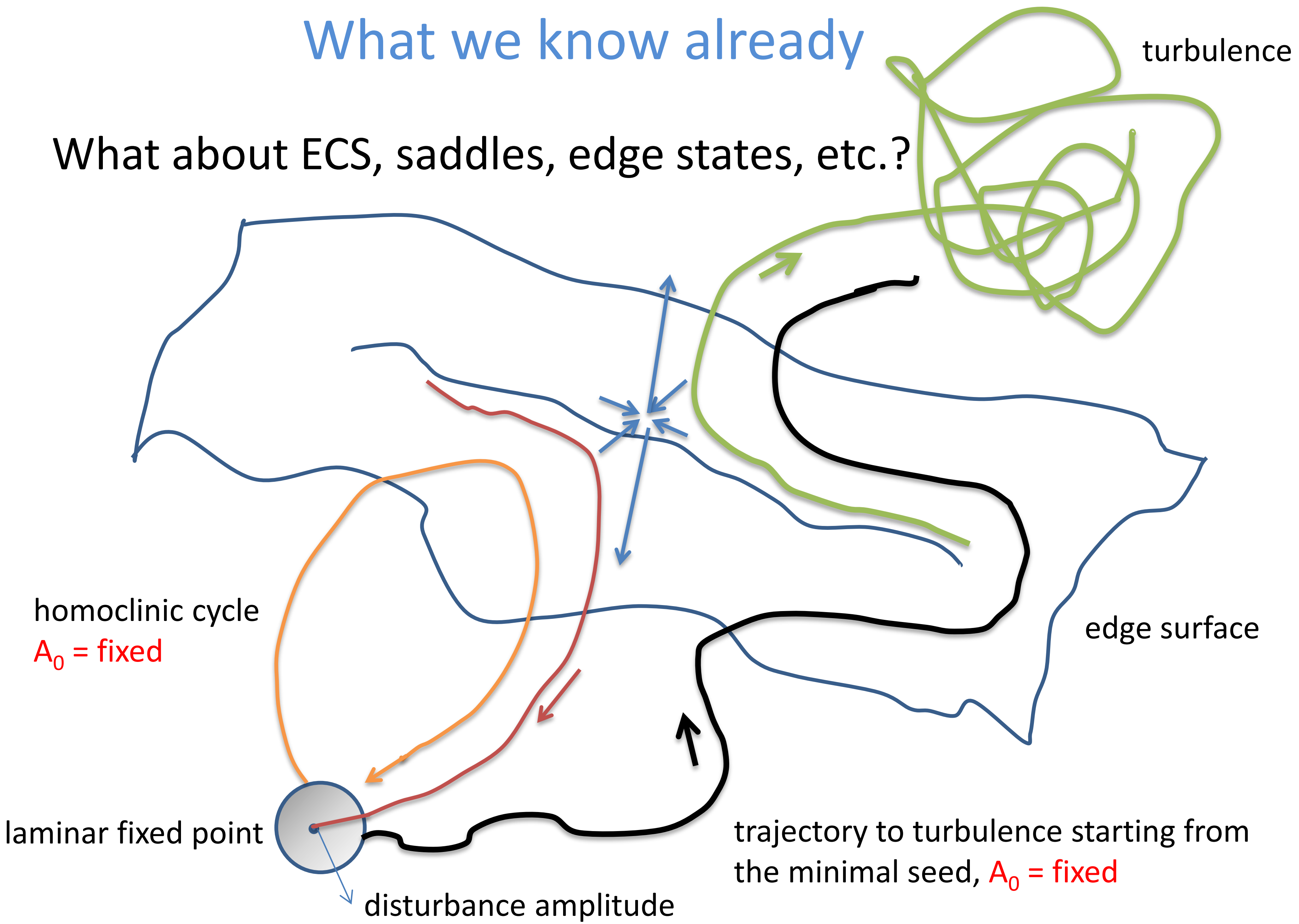
What we know already

What about ECS, saddles, edge states, etc.?



What we know already

What about ECS, saddles, edge states, etc.?



Non-normality

- A linear operator is *non-normal* if it does not commute with its adjoint:

  take the linear system $du/dt = L\,u$,

  with adjoint $-\,dv/dt = L^{\dagger}\,v$;

  if $L\,L^{\dagger} \neq L^{\dagger}\,L$ the operator L is *non-normal*

- Most hydrodynamic stability operators are non-normal

Non-normality

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if $L\,L^{\dagger} \neq L^{\dagger}\,L$ the operator L is *non-normal*

- Most hydrodynamic stability operators are non-normal

And so what?

Effects of non-normality

(let us move to finite dimensional – i.e. computational – space)

- Eigenvectors may form a complete set but are not orthogonal
- This allows for *transient energy growth* even when all eigenvalues are damped

Ex. $d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u}$ $\quad \mathbf{A} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -2\varepsilon \end{pmatrix}$ $\quad \mathbf{A}^* = \begin{pmatrix} -\varepsilon & 0 \\ 1 & -2\varepsilon \end{pmatrix}$

$0 < \varepsilon \ll 1$

$$\mathbf{u} = \Sigma_k \; c_k \, \mathbf{u_k} \, e^{\lambda_k t}$$

2 damped e-values

$\lambda_1 = -\varepsilon \qquad \mathbf{u}_1 = (1 \quad 0)^T$

$\lambda_2 = -2\varepsilon \qquad \mathbf{u}_2 = (1 \quad -\varepsilon)^T$ $\qquad$ (**A** is a disturbance of a Jordan block)

Energy $\qquad E = \mathbf{u} \cdot \mathbf{u} = \Sigma_k \, \Sigma_h \, \overline{c_k} \, c_h \, e^{(\overline{\lambda_k} + \lambda_h)\,t} \; \mathbf{u}_k \cdot \mathbf{u}_h$

Effects of non-normality

(let us move to finite dimensional – i.e. computational – space)

- If e-vectors are orthogonal (and orthonormal, so that $\mathbf{u}_k \cdot \mathbf{u}_h = \delta_{hk}$):

$$E = \mathbf{u} \cdot \mathbf{u} = \Sigma_k \Sigma_h \; \overline{c_k} \, c_h \, e^{(\overline{\lambda_k} + \lambda_h)\, t} \; \mathbf{u}_k \cdot \mathbf{u}_h = \Sigma_k \; \|c_k\|^2 \, e^{2\, Re(\lambda_k)\, t}$$

Energy is the sum of the energy of all individual eigenmodes;

if all modes are damped, then any arbitrary disturbance is damped

Effects of non-normality

(let us move to finite dimensional – i.e. computational – space)

- If e-vectors are orthogonal (and orthonormal, so that $\mathbf{u}_k \cdot \mathbf{u}_h = \delta_{hk}$):

$$E = \mathbf{u} \cdot \mathbf{u} = \Sigma_k \Sigma_h \; \overline{c_k} \, c_h \, e^{(\overline{\lambda_k} + \lambda_h)\, t} \; \mathbf{u}_k \cdot \mathbf{u}_h = \Sigma_k \; \|c_k\|^2 \, e^{2\,\mathrm{Re}(\lambda_k)\, t}$$

Energy is the sum of the energy of all individual eigenmodes;

if all modes are damped, then any arbitrary disturbance is damped

- If e-vectors are not orthogonal (as in the simple 2 x 2 example for which $\mathbf{u}_1 \cdot \mathbf{u}_2 = 1$ and the e-vectors are almost parallel):

$$\mathbf{u} = c_1 \, \mathbf{u}_1 \, e^{-\varepsilon t} + c_2 \, \mathbf{u}_2 \, e^{-2\varepsilon t}$$

$$E = \mathbf{u} \cdot \mathbf{u} = c_1^2 \, e^{-2\varepsilon\, t} + c_2^2 \, (1 + \varepsilon^2) \, e^{-4\varepsilon\, t} + 2\, c_1 \, c_2 \, e^{-3\varepsilon\, t}$$

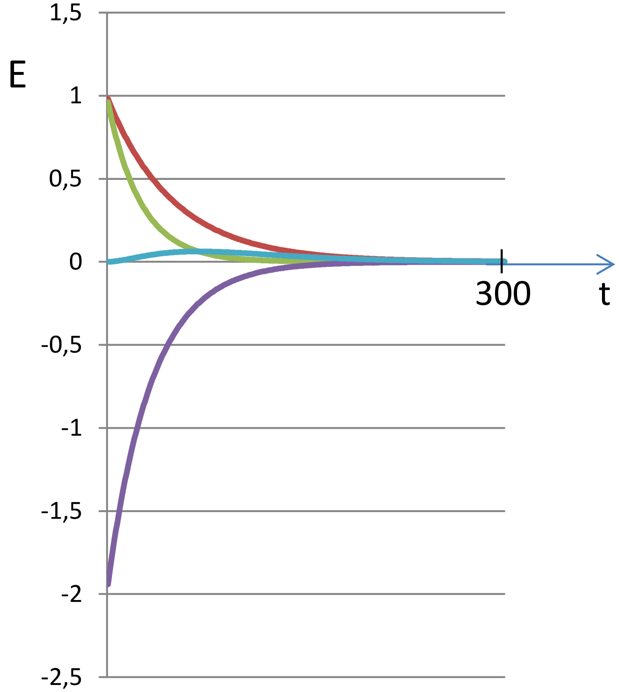
Effects of non-normality

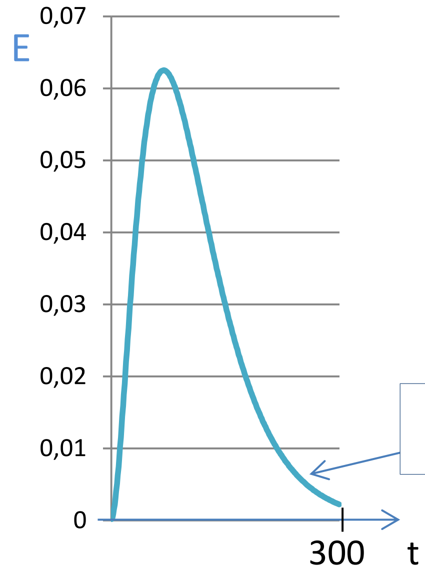
- e-vectors are not orthogonal:

$$\mathbf{u} = c_1\, \mathbf{u}_1\, e^{-\varepsilon t} + c_2\, \mathbf{u}_2\, e^{-2\varepsilon t}$$

$$E = \mathbf{u} \cdot \mathbf{u} \;=\; c_1^2\; e^{-2\varepsilon\, t} \;+\; c_2^2\; (1 + \varepsilon^2)\, e^{-4\varepsilon\, t} \;+\; \underbrace{2\, c_1 c_2}\; e^{-3\varepsilon\, t}$$

the product $c_1 c_2$ can be negative ...





$c_1 = 1$

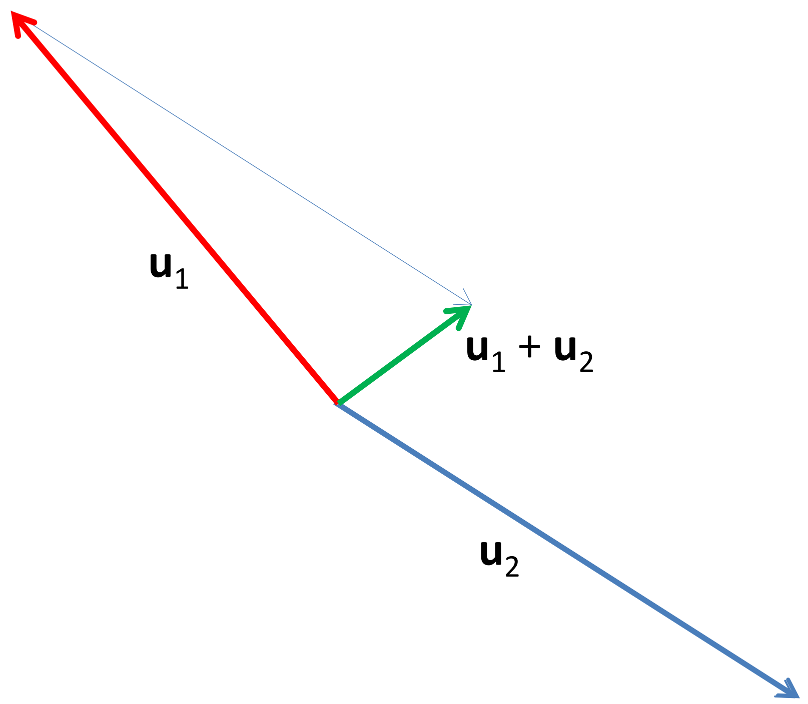
$c_2 = -1$

$\varepsilon = 0.01$

exponential decay for $t \rightarrow \infty$

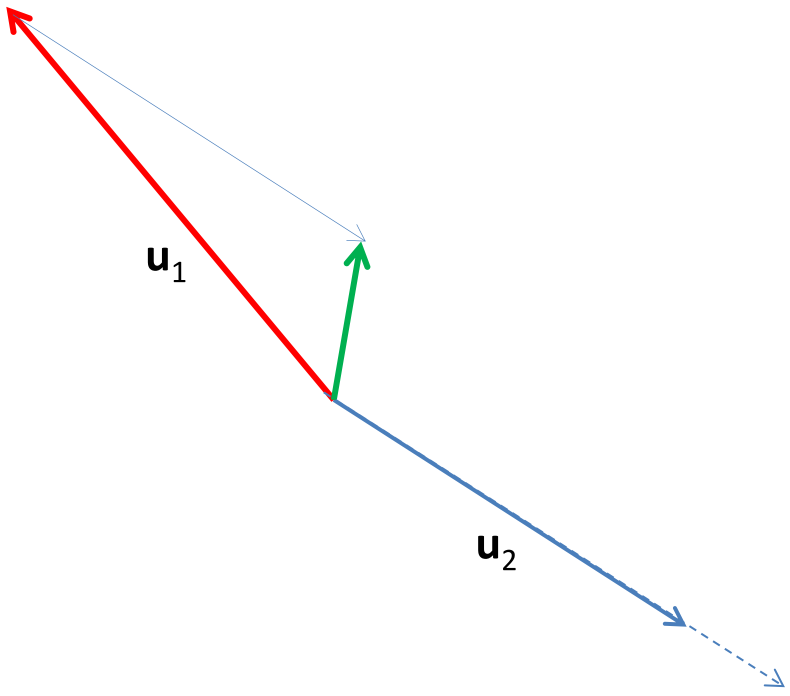
Effects of non-normality

- e-vectors are not orthogonal:



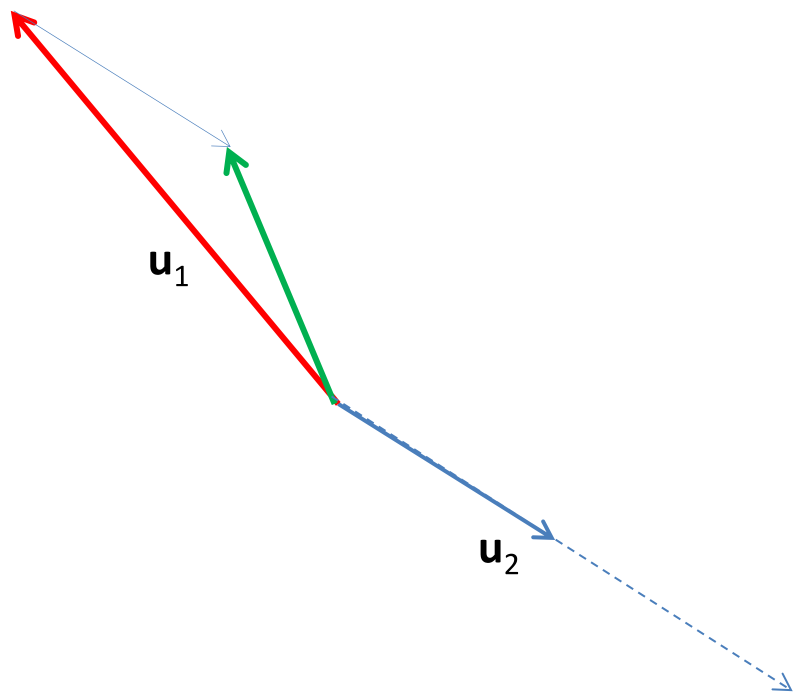
Effects of non-normality

- e-vectors are not orthogonal:



Effects of non-normality

- e-vectors are not orthogonal:

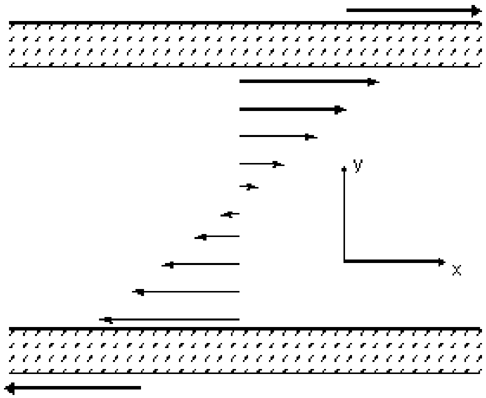


Interesting conclusion

- **In a linear system which is non-normal an arbitrary disturbance can grow transiently (for early times) even when all eigenmodes are damped**
- Bearings onto hydrodynamic stability problems ruled by non-normal operators → much work has gone on in the past twenty years on transient growth of perturbations (particularly for parallel and quasi-parallel shear flows, but recently also for other types of instabilities, including thermoacoustics, *cf.* the recent review by R. I. Sujith, M. P. Juniper & P. J. Schmid, "Non-Normality and Nonlinearity in Thermoacoustic Instabilities", *International Journal of Spray and Combustion Dynamics*, in press, 2015.)

Use variational analysis to find **optimal initial disturbances** which grow the most over a given time stretch

Prototype problem: **plane Couette flow**



$$P_x = \frac{1}{Re} U_{yy}$$

base flow: $\begin{bmatrix} U(y) \\ 0 \\ 0 \\ P(x) \end{bmatrix}$ = (y, 0, 0, 0)^T

disturbance: $\epsilon \begin{bmatrix} u_{11}(y,t) \\ v_{11}(y,t) \\ w_{11}(y,t) \\ p_{11}(y,t) \end{bmatrix} e^{i(\alpha x + \beta z)}$ + c.c.

The disturbance equations are:

$$i\alpha u_{11} + v_{11_y} + i\beta w_{11} = 0,$$

$$u_{11_t} + i\alpha(U \qquad)u_{11} + v_{11}(U \qquad)_y + \qquad i\alpha p_{11} = \frac{1}{Re}\Delta_k u_{11},$$

$$v_{11_t} + i\alpha(U \qquad)v_{11} \qquad + p_{11_y} = \frac{1}{Re}\Delta_k v_{11},$$

$$w_{11_t} + i\alpha(U \qquad)w_{11} + \qquad i\beta p_{11} = \frac{1}{Re}\Delta_k w_{11},$$

$$\Delta_k = \partial^2/\partial y^2 - \alpha^2 - \beta^2$$

Objective: $$e(T) = \frac{\epsilon^2}{2}\int_{-1}^{1}(u_{11}\bar{u}_{11} + v_{11}\bar{v}_{11} + w_{11}\bar{w}_{11})dy\Big|_{t=T}$$

$$G(\epsilon, Re, \alpha, \beta, T) = \frac{e(T)}{e(0)}$$

Iterative optimization (adjoint looping)

- Direct problem: $du/dt = \mathbf{A}\,\mathbf{u}$ + maximize $G = \dfrac{\mathbf{u}(T)\,.\,\mathbf{u}(T)}{\mathbf{u}(0)\,.\,\mathbf{u}(0)}$

$$\mathbf{A} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -2\varepsilon \end{pmatrix} \qquad \mathbf{A^*} = \begin{pmatrix} -\varepsilon & 0 \\ 1 & -2\varepsilon \end{pmatrix} = \bar{\mathbf{A}}^T$$

$$\lambda_1 = -\varepsilon \qquad \mathbf{u}_1 = (1 \quad 0)^T \qquad \mathbf{v}_1 = (\varepsilon \quad 1)^T$$

$$\lambda_2 = -2\varepsilon \qquad \mathbf{u}_2 = (1 \quad -\varepsilon)^T \qquad \mathbf{v}_2 = (0 \quad 1)^T$$

$$\mathbf{U} = \left(\begin{array}{c|c} 1 & 1 \\ 0 & -\varepsilon \end{array}\right) \qquad \mathbf{V} = \left(\begin{array}{c|c} \varepsilon & 0 \\ 1 & 1 \end{array}\right)$$

Iterative optimization (adjoint looping)

- Direct problem: $d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u}$ + maximize $G = \dfrac{\mathbf{u}(T)\cdot\mathbf{u}(T)}{\mathbf{u}(0)\cdot\mathbf{u}(0)}$

We can easily show that $\mathbf{u}(t) = \mathbf{P}(t)\,\mathbf{u}(0)$

P *propagator* of the initial condition defined by:

$$\mathbf{P}(t) = \mathbf{U}\, e^{\Lambda t}\, \mathbf{U}^{-1} = \mathbf{U}\, e^{\Lambda t}\, \mathbf{V}^{T}$$

so that $$G = \frac{\mathbf{P}\,\mathbf{u}(0)\cdot\mathbf{P}\,\mathbf{u}(0)}{\mathbf{u}(0)\cdot\mathbf{u}(0)} = \frac{\overline{\mathbf{u}}(0)^{T}\,\overline{\mathbf{P}}^{T}\,\mathbf{P}\,\mathbf{u}(0)}{\mathbf{u}(0)\cdot\mathbf{u}(0)} = \frac{\mathbf{u}(0)\cdot\overline{\mathbf{P}}^{T}\,\mathbf{P}\,\mathbf{u}(0)}{\mathbf{u}(0)\cdot\mathbf{u}(0)}$$

Iterative optimization (adjoint looping)

- Direct problem: $d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u}$ + maximize $G = \dfrac{\mathbf{u}(T)\cdot\mathbf{u}(T)}{\mathbf{u}(0)\cdot\mathbf{u}(0)}$

  The Rayleigh quotient $G = \dfrac{\mathbf{u}(0)\cdot\bar{\mathbf{P}}^T\mathbf{P}\,\mathbf{u}(0)}{\mathbf{u}(0)\cdot\mathbf{u}(0)}$ yields the largest

  gain G as the largest (real) e-value of the problem:

  $$\bar{\mathbf{P}}^T\mathbf{P}\,\mathbf{u}_0 = G\,\mathbf{u}_0$$

  $\bar{\mathbf{P}}^T\mathbf{P}$ symmetric

  The corresponding $\mathbf{u}_0$ is the optimal (initial) perturbation

  The optimal output at time T is: $\mathbf{u}(T) = \mathbf{P}(T)\,\mathbf{u}_0 = G\,(\mathbf{P}^*)^{-1}\,\mathbf{u}_0$

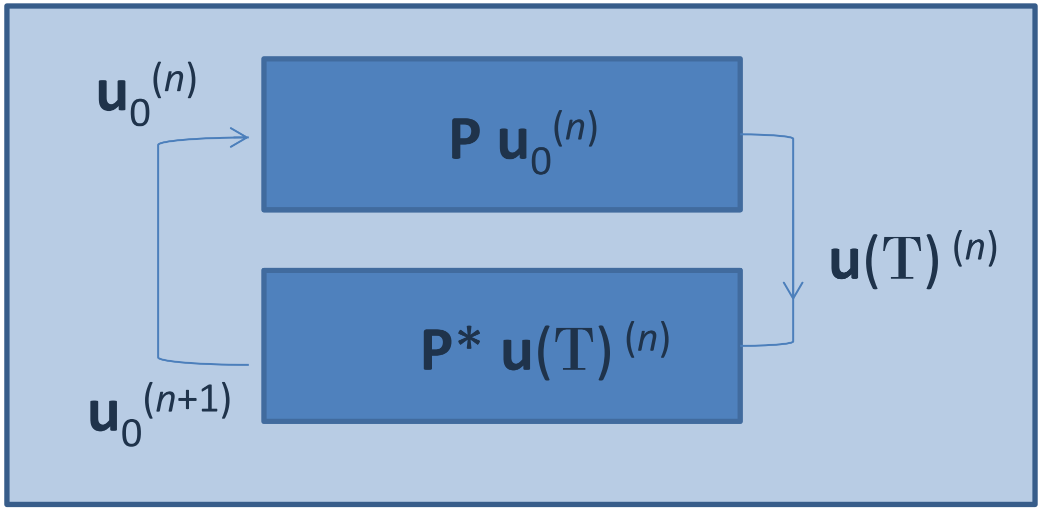
Iterative optimization (**adjoint looping**)

- Direct problem: $d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u}$ + maximize $G = \dfrac{\mathbf{u}(T)\,.\,\mathbf{u}(T)}{\mathbf{u}(0)\,.\,\mathbf{u}(0)}$

- Adjoint problem: $-\,d\mathbf{v}/dt = \bar{\mathbf{A}}^{T}\,\mathbf{v} = \mathbf{A}^{*}\,\mathbf{v}$

$$\mathbf{P}^{*}\,\mathbf{P}\,\mathbf{u}_0 = G\,\mathbf{u}_0$$

(most often it is not easy to compute **P** ...)



back to Couette flow …

- The adjoint equations are linear and can be easily obtained from the direct equations via integrations by parts
- Optimal disturbances are quasi-streamwise vortices, with $G = 0.00118\ Re^2$, $\alpha = 35/Re$, $\beta = 1.60$, at $T = 0.117\ Re$, which transform into elongated streaks downstream.

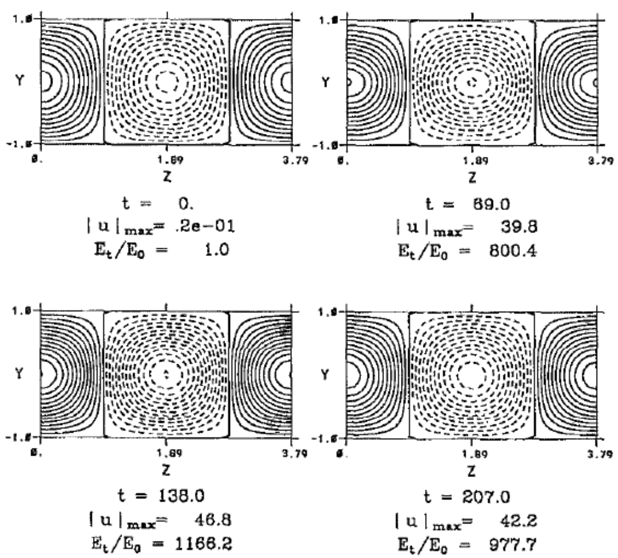


FIG. 4. Development of the perturbation streamwise velocity u for the best growing perturbation independent of x in Couette flow with $R = 1000$, located at $\beta = 1.66$, $\tau = 138$. Values are normalized by the maximum value of v at time $t = 0$.

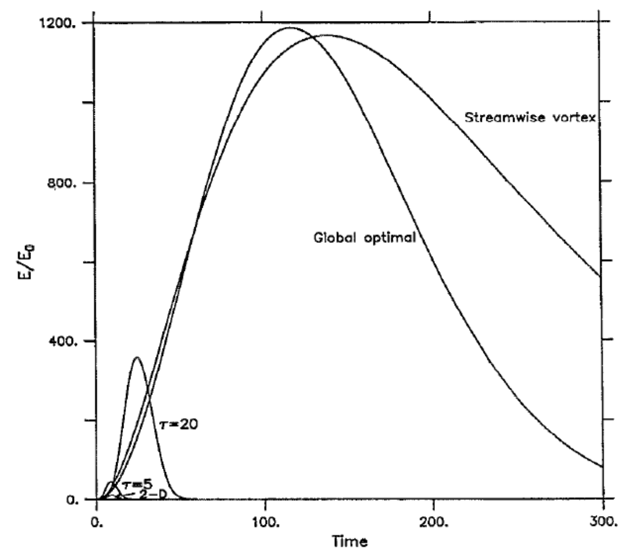


FIG. 8. Energy growth versus time for the global optimal, the streamwise vortex, and 2-D perturbation which grow the most, and perturbations which grow the most in 5 and 20 advective time units in Couette flow with $R = 1000$.

Three-dimensional optimal perturbations in viscous shear flow

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Problem is … this does not work!

- The optimal perturbations DO NOT set up a flow field which undergoes transition easily. Other (suboptimal) disturbances undergo transition at much smaller initial energy levels E_0 …

Another way to look at it: Lagrange multipliers

$d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u} \quad + \quad$ b.c.

Functional: $\mathcal{L} = \mathbf{u}(T) \cdot \mathbf{u}(T)$

Constraint: $\mathbf{u}(0) \cdot \mathbf{u}(0) = E_0$ (imposed)

Scalar product: $\mathbf{a} \cdot \mathbf{b} = \bar{\mathbf{a}}^T \mathbf{b}$

$$\max \mathcal{L} \;\rightarrow\; \max \mathcal{F} = \mathcal{L} + \int_0^T \mathbf{v} \cdot (d\mathbf{u}/dt - \mathbf{A}\,\mathbf{u})\, dt + \\ + a\,(\mathbf{u}(0) \cdot \mathbf{u}(0) - E_0)$$

v and a are Lagrange multipliers

Another way to look at it: Lagrange multipliers

$$\max \mathcal{L} \;\rightarrow\; \max \mathcal{F} \;=\; \mathcal{L} + \int_0^T \left(\bar{\mathbf{v}}^T d\mathbf{u}/dt - \bar{\mathbf{v}}^T \mathbf{A}\, \mathbf{u} \right) dt \; + $$

$$+\; a \left(\bar{\mathbf{u}}_0^T \mathbf{u}_0 - E_0 \right)$$

$$\delta\mathcal{F} = 0$$

$$\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\, \delta\mathbf{u} = 0 =$$

$$= \bar{\mathbf{u}}_T^T\, \delta\, \mathbf{u}_T + \int_0^T \left(\bar{\mathbf{v}}^T d\, \delta\, \mathbf{u}/dt - \bar{\mathbf{v}}^T \mathbf{A}\, \delta\mathbf{u} \right) dt + a\, \bar{\mathbf{u}}_0^T\, \delta\mathbf{u}_0 =$$

$$= \mathbf{u}_T . \delta\, \mathbf{u}_T + \int_0^T \left(-\, d\mathbf{v}/dt . \delta\, \mathbf{u} - \bar{\mathbf{A}}^T \mathbf{v} . \delta\mathbf{u} \right) dt + a\, \mathbf{u}_0 . \delta\mathbf{u}_0 + \left[\mathbf{v} . \delta\mathbf{u} \right]_0^T$$

Another way to look at it: Lagrange multipliers

Adjoint problem: $-d\mathbf{v}/dt = \bar{\mathbf{A}}^T\mathbf{v}$

Initial condition: $\mathbf{v}_T = -\mathbf{u}_T$

Initial condition direct problem: $\mathbf{u}_0 = \mathbf{v}_0/a$

$(d\mathbf{u}/dt = \mathbf{A}\,\mathbf{u})$

with the scalar a chosen so that $\mathbf{u}_0 \cdot \mathbf{u}_0 = E_0$

Direct-adjoint loop typically converges fast; it is stopped when $(\mathbf{u}_T \cdot \mathbf{u}_T)^{n+1} - (\mathbf{u}_T \cdot \mathbf{u}_T)^n <$ tolerance set

Another way to look at it: Lagrange multipliers

What are **adjoints** good for?

THEY PROVIDE **SENSITIVITY MAPS** (crucial for sensitivity, receptivity, data assimilation, optimal and robust control, etc.)

An Introduction to Adjoint Problems

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Supplemental appendix to Adjoint Equations in Stability Analysis, *Annu. Rev. Fluid Mech.* **46**:493–517 (2014)

Another way to look at it: Lagrange multipliers

IMPORTANT

Lagrangian approach can be easily extended to perform **NONLINEAR** optimization.

Adding nonlinear terms …

Model problem: $du/dt = \mathbf{A}\,\mathbf{u} + \|\mathbf{u}\|\,\mathbf{B}\,\mathbf{u}$

$\mathbf{A} = \begin{pmatrix} -\varepsilon & 1 \\ 0 & -2\varepsilon \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ where **B** is chosen to be energy preserving, *i.e.* $\overline{\mathbf{u}}^T\mathbf{B}\mathbf{u} = 0$ (to mimick the nonlinear terms of NS eqs.)

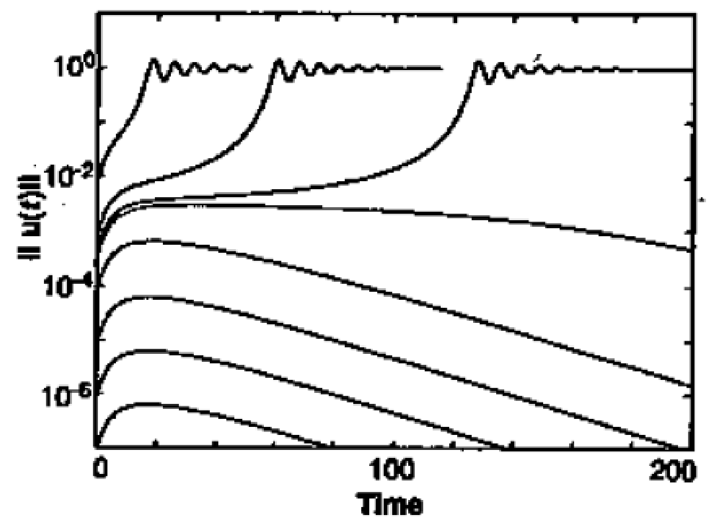


Fig. 10. $\|\mathbf{u}(t)\|$ for solutions to the nonlinear 2 × 2 model problem of Eq. 14 with initial amplitudes $\|\mathbf{u}(0)\| = 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}, 4 \times 10^{-4}, 5 \times 10^{-4}, 10^{-3}$, and 10^{-2}. The threshold amplitude is $\|\mathbf{u}(0)\| = 4.22 \times 10^{-4}$.

Hydrodynamic Stability Without Eigenvalues

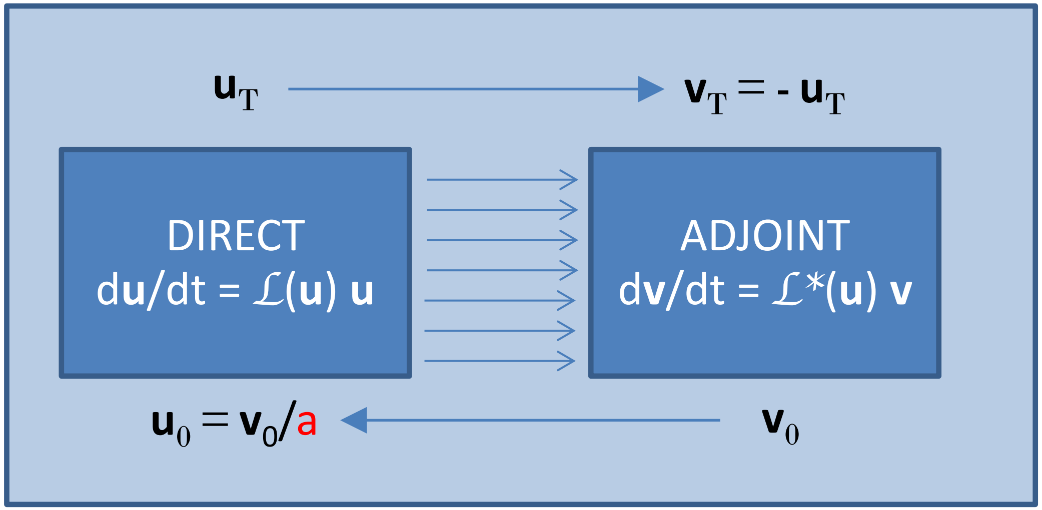
Lloyd N. Trefethen, Anne E. Trefethen, Satish C. Reddy, Tobin A. Driscoll

Adding nonlinear terms …

The adjoint equation becomes:

$$- d\mathbf{v}/dt = \mathbf{A}^* \, \mathbf{u} - \|\mathbf{u}\| \, \mathbf{B}^* \, \mathbf{v}$$

and optimization can be carried out just like in the nonlinear case, i.e. adjoint looping



Couette flow

$$\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v} \otimes \vec{v}) + \nabla p = \frac{1}{Re} \nabla^2 \vec{v} + \vec{f},$$
$$\nabla \cdot \vec{v} = \dot{m}.$$

$$\mathscr{L} = \frac{E(T)}{E(0)} - \int_0^T \left\langle \boldsymbol{u}^\dagger, \left\{ \frac{\partial \boldsymbol{u}'}{\partial t} + \boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{U} + \boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{u}' + \boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}' + \boldsymbol{\nabla} p' - \frac{\nabla^2 \boldsymbol{u}'}{Re} \right\} \right\rangle \mathrm{d}t$$
$$- \int_0^T \left\langle p^\dagger, \boldsymbol{\nabla} \cdot \boldsymbol{u}' \right\rangle \mathrm{d}t - \lambda \left(\frac{E_0}{E(0)} - 1 \right).$$

$$\frac{\partial \vec{f}^\dagger}{\partial t} + \vec{v} \cdot \left(\nabla \vec{f}^\dagger + \nabla \vec{f}^{\dagger\mathrm{T}} \right) + \nabla \dot{m}^\dagger + \frac{1}{Re} \nabla^2 \vec{f}^\dagger = -\mathcal{D}_{\vec{v}} J,$$
$$\nabla \cdot \vec{f}^\dagger = -\mathcal{D}_p J,$$

Couette flow

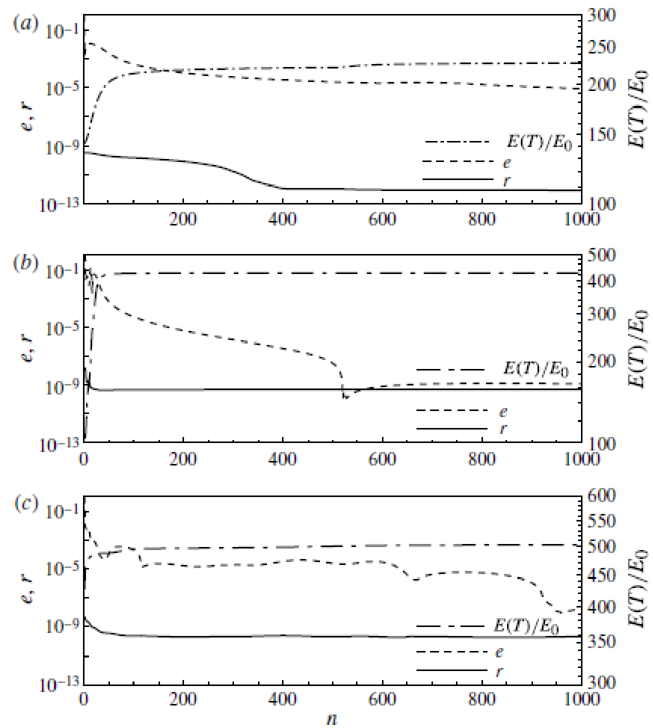


FIGURE 1. Convergence history for a nonlinear optimization at target time (*a*) $T = 50$ with $E_0 = 0.005$, (*b*) $T = 30$ with $E_0 = 0.025$, and (*c*) $T = 30$ with $E_0 = 0.1$. Solid line, residual; dashed line, error on the objective function; dot-dashed line, value of the energy gain.

Nonlinear optimal perturbations in a Couette flow: bursting and transition

S. Cherubini[1,2,†] and P. De Palma[2]

Couette flow

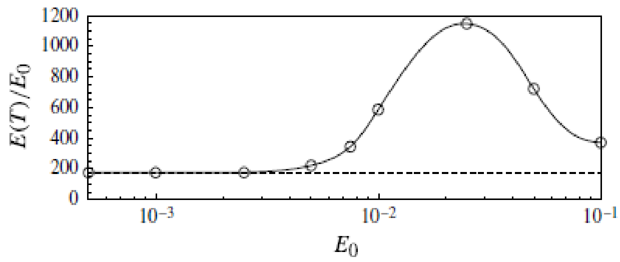


FIGURE 3. Optimal energy gain at target time $T = 50$ for different values of the initial energy E_0. The dashed line represents the linear optimization result.

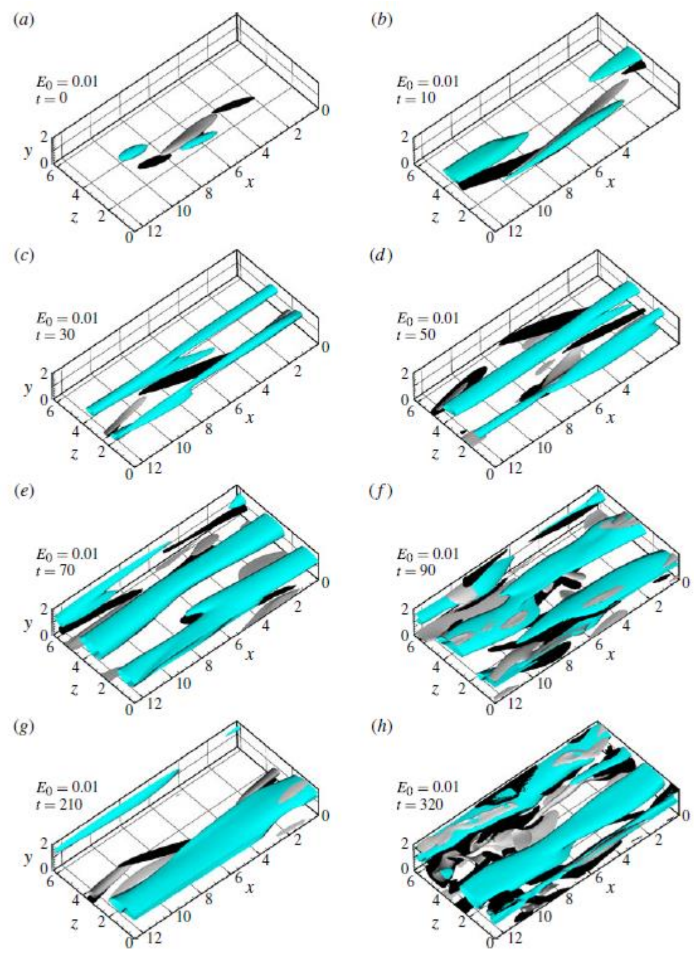


FIGURE 10. (Colour online) Snapshots of the time evolution of the nonlinear optimal perturbation obtained for $E_0 = 0.01$ and $T = 50$: iso-surfaces of the perturbations (grey, blue online, for the negative streamwise component of the velocity; black and pale grey for negative and positive streamwise vorticity, respectively) at (a) $t = 0$, (b) $t = 10$, (c) $t = 30$, (d) $t = 50$, (e) $t = 70$, (f) $t = 90$, (g) $t = 210$ and (h) $t = 320$. Surfaces for (a,b) $u' = -0.015$, $\omega'_x = \pm 0.5$, (c) $u' = -0.025$, $\omega'_x = \pm 0.75$, (d–h) $u' = -0.035$, $\omega'_x = \pm 0.75$.

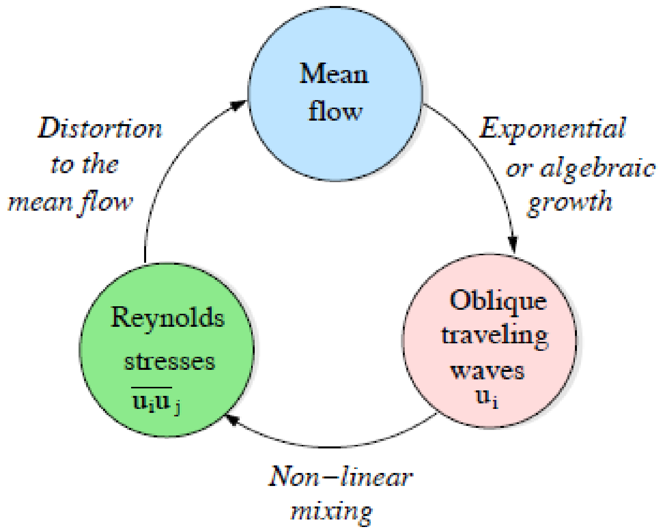
Couette flow

KEY POINTS:

1. Nonlinear optimals are very different from linear ones
2. A complete parametric study is impossible,because of
    - large parametric space
    - each direct problem is a full DNS

Alternative: **weakly nonlinear optimization of transition**

WNL optimals



The simplest possible triple development

$$\begin{bmatrix} U(y) \\ 0 \\ 0 \\ P(x) \end{bmatrix} + \epsilon \begin{bmatrix} \tilde{u}(x,y,z,t) \\ \tilde{v}(x,y,z,t) \\ \tilde{w}(x,y,z,t) \\ \tilde{p}(x,y,z,t) \end{bmatrix} + \epsilon^2 \begin{bmatrix} u_{00}(y,t) \\ v_{00}(y,t) \\ w_{00}(y,t) \\ p_{00}(y,t) \end{bmatrix}$$

$$(\tilde{\mathbf{u}}, \tilde{p})(x,y,z,t) = (\mathbf{u}_{11}, p_{11})(y,t)e^{i(\alpha x + \beta z)} + (\overline{\mathbf{u_{11}}}\ ,\overline{p_{11}})(y,t)e^{-i(\alpha x + \beta z)},$$

WNL optimals

$\mathcal{O}(\epsilon)$

$$i\alpha u_{11} + v_{11_y} + i\beta w_{11} = 0,$$

$$u_{11t} + i\alpha(U + \epsilon^2 u_{00})u_{11} + v_{11}(U + \epsilon^2 u_{00})_y + i\beta(\epsilon^2 w_{00})u_{11} + i\alpha p_{11} = \frac{1}{Re}\Delta_k u_{11},$$

$$v_{11t} + i\alpha(U + \epsilon^2 u_{00})v_{11} + i\beta(\epsilon^2 w_{00})v_{11} + p_{11_y} = \frac{1}{Re}\Delta_k v_{11},$$

$$w_{11t} + i\alpha(U + \epsilon^2 u_{00})w_{11} + v_{11}(\epsilon^2 w_{00_y}) + i\beta(\epsilon^2 w_{00})w_{11} + i\beta p_{11} = \frac{1}{Re}\Delta_k w_{11},$$

$$\Delta_k = \partial^2/\partial y^2 - k^2 \text{ and } k^2 = \alpha^2 + \beta^2$$

WNL optimals

$\mathcal{O}(\epsilon^2)$

$$v_{00} = 0,$$

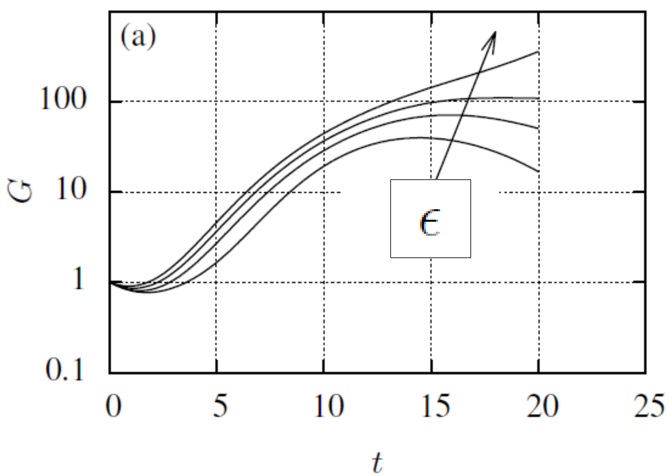
$$u_{00t} - \frac{1}{Re} u_{00yy} = -[v_{11}\overline{u}_{11y} + i\beta w_{11}\overline{u}_{11} + c.c.],$$

$$p_{00y} = -[i\alpha u_{11}\overline{v}_{11} + v_{11}\overline{v}_{11y} + i\beta w_{11}\overline{v}_{11} + c.c.],$$

$$w_{00t} - \frac{1}{Re} w_{00yy} = -[i\alpha u_{11}\overline{w}_{11} + v_{11}\overline{w}_{11y} + c.c.],$$

WNL optimals

$$e(T) = \frac{\epsilon^2}{2} \int_{-1}^{1} (u_{11}\overline{u_{11}} + v_{11}\overline{v_{11}} + w_{11}\overline{w_{11}})dy \Big|_{t=T}$$



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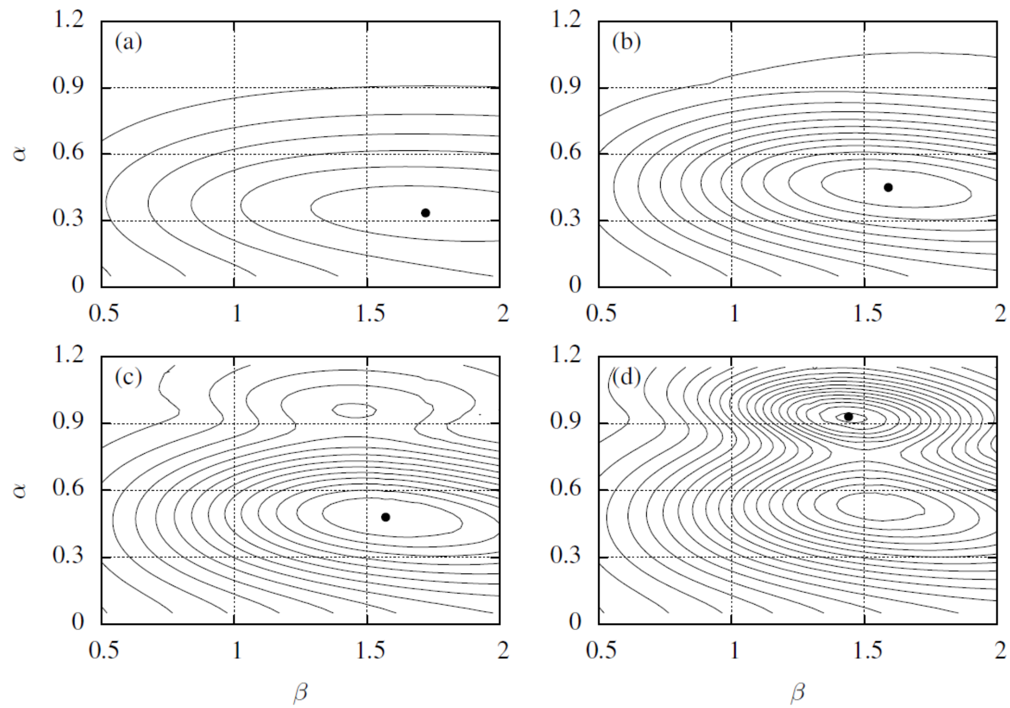

FIGURE 6. Gain G in the $\alpha - \beta$ plane for $Re = 400$ and $T = 20$; (a) $\epsilon \to 0$ with contour values from 20 to 120, (b) $\epsilon = 0.0145$ with contour values from 20 to 240, (c) $\epsilon = 0.0153$ with contour values from 20 to 300, (d) $\epsilon = 0.01603$ with contour values from 20 to 380. The interval among adjacent isolines is ΔG=20 in frames. The maximum value of G for each ϵ is denoted by a filled circle.

What of industrial aspects of thermoacoustics??????????

What of thermoacoustic instabilities in combustion chambers**???**

Non-normality and Nonlinearity in Thermoacoustic Instabilities

R. I. Sujith[1], M. P. Juniper[2] & P. J. Schmid[3]

Thermoacoustics

- Can a combustor sustain limit-cycle oscillations even when its base flow is linearly stable?

Thermoacoustics

- Can a combustor sustain limit-cycle oscillations even when its base flow is linearly stable?
- "***Triggering***" is driven by nonlinearities and nonnormality!

Thermoacoustics

- **NN** and **NL** stems from the **convective terms** in the governing equations and it has been known since Dowling (1996) that convective terms should not be discarded (Ma → 0 limit is questionable)
- **NN** and **NL** effects are present also in the **flame-acoustic interaction term**
- Any **model** of annular (or other) combustor must account for nonlinearity and modal interactions

Work in progress

- Much has been accomplished, in particular by the groups of Sujith and Juniper, for the Rjike tube
- There is much scope for investigating more complex configurations, with low order models like LOTAN/LOMTI or higher fidelity simulations (COMSOL, OpenFOAM)
- *Bottleneck*: unsteady heat release model (progress along the lines of Maria's FDF)

A new perspective on the flame describing function of a matrix flame

Maria Heckl