Chapter 3: Potential flow theory

Ideal-fluid flow

Ideal fluids are inviscid and incompressible

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \mathbf{f}$$

$\mathbf{v} \cdot \mathbf{n} = \mathbf{U}_{\text{body}} \cdot \mathbf{n}$ on solid boundaries,

i.e. the body surface is a streamline.

Ideal-fluid flow

KCT: if the flow of an *ideal* fluid is initially irrotational (say, the flow upstream of a body is uniform) it will remain irrotational once the fluid particles are near the body, i.e.

$$\zeta = 0 \quad \text{everywhere in the fluid.}$$

Since $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}$ for any scalar function ϕ, the condition of irrotationality is satisfied by

$$\mathbf{v} = \boldsymbol{\nabla}\phi$$

Velocity potential

ϕ: *velocity potential*

irrotational flows $\Longleftrightarrow$ potential flows

$$\nabla^2 \phi = 0$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2} \nabla \phi \cdot \nabla \phi - G = F(t)$$

Two-dimensional potential flows

The function ϕ satisfies the irrotationality constraint. In 2D the streamfunction ψ can be introduced to satisfy automatically the equation of continuity.

In Cartesian coordinates: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and ψ is defined from: $$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

(valid both for rotational and irrotational flows)

$$\zeta = \partial v / \partial x - \partial u / \partial y \quad \Longrightarrow \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Two-dimensional potential flows

Both ϕ and ψ are thus harmonic functions, with *streamlines* and *equipotential lines* orthogonal to one another. Furthermore, we have:

$$\begin{cases} u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} \\ v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

Cauchy-Riemann conditions for $\phi(x, y)$ and $\psi(x, y)$

Complex analysis: let us introduce the ***complex potential*** $F(z)$ defined as:

$$F(z) = \phi(x, y) + i\psi(x\, y)$$

with $z = x + iy$

Analytic functions

$F(z)$ is analytic at $z = z_0 \in \mathbb{C}$ if it admits a power series expansion which converges for all z sufficiently close to z_0.

$$F(z) = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$F(z)$ analytic function $\Longleftrightarrow$ C-R conditions satisfied

$F(z)$ analytic function $\Longleftrightarrow$ dF/dz is a point function which is *independent* of the direction along which it is calculated

Analytic functions

$$W(z) = \frac{dF}{dz} = \frac{\partial F}{\partial x}$$

$$= \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$W(z) = \frac{dF}{dz} = \frac{\partial F}{i\partial y}$$

$$= -i\frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial y}$$

$$\Rightarrow \boxed{W(z) = \frac{dF}{dz} = u - iv}$$

$W(z)$: ***complex velocity***

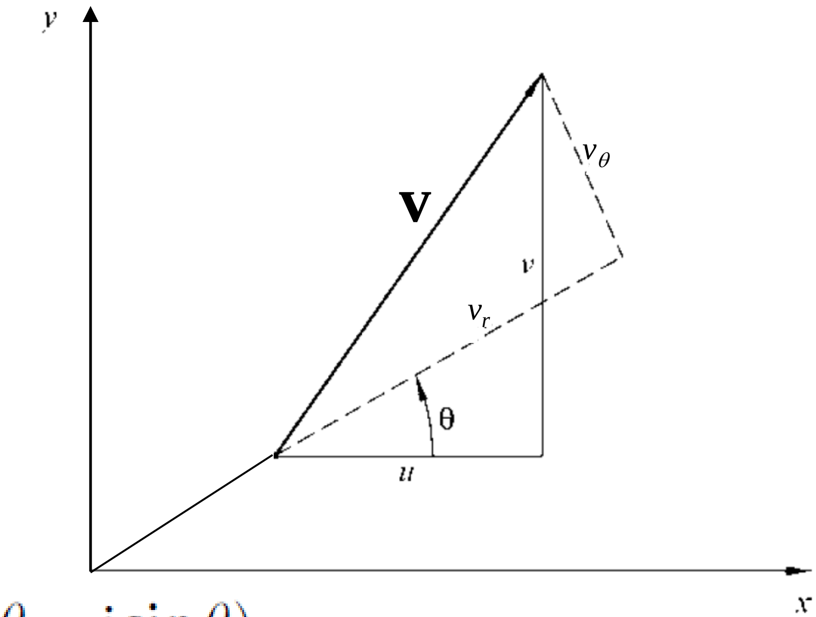
$$W\bar{W} = (u - iv)(u + iv)$$

$$= u^2 + v^2 = \mathbf{v} \cdot \mathbf{v}$$

Analytic functions

Cylindrical coordinates

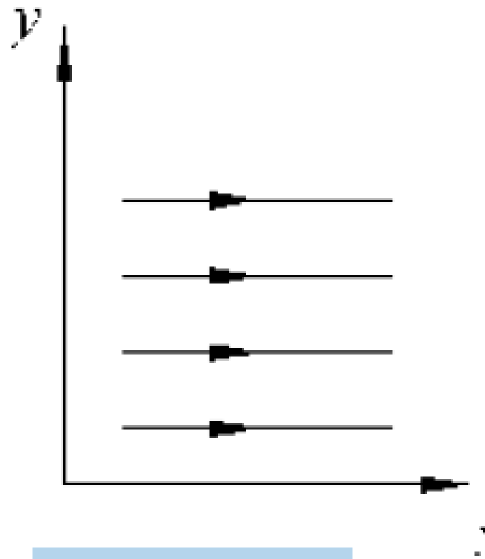
$$u = v_r \cos\theta - v_\theta \sin\theta$$
$$v = v_r \sin\theta + v_\theta \cos\theta$$

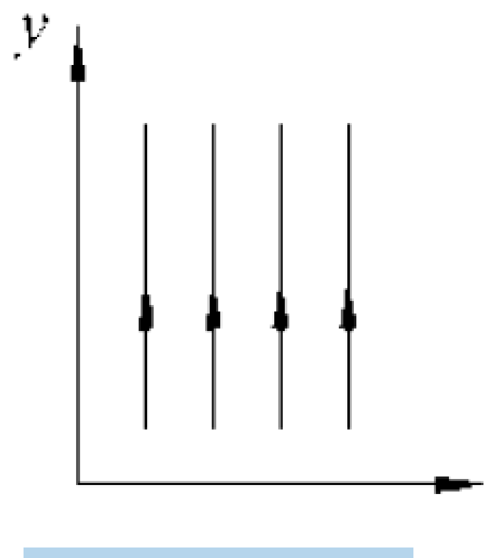


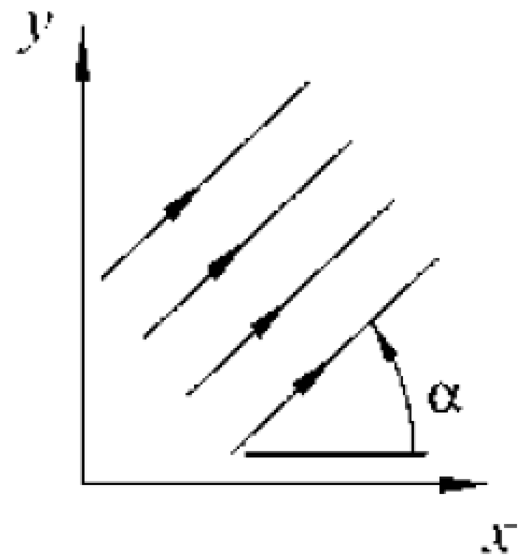
$$W = v_r(\cos\theta - i\sin\theta) - i v_\theta(\cos\theta - i\sin\theta)$$
$$= (v_r - i\, v_\theta)e^{-i\theta}$$

These results are sufficient to establish flow fields represented by simple analytic functions.

Uniform flow







$$F(z) = Uz$$

$$W(z) = u - iv = U$$

$$F(z) = -iVz$$

$$W(z) = u - iv = -iV$$

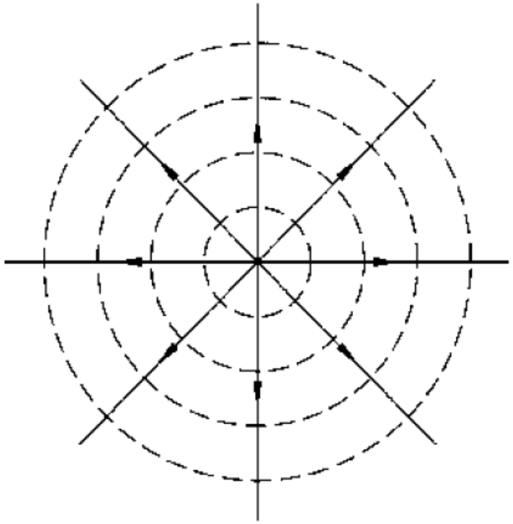
$$F(z) = ce^{-i\alpha}z$$

$$u = c \cos \alpha$$

$$v = c \sin \alpha$$

(U, V, c and α real)

Source, sink and vortex flows



$$F(z) = c\,\log z = c\log re^{i\theta} = c\,\log r + i\,c\,\theta$$

$$\phi = c\log r \qquad \psi = c\theta$$

$$W(z) = \frac{c}{z} = \frac{c}{r}e^{-i\theta}$$

log z multivalued function

$\longrightarrow \quad 0 \le \theta < 2\pi$

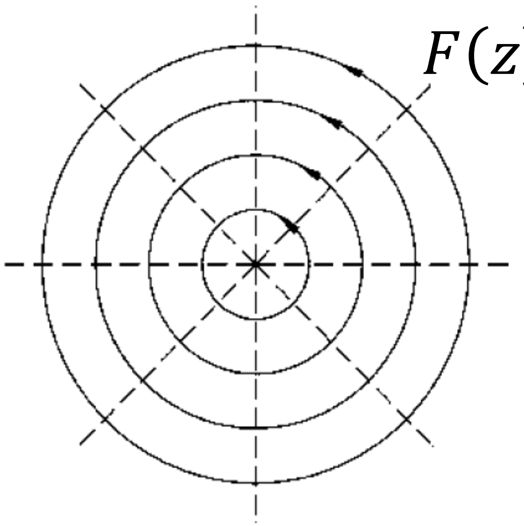
$$v_r = \frac{c}{r} \qquad v_\theta = 0$$

(origin: singular point of ∞ velocity)

$$\dot{V}/_L = \int_0^{2\pi} v_r\, r\, \mathrm{d}\theta = 2\pi c$$

$$F(z) = \frac{\dot{V}/_L}{2\pi}\log z$$

Source, sink and ***vortex*** flows



$$F(z) = -ic \log z = -ic \log re^{i\theta} = -ic \log r + c\,\theta$$

$$\phi = c\theta \qquad \psi = -c \log r$$

$$W(z) = -i\frac{c}{z} = -i\frac{c}{r}\,e^{-i\theta}$$

$$v_r = 0 \qquad v_\theta = \frac{c}{r}$$

(origin: singular point of ∞ velocity)

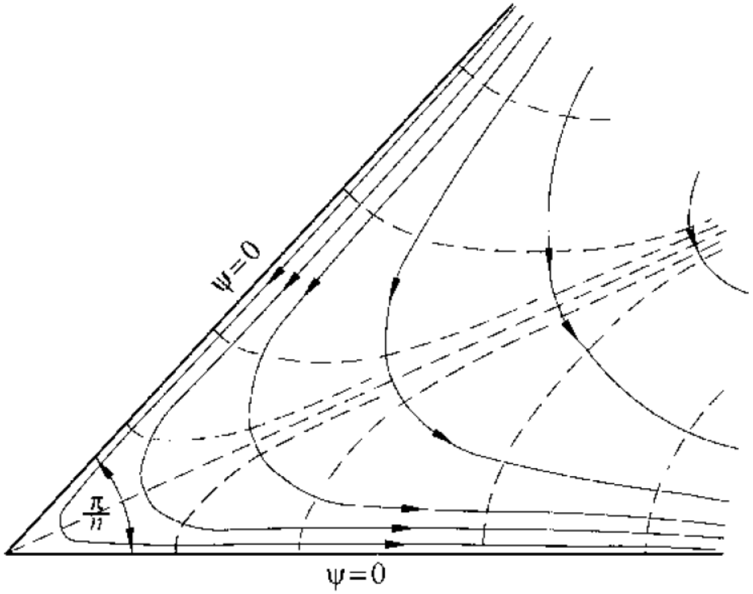
log z multivalued function

$$0 \le \theta < 2\pi$$

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l} = \int_0^{2\pi} v_\theta\, r\, \mathrm{d}\theta = 2\pi c$$

$$F(z) = -i\frac{\Gamma}{2\pi}\log z$$

Flow in a sector



$$F(z) = c\, z^n, \qquad n \geq 1$$

($F(z)$ is a harmonic function)

$$F(z) = c(re^{i\theta})^n = cr^n \cos n\theta + \\ + icr^n \sin n\theta = \\ = \phi + i\psi$$

for $\theta = 0, \pi/n \rightarrow \psi = 0$

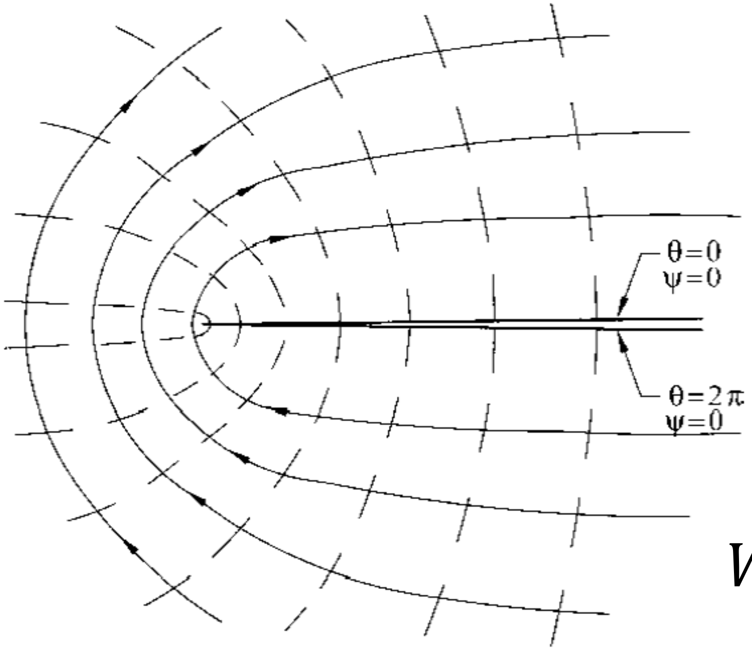
$$W(z) = ncz^{n-1} = (ncr^{n-1} \cos n\theta + incr^{n-1} \sin n\theta) e^{-i\theta}$$

$$\begin{cases} v_r = ncr^{n-1} \cos n\theta \\ v_\theta = -ncr^{n-1} \sin n\theta \end{cases}$$

n = 1 uniform rectilinear flow

n = 2 right-angled corner

Flow around a sharp edge



$$F(z) = c\, z^{1/2} \qquad c \in \Re \quad 0 \le \theta < 2\pi$$

($F(z)$ is a harmonic function)

$$F(z) = c\,(r\,e^{i\theta})^{1/2} = c\,r^{1/2}\cos{\theta}/{2} + $$
$$+\, i\,c\,r^{1/2}\sin{\theta}/{2} = \phi + i\psi$$

$$W(z) = \frac{dF}{dz} = \frac{1}{2}c\,z^{-1/2} = \frac{1}{2}c\,r^{-1/2}e^{-i\theta/2} =$$
$$= \frac{1}{2}c\,r^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)e^{-i\theta}$$

$$\begin{cases} v_r = \dfrac{1}{2}cr^{-1/2}\cos\dfrac{\theta}{2} \\ v_\theta = -\dfrac{1}{2}cr^{-1/2}\sin\dfrac{\theta}{2} \end{cases}$$

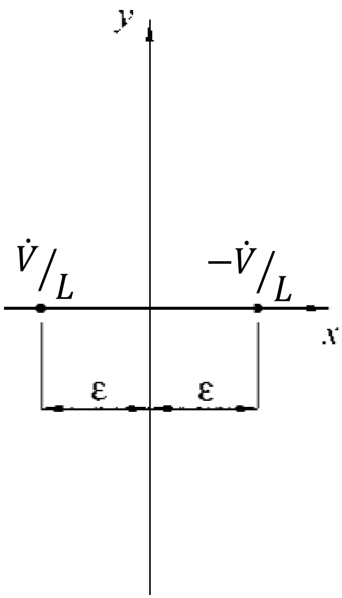
The corner ($r=0$) is a singular point, and the velocity is singular as the square root of the distance from the edge (*Kutta!*).

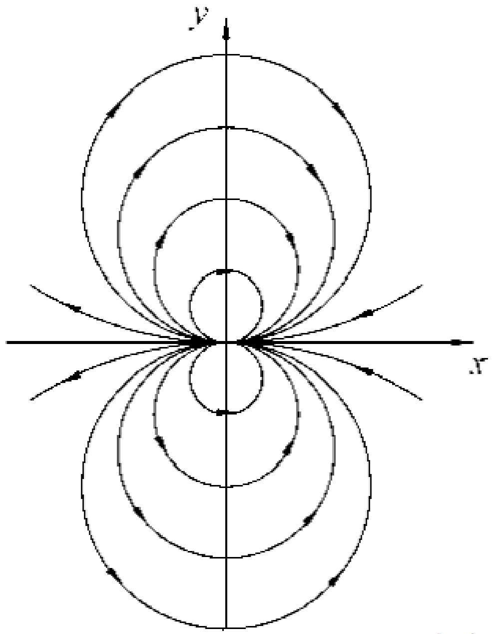
Superposition principle

Linearity of the equations allows superposition of elementary flows to create more complicated flow patterns:

https://youtu.be/4x2g676GgNQ

Doublet





$$F(z) = \frac{\dot{V}/L}{2\pi}\log(z+\varepsilon) - \frac{\dot{V}/L}{2\pi}\log(z-\varepsilon)$$

$$= \frac{\dot{V}/L}{2\pi}\log\left(\frac{z+\varepsilon}{z-\varepsilon}\right)$$

$$= \frac{\dot{V}/L}{2\pi}\log\left(\frac{1+\varepsilon/z}{1-\varepsilon/z}\right)$$

$$F(z) = \frac{\dot{V}/L}{2\pi}\log\left\{\left(1+\frac{\varepsilon}{z}\right)\left[1+\frac{\varepsilon}{z}+O\left(\frac{\varepsilon^2}{z^2}\right)\right]\right\}$$

$$= \frac{\dot{V}/L}{2\pi}\log\left[1+2\frac{\varepsilon}{z}+O\left(\frac{\varepsilon^2}{z^2}\right)\right]$$

Doublet

$$\log(1+\gamma) = \gamma + O(\gamma^2)$$

$$F(z) = \frac{\dot{V}/L}{2\pi}\left[2\frac{\varepsilon}{z} + O\left(\frac{\varepsilon^2}{z^2}\right)\right]$$

$$\lim_{\varepsilon \to 0} \varepsilon \dot{V}/L = \pi\mu,$$ with μ a finite constant

$$F(z) = \frac{\mu}{z}$$

($F(z)$ is another harmonic function)

Doublet

$$F(z) = \frac{\mu}{x+iy}$$

$$= \mu \frac{x-iy}{x^2+y^2}$$

$$\therefore \psi = -\mu \frac{y}{x^2+y^2}$$

Streamlines: ψ = constant

$$x^2 + y^2 + \frac{\mu}{\psi} y = 0$$

$$x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$

Circle of radius $\mu/(2\psi)$ centered in $x = 0$, $y = -\mu/(2\psi)$

$$W(z) = -\frac{\mu}{z^2} = -\frac{\mu}{r^2} e^{-2i\theta} = -\frac{\mu}{r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$$v_r = -\frac{\mu}{r^2}\cos\theta \qquad v_\theta = -\frac{\mu}{r^2}\sin\theta$$

Flow past a circular cylinder

Let us superpose a uniform rectilinear flow to a doublet in the origin

$$F(z) = Uz + \frac{\mu}{z}$$

On a circle of radius $r = a$ we have $z = ae^{i\theta}$ and the complex potential on this circle is

$$F(z) = Uae^{i\theta} + \frac{\mu}{a}e^{-i\theta}$$
$$= \left(Ua + \frac{\mu}{a}\right)\cos\theta + i\left(Ua - \frac{\mu}{a}\right)\sin\theta$$

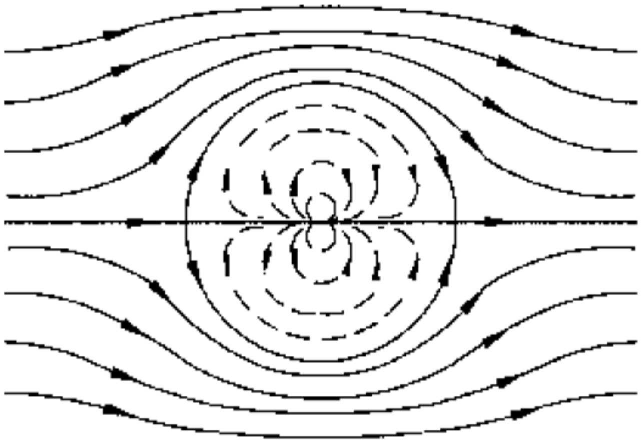
so that the streamfunction on the circle is $\psi = \left(Ua - \frac{\mu}{a}\right)\sin\theta$

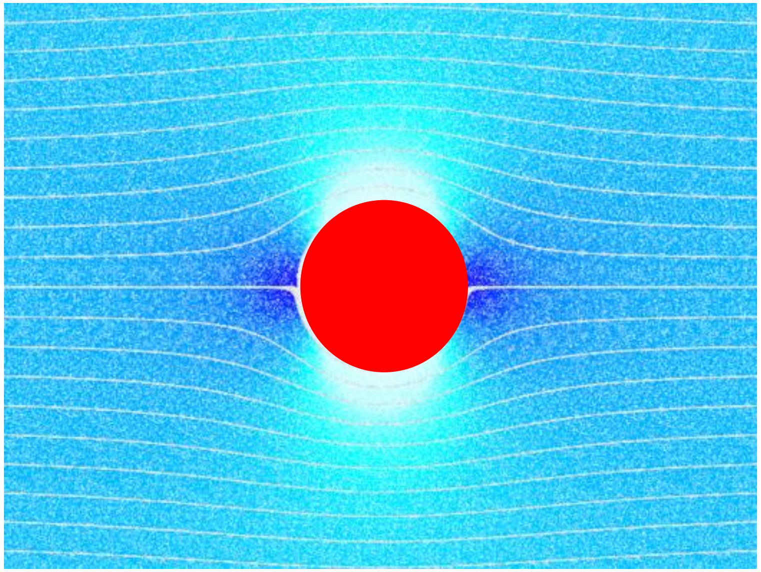
Flow past a circular cylinder

$$\psi = \left(Ua - \frac{\mu}{a} \right) \sin\theta$$

Let us choose the strength of the doublet

$\mu = Ua^2 \longrightarrow \psi(a) = 0$

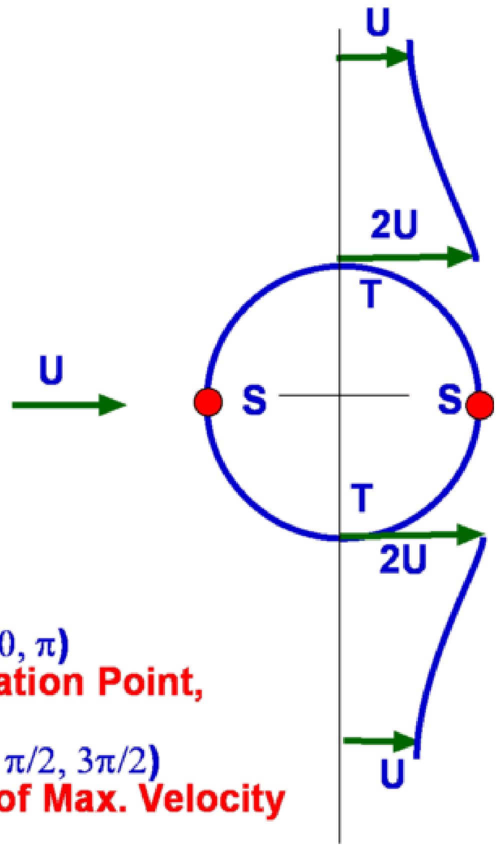




$$F(z) = U(z + a^2/z)$$

Fields are symmetric,
no lift nor drag on cylinder!

Flow past a circular cylinder



$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

From Bernoulli's equation, the surface pressure (in $r = a$) is:

$$p_s = p_\infty + \frac{1}{2}\rho U^2 \underbrace{(1 - 4\sin^2\theta)}_{c_p}$$

Circular cylinder with circulation

Circulation implies lift: let us add a vortex, centered in the origin, to the previous solution:

$$F(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log\frac{z}{a}$$

so that $\psi(a) = 0$, as before.

$$W(z) = U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi}\frac{1}{z} = U\left(1 - \frac{a^2}{r^2}e^{-2i\theta}\right) - \frac{i\Gamma}{2\pi r}e^{-i\theta}$$

$$= \left[U\left(e^{i\theta} - \frac{a^2}{r^2}e^{-i\theta}\right) - \frac{i\Gamma}{2\pi r}\right]e^{-i\theta}$$

Circular cylinder with circulation

$$\ldots = \left\{ U\left(1 - \frac{a^2}{r^2}\right)\cos\theta + i\left[U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi r}\right]\right\} e^{-i\theta}$$

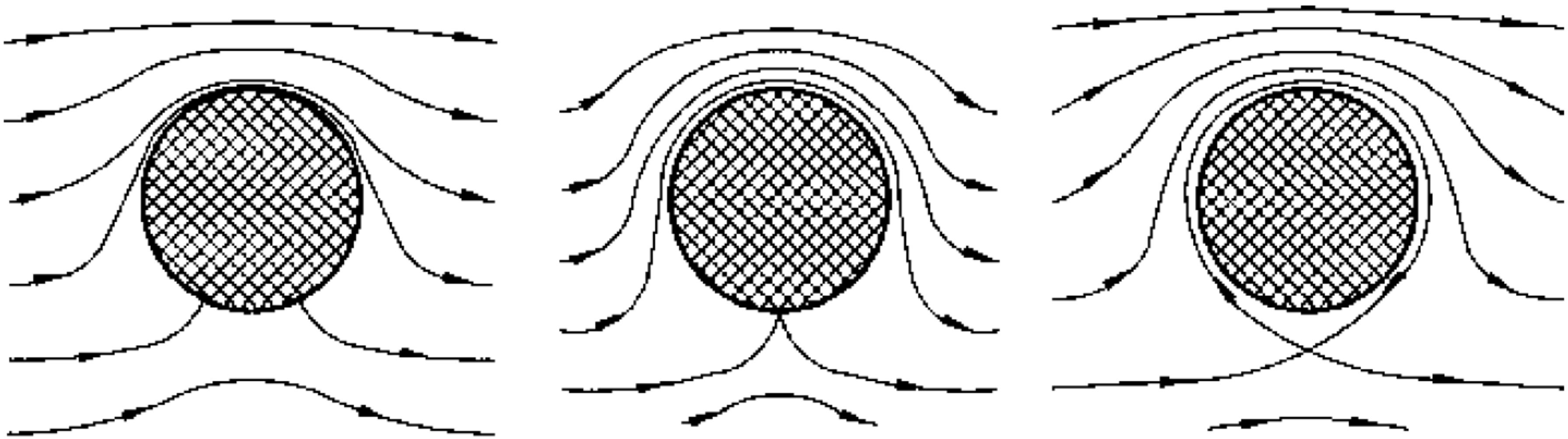
$$\begin{cases} v_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \\ v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r} \end{cases} \Rightarrow \begin{cases} v_r(a) = 0 \\ v_\theta(a) = -2U\sin\theta + \frac{\Gamma}{2\pi a} \end{cases}$$

stagnation points on the cylinder:

$$\sin\theta_s = \frac{\Gamma}{4\pi U a}$$

Circular cylinder with circulation

Negative (clockwise) circulation of magnitude Γ



$$-1 < \frac{\Gamma}{4\pi a U} < 0 \qquad \frac{\Gamma}{4\pi a U} = -1 \qquad \frac{\Gamma}{4\pi a U} < -1$$

https://youtu.be/wxdXB7N5pbQ

Circular cylinder with circulation

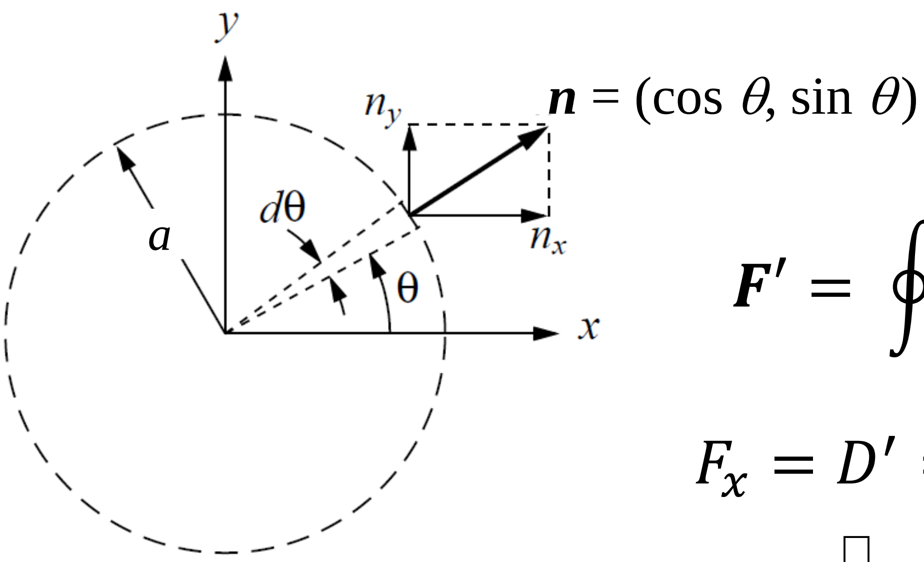
In the last case the stagnation point has coordinates (*check!*):

$$\begin{cases} \theta_s = \dfrac{3\pi}{2} \\ \dfrac{r_s}{a} = \dfrac{-\Gamma}{4\pi U a}\left[1 + \sqrt{1 - \left(\dfrac{4\pi U a}{\Gamma}\right)^2}\,\right] \end{cases}$$

No drag (y symmetry!) but lift appears on the cylinder. From Bernoulli it is easy to find the surface pressure:

$$p_s = p_\infty + \frac{1}{2}\rho U^2 \underbrace{\left[1 - \left(2\,\sin\theta - \frac{\Gamma}{2\pi U a}\right)^2\right]}_{c_p}$$

Circular cylinder with circulation: lift force



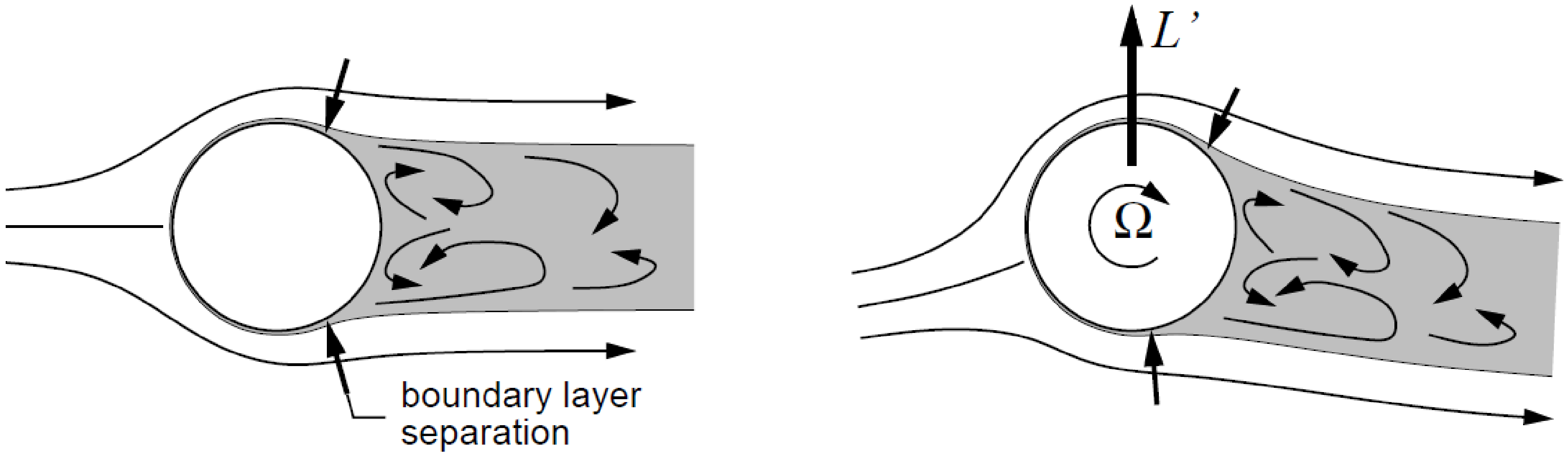
$$\boldsymbol{F}' = \oint -p_s \boldsymbol{n}\, \mathrm{d}l = \int_0^{2\pi} -\, p_s\, \boldsymbol{n}\, a\, \mathrm{d}\theta$$

$$F_x = D' = 0 \qquad\qquad F_y = L' = -\rho\, U\, \Gamma$$

⇩ D'Alembert paradox

⇩ Kutta-Joukowski theorem

Real life



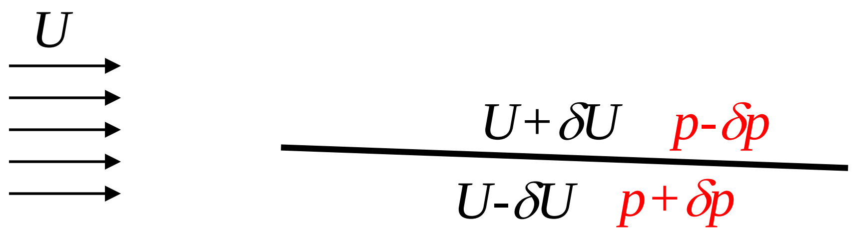
Magnus effect. Large lift, however the cylinder is not a satisfactory lifting device because of the large drag.

Kutta-Joukowski theorem

The Kutta-Joukowski theorem, $L' = -\rho \, U \, \Gamma$, with L' acting always perpendicular to the direction of U, applies not just to a cylinder, but to **2D bodies** of any shape, in unbounded domains.

We can show that K-J theorem applies by using a simple heuristic argument or we can demonstrate it in a more rigorous way. For the latter we need to resort to complex variable theory and to the so-called *Blasius formula* …

K-J: the qualitative argument



This airfoil (flat plate) of chord c at small angle of attack

$$\Gamma = (U - \delta U)\, c - (U + \delta U)\, c = -2\, \delta U\, c$$

Bernoulli: $(p + \delta p) + \rho (U - \delta U)^2/2 = (p - \delta p) + \rho (U + \delta U)^2/2$

$$2\, \delta p = 2\, \rho\, U\, \delta U + \rho\, (\delta U)^2$$

smaller order, can neglect

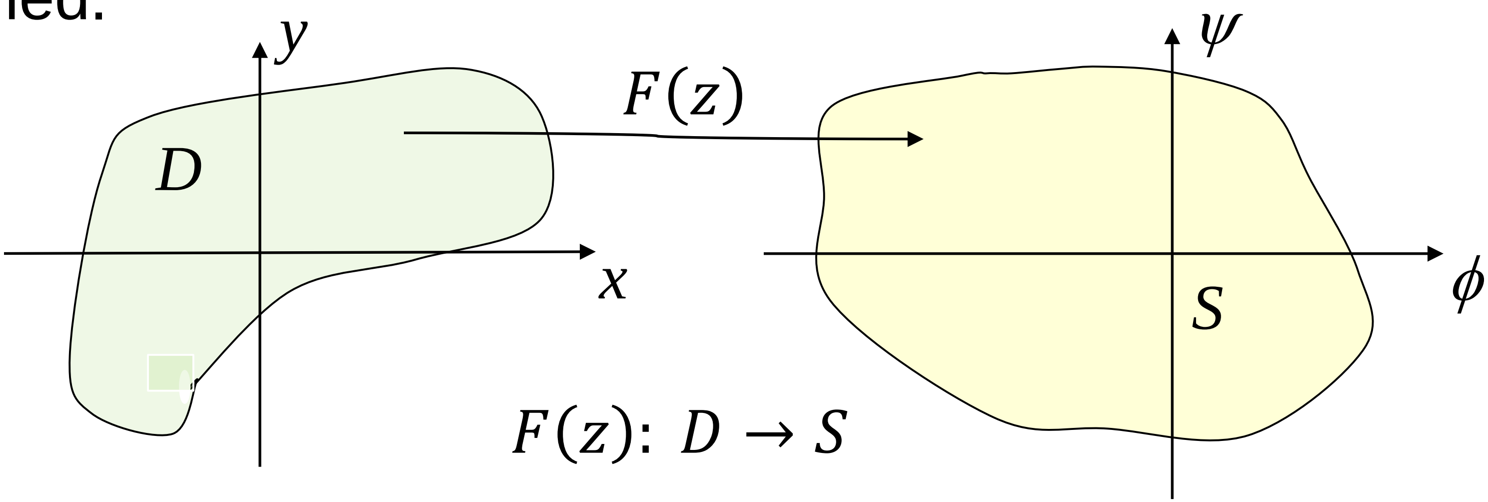
force on airfoil per unit span:
(acting ⊥ to airfoil …)

$$F_p = c\, 2\, \delta p \approx 2\, c\, \rho\, U\, \delta U = -\rho\, U\, \Gamma$$

A quick recap on complex analysis

Complex *analytic functions* have been defined in slide 7.

Also: a complex, single-valued function $F(z)$ which is differentiable in z_0 **and** in a neighborhood of z_0 is said to be analytic (or holomorphic) at z_0. As already stated a sufficient condition for differentiability is that C-R are satisfied.



A quick recap on complex analysis

If $F(z)$ is analytic inside and on a circle C centered in $z = z_0$, then $F(z)$ admits a **Taylor series** representation for any point z inside C:

$$F(z) = F(z_0) + F'(z_0)(z - z_0) + \frac{F''(z_0)}{2!}(z - z_0)^2 + \ ...$$

$\rightarrow$ all complex functions, analytic in a neighborhood of z_0, are *infinitely differentiable* in a neighborhood of z_0.

A quick recap on complex analysis

Singularities

There are three possible types of singularities of the complex function $F(z)$: *poles*, *branch points* and *essential singularities.* We will mostly be concerned with the first type.

$\boldsymbol{Pole}$: a singular point $z = z_0$ is called a *pole of order* n $(n > 0,\ n \in \mathbb{Z})$ if and only if

$$F(z) = \frac{h(z)}{(z - z_0)^n}$$

where $h(z)$ is analytic at $z = z_0, h(z_0) \neq 0$.

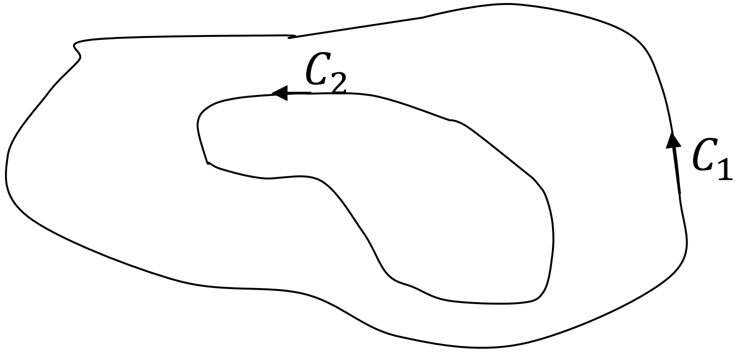
The simplest example of the case above is $F(z) = \frac{a}{(z-z_0)^n}$ with $a \neq 0$ a complex constant.

A quick recap on complex analysis

Cauchy theorem (or Cauchy-Goursat theorem)
If $F(z)$ is analytic inside **and** on a closed curve C, then

$$\oint_C F(z)\,\mathrm{d}z = 0$$

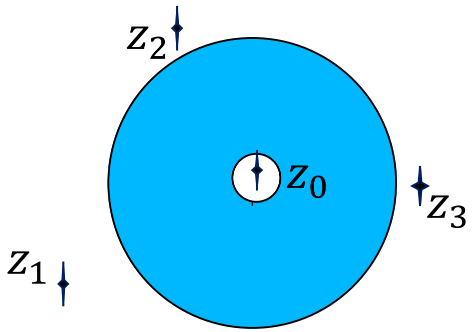
This implies that the contour can be deformed provided we do not cross *singularities*.



$$\oint_{C_1} F(z)\,\mathrm{d}z = \oint_{C_2} F(z)\,\mathrm{d}z = 0$$

A quick recap on complex analysis

If $F(z)$ is analytic in an annulus centered around some point $z = z_0$ (and $z = z_0$ can be a *singularity* of $F(z)$) a **Laurent series** is defined in the annulus as:



$$F(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

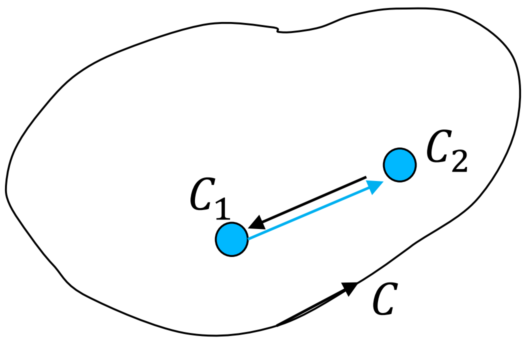
(keep an eye on $n = -1$!)

(if $z = z_0$ is not singular → $a_n = 0$ for $n = -1, -2, -3$ ...
→ the Laurent series coincides with the Taylor series!)

A quick recap on complex analysis

The residue theorem

Assume that z_1 and z_2 are two singularities, contained within a closed contour C.



We deform the contour C until we get to the situation in the figure. Then:

$$\oint_C F(z)\,\mathrm{d}z = \oint_{C_1} F(z)\,\mathrm{d}z + \oint_{C_2} F(z)\,\mathrm{d}z = 2\pi i\left[a_{-1}^{(z_1)} + a_{-1}^{(z_2)}\right]$$

a_{-1} is the $n=-1$ coefficient of the Laurent series around each singularity; a_{-1} **is the residue of** $F(z)$ **at the singular point.**

A quick recap on complex analysis

Examples:

$$\oint_C k\, z^n \mathrm{d}z = \left.\frac{k\, z^{n+1}}{n+1}\right|_{start}^{end} = 0 \qquad n = 0, 1, 2\ \ldots$$

starting point and end point coincide for closed curve!

$$\oint_C \frac{k}{z}\,\mathrm{d}z = 2\pi i\, a_{-1} = 2\pi i k$$

from residue theorem

Check: $$\oint_C \frac{k}{z}\,\mathrm{d}z = k\,\log(z)\Big|_{start}^{end} = k\,\log(r)\Big|_{start}^{end} + \ i\,k\,\theta\Big|_{start}^{end} = 2\pi i k$$

A quick recap on complex analysis

At this point you **should** watch – in the given order - the short videos by Prof. Michael Barrus (URI), to review and better understand the basics of complex analysis (i.e. all that we really need to know, at least until now):

https://youtu.be/PNnpcTe0uAY
https://youtu.be/Xp1Q9SHe6NU
https://youtu.be/S-_bMON1mzQ
https://youtu.be/oMpWn90ETno
https://youtu.be/xZ0S8Ywwc9o
https://youtu.be/GPqVd30eHrg
https://youtu.be/eW0ArgJ3Isk

A quick recap on complex analysis

Exercises (first set)

For the complex functions which follow find the singular points (poles) and calculate the residues in the poles.

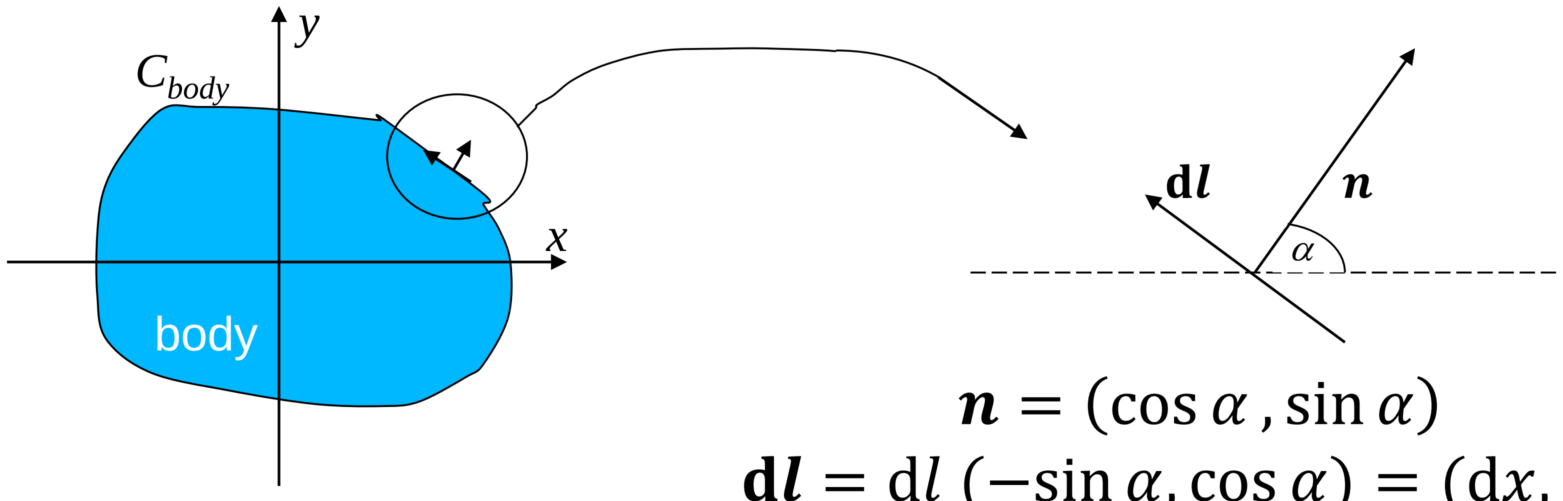
$f(z) = \frac{1}{z-1}$ (first order pole in $z = 1$, residue: $a_{-1} = 1$)

$$f(z) = \frac{\cos z}{z}$$ (first order pole in $z = 0$, residue: $a_{-1} = 1$)

$$f(z) = \frac{2z+3}{z^2-4z}$$ (two first order poles ...)

$f(z) = \frac{z^2}{(z^2+4)^2}$ (two poles of order two ...)

Blasius formula



$$\boldsymbol{n} = (\cos\alpha\,, \sin\alpha)$$

$$\mathbf{d}\boldsymbol{l} = \mathrm{d}l\,(-\sin\alpha, \cos\alpha) = (\mathrm{d}x, \mathrm{d}y)$$

$$\boldsymbol{n}\,\mathrm{d}l = \mathrm{d}l\,(\cos\alpha\,, \sin\alpha) = (\mathrm{d}y, -\mathrm{d}x)$$

$$\boldsymbol{F}' = \oint_{C_{body}} -p_s\boldsymbol{n}\,\mathrm{d}l = (D', L')$$

Blasius formula

$$D' = \oint_{C_{body}} -p_s \, \mathrm{d}y \qquad L' = \oint_{C_{body}} p_s \, \mathrm{d}x$$

$$D' - iL' = -\oint_{C_{body}} p_s \, (\mathrm{d}y + i \, \mathrm{d}x) = -i \oint_{C_{body}} p_s \, \overline{\mathrm{d}z}$$

In steady flow: $p + \frac{1}{2}\rho W\overline{W} = \text{constant}$

$$D' - iL' = i\frac{\rho}{2} \oint_{C_{body}} W\overline{W} \, \overline{\mathrm{d}z}$$

Blasius formula

$$D' - iL' = i\frac{\rho}{2}\oint_{C_{body}} W\bar{W}\,\overline{\mathrm{d}z}$$

$$\bar{W}\,\overline{\mathrm{d}z} = \overline{\mathrm{d}F} = \mathrm{d}\phi - i\,\mathrm{d}\psi$$

$$W\mathrm{d}z = \mathrm{d}F = \mathrm{d}\phi + i\,\mathrm{d}\psi$$

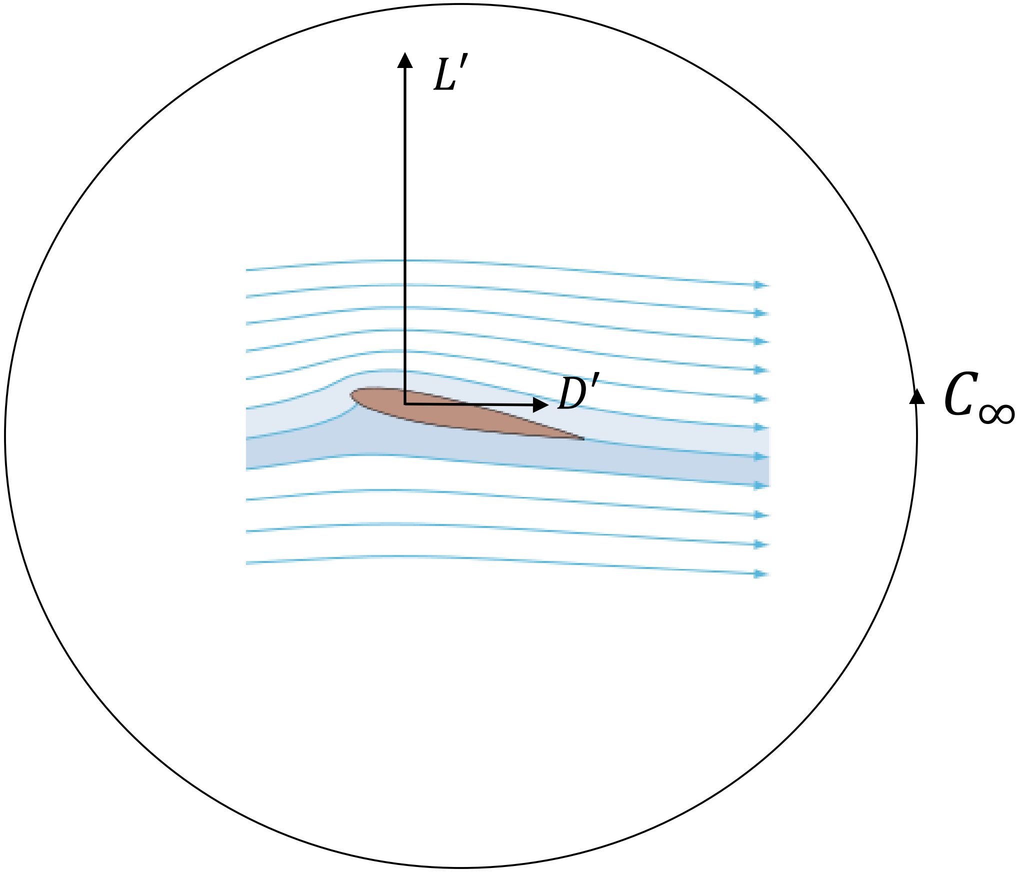
since ψ = constant is a streamline on the body

On the body: $\mathrm{d}F = \overline{\mathrm{d}F} = \bar{W}\,\overline{\mathrm{d}z} = W\mathrm{d}z$

$$D' - iL' = i\frac{\rho}{2}\oint_{C_{body}} W^2\,\mathrm{d}z$$

Blasius formula

Blasius formula: application



The integration path can be deformed, *provided that we do not cross singularities.*

Blasius integral formula can thus be used, integrating W^2 on the complex plane around the closed path C_∞, very far away from the airfoil On this path we have $\frac{1}{z} \to 0$.

Blasius formula: application

$$F(z) = Uz + \frac{\Gamma}{2\pi i}\log z + O\left(\frac{1}{z}\right) \longrightarrow W = \frac{dF}{dz} = U + \frac{\Gamma}{2\pi i}\frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

$$W^2 = U^2 + \frac{\Gamma U}{\pi i}\frac{1}{z} + O\left(\frac{1}{z^2}\right)$$

residue

$$\oint_{C_{body}} W^2\,\mathrm{dz} = \oint_{C_\infty} W^2\,\mathrm{dz} = 2\pi i\, a_{-1} = 2\,\Gamma\, U$$

From Blasius formula: $D' - iL' = i\frac{\rho}{2}(2\,\Gamma\,U) = i\,\rho\,U\,\Gamma$

$\boldsymbol{D' = 0}$

D'Alembert paradox

$\boldsymbol{L' = -\rho\,U\,\Gamma}$

and for $\Gamma < 0$ (clockwise circulation) we have upward lift

Blasius formula: application

Exercises (second set)

1. Consider a 2D body which moves in a motionless fluid. The fluid remains at rest far from the body. Show that the circulation around the body is

$$\Gamma = \oint_{C_{body}} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \mathrm{Real} \oint_{C} \mathrm{d}F ,$$

with F the complex potential, and C a contour, to be followed counterclockwise, taken at a large distance from the body (provided - of course! - that singularities are not crossed as the contour is deformed). Then, show that the volumetric flow rate (per unit depth) computed around a contour fixed on the body is

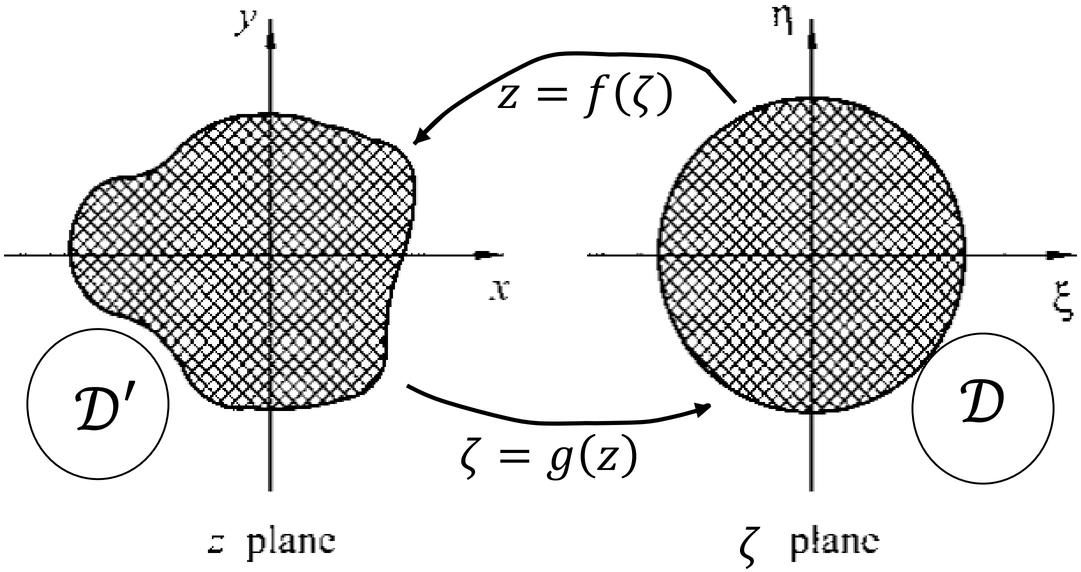
$$\acute{V}/_{L} = \oint_{C_{body}} \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}l = \mathrm{Imag} \oint_{C} \mathrm{d}F .$$

Under which conditions the integral above does not vanish?

2. Use Blasius formula and the residue theorem to compute the components D' and L' of the aerodynamic force on the Rankine body (uniform flow + source in $z = 0$ + sink in $z = e$).

Conformal mapping

Complex variable theory is a powerful tool for the solution of 2D incompressible potential flow problems through its mapping properties. A conformal mapping creates a geometrical correspondence between two planes, by the use of the analytic function $f(\zeta)$.

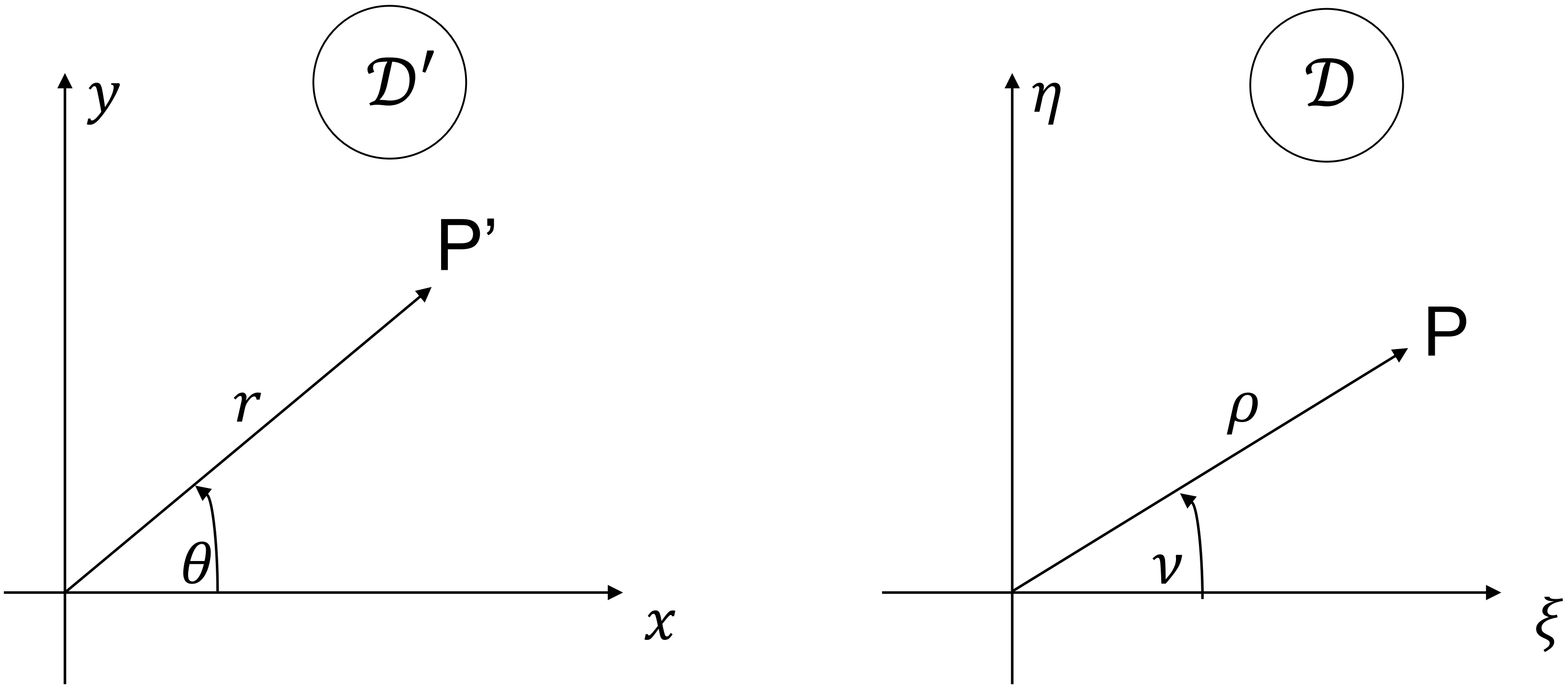


$$z = f(\zeta)$$
$$\zeta = g(z)$$

$g = f^{-1}$

inverse function

Conformal mapping



$$z = r\, e^{i\theta}$$

$$\zeta = \rho\, e^{i\nu}$$

Conformal mapping

Theorem

The analytic, single-valued function $f(\zeta)$ in the domain $\mathcal{D}$ admits an inverse analytic function in $\mathcal{D}'$ whose derivative is $\frac{1}{f'(\zeta)}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[f^{-1}(z)\right] = \frac{1}{f'(\zeta)} \qquad \text{in } z = f(\zeta)$$

provided it is $f'(\zeta) \neq 0$ at every ζ point, so that the inverse function is differentiable. The points where $f'(\zeta) = 0$ are called **critical points**. Thus, the analytic inverse function $g(z) = f^{-1}(z)$ exists away from critical points in $\mathcal{D}$.

On critical points the transformation is *non-conformal*.

Conformal mapping

$$z = f(\zeta) = x + iy$$

$$\zeta = f^{-1}(z) = g(z) = \xi(x, y) + i\eta(x, y)$$

If the function $g(z)$ is analytic, it must satisfy C-R, i.e.

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}$$

$$\nabla^2 \xi = \nabla^2 \eta = 0$$

Conformal mapping

First question: what is the effect of the transformation $z = f(\zeta)$ on the potential function and on the streamfunction?

If $\phi(x, y)$ is harmonic on $\mathcal{D}$' it can be shown that the transformed potential $\tilde{\phi}(\xi, \eta)$ is also harmonic (in $\mathcal{D}$), i.e. the transformed motion is also a potential motion.

Same applies to $\psi(x, y)$ and $\tilde{\psi}(\xi, \eta)$.

(shown, for example, in the book by Currie, 3rd edition, pages 105-108)

Conformal mapping

First question: what is the effect of the transformation $z = f(\zeta)$ on the potential function and on the streamfunction?

$$F = F(z) = \phi(x, y) + i\psi(x, y)$$

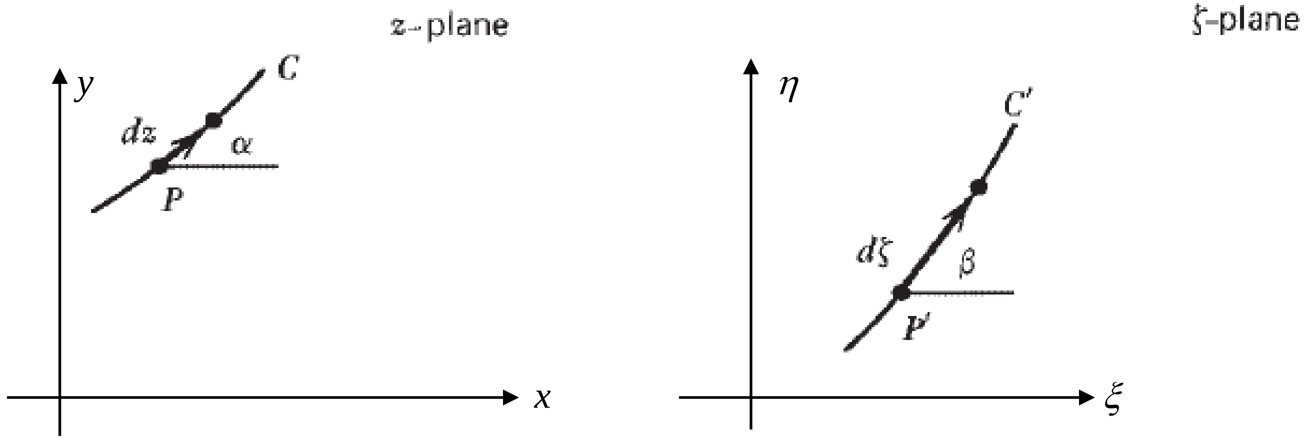
$$F = F[f(\zeta)] = \tilde{F}(\zeta) = \tilde{\phi}(\xi, \eta) + i\tilde{\psi}(\xi, \eta)$$

and viceversa if starting from $\tilde{F}(\zeta)$

If the solution for a simple body is known, e.g. in $\mathcal{D}$, then the solution for the more complex body in $\mathcal{D}$' is found by substituting $\zeta = g(z)$ in the complex potential $\tilde{F}(\zeta)$.

Conformal mapping

Second question: what is the effect of the transformation $z = f(\zeta)$ on infinitesimal linear elements?



$$\mathrm{d}z = |\mathrm{d}z|\exp(i\alpha) \qquad \mathrm{d}\zeta = |\mathrm{d}\zeta|\exp(i\beta)$$

$$\frac{\mathrm{d}z}{\mathrm{d}\zeta} = f'(\zeta) = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right| \exp[i(\alpha - \beta)]$$

Conformal mapping

Second question: what is the effect of the transformation $z = f(\zeta)$ on infinitesimal linear elements?

Since $f(\zeta)$ is analytic on $\mathcal{D}$, the derivative $f'(\zeta)$ does not depend on the direction $\mathrm{d}\zeta$; all lines through a point are *stretched* and *rotated* by the same amount.

$$\mathrm{d}z = \mathrm{d}\zeta \; |f'(\zeta)| \exp[i(\alpha - \beta)]$$

$$\begin{cases} |\mathrm{d}z| = |\mathrm{d}\zeta| \; |f'(\zeta)| \\ \arg(\mathrm{d}z) = \arg(\mathrm{d}\zeta) + \underbrace{(\alpha - \beta)}_{\arg(f')} \end{cases}$$

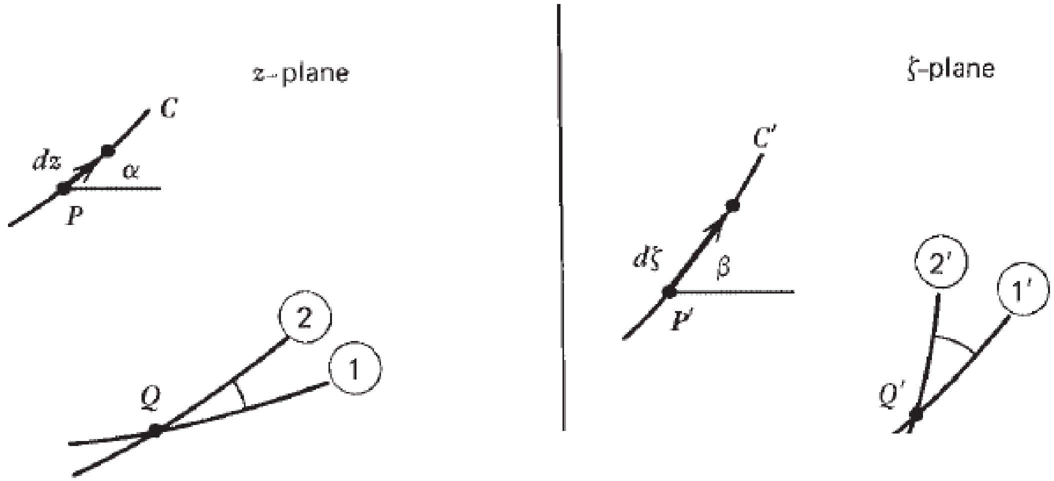
stretching factor: $|f'(\zeta)|$

rotation by $\alpha - \beta$

Conformal mapping

Second question: what is the effect of the transformation $z = f(\zeta)$ on infinitesimal linear elements?

For points where $|f'(\zeta)| \neq 0$ the transformation preserves angles between pairs of corresponding infinitesimal elements.



$$\arg(dz_2) - \arg(dz_1) = \arg(d\zeta_2) - \arg(d\zeta_1)$$

Conformal mapping

Third question: what is the effect of the transformation $z = f(\zeta)$ on the complex velocity?

$$\widetilde{W}(\zeta) = \frac{\mathrm{d}\widetilde{F}}{\mathrm{d}\zeta} = \frac{\mathrm{d}F}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\zeta} = f'\frac{\mathrm{d}F}{\mathrm{d}z} = f'\,W(z)$$

i.e. complex velocities are proportional to one another, and the proportionality constant is the derivative of the transformation.

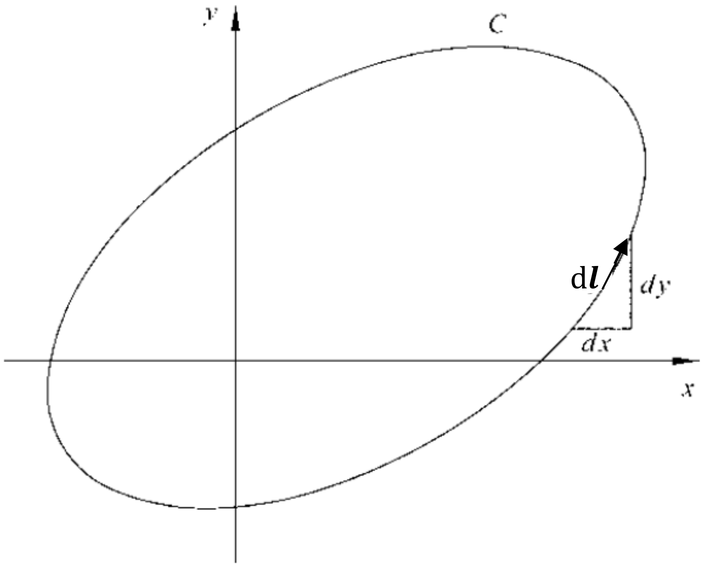
$$\left|\widetilde{W}(\zeta)\right| = \left|\frac{\mathrm{d}f}{\mathrm{d}\zeta}\right|\,|W(z)|$$

Critical points ($|f'|=0$) *are stagnation points on the* ζ*-plane, i.e. stagnation points in* $\mathcal{D}$ *are not necessarily stagn. pts in* $\mathcal{D}'$*.*

Conformal mapping

Fourth question: what do sources, sinks and vortices in one plane become on the other plane upon transforming?

Let us integrate the complex velocity around a closed contour c in the z-plane.



$$d\boldsymbol{l} = (dx, dy)$$

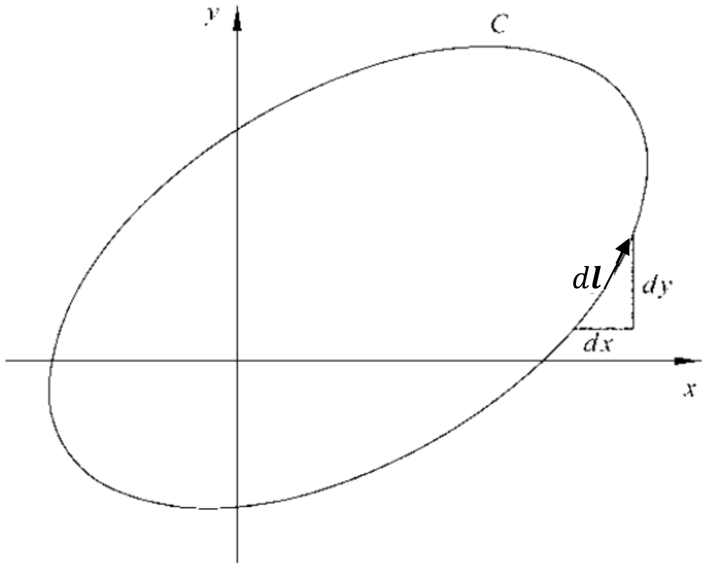
$$\boldsymbol{n} \perp d\boldsymbol{l}$$

$$\boldsymbol{n} = \frac{(dy, -dx)}{dl} \qquad (\boldsymbol{n} \cdot d\boldsymbol{l} = 0)$$

Conformal mapping

Fourth question: what do sources, sinks and vortices in one plane become on the other plane upon transforming?

We already know that: $$\frac{\dot{V}}{L} = \oint_c \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}l = \oint_c u \, \mathrm{d}y - v \, \mathrm{d}x$$



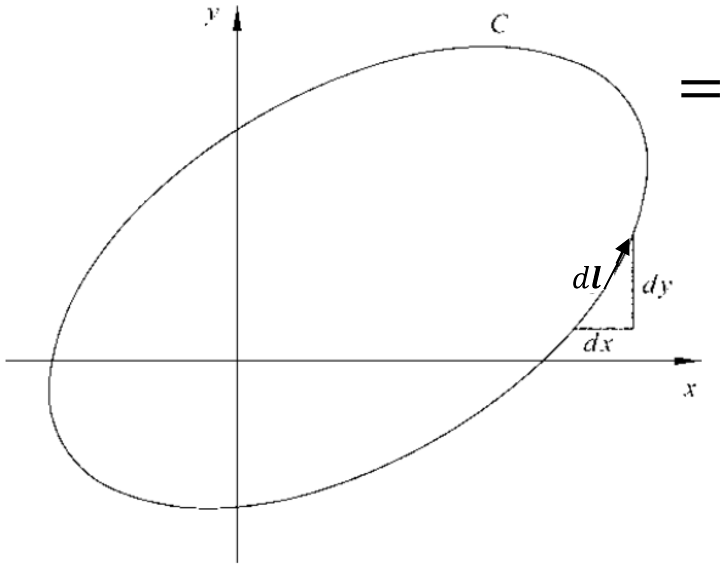
and

$$\Gamma = \oint_c \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{l} = \oint_c u \, \mathrm{d}x + v \, \mathrm{d}y$$

Conformal mapping

Fourth question: what do sources, sinks and vortices in one plane become on the other plane upon transforming?

$$\oint_c W(z)\, dz = \oint_c (u - iv)\,(\mathrm{d}x + i\,\mathrm{d}y)$$



$$= \oint_c u\,\mathrm{d}x + v\,\mathrm{d}y + i\oint_c u\,\mathrm{d}y - v\,\mathrm{d}x$$

$$= \Gamma + i\frac{\dot{V}}{L}$$

(assuming a single source or sink and a single vortex within the contour)

Conformal mapping

Fourth question: what do sources, sinks and vortices in one plane become on the other plane upon transforming?

We thus have:

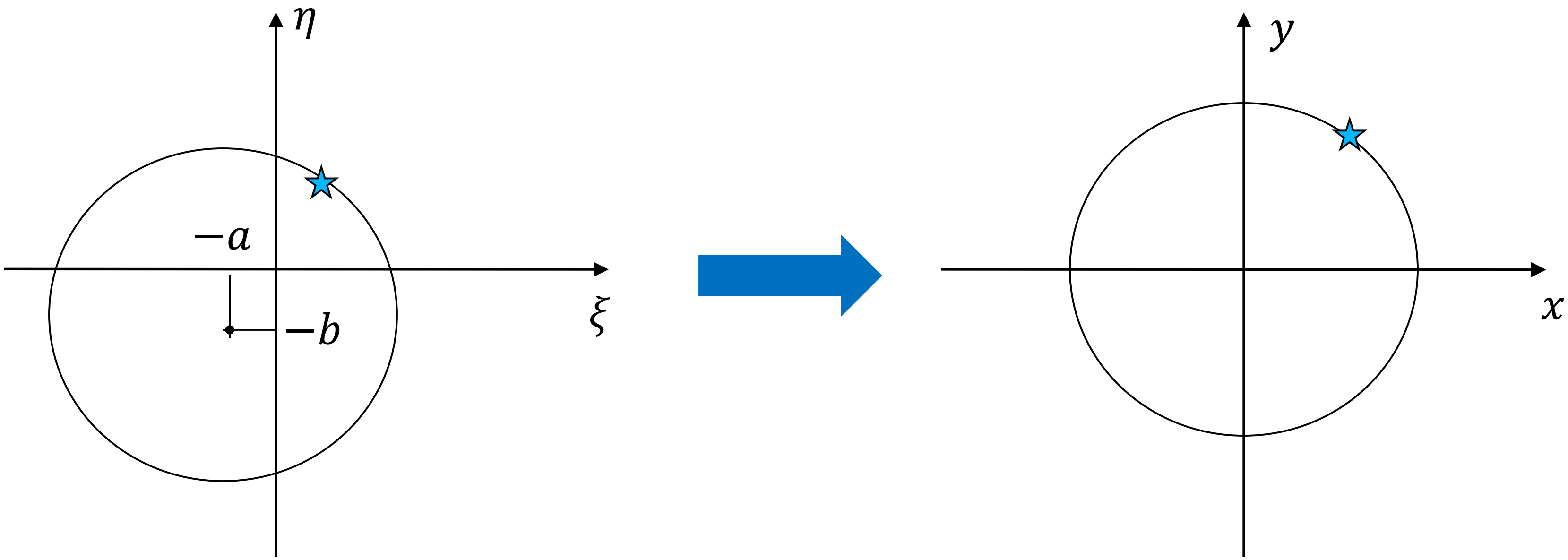
$$\Gamma + i\frac{\dot{V}}{L} = \oint_c W(z)\,dz = \oint_c \left[\widetilde{W}(\zeta)\frac{d\zeta}{dz}\right] dz =$$

$$= \oint_{\tilde{c}} \widetilde{W}(\zeta)\; d\zeta = \tilde{\Gamma} + i\frac{\dot{\tilde{V}}}{\tilde{L}}$$

A conformal mapping transforms sources, sinks and vortices in one plane (ex. the ζ-plane) into sources, sinks and vortices of equal strength in the other plane (ex. the z-plane).

Some elementary mappings $z = f(\zeta)$

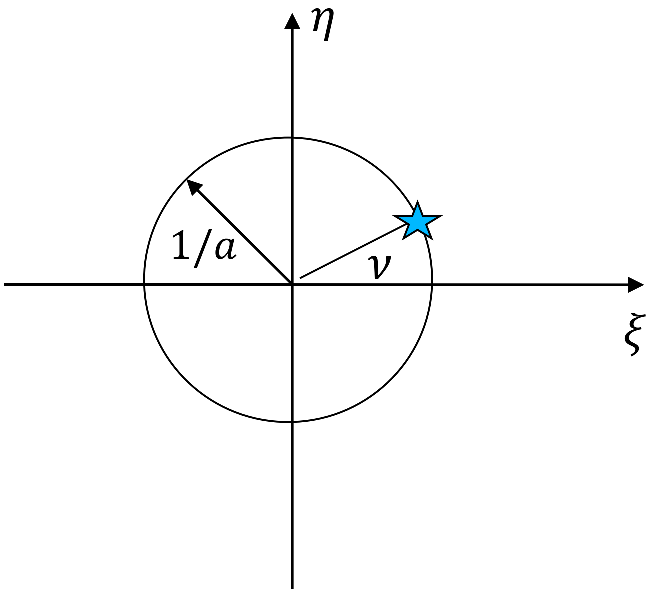
Translation: $z = \zeta + \zeta_0 = (\xi + a) + i(\eta + b)$

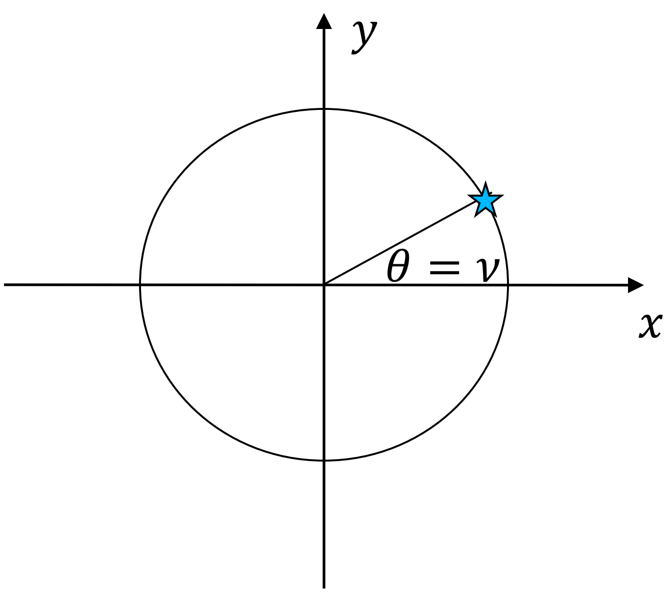


circle centered in $-\zeta_0$ → circle of same radius centered in the origin

Some elementary mappings $z = f(\zeta)$

Scaling: $z = a\,\zeta = a\,\xi + i\,a\,\eta$ $\qquad a \neq 0,\ \ a \in \mathcal{R}$

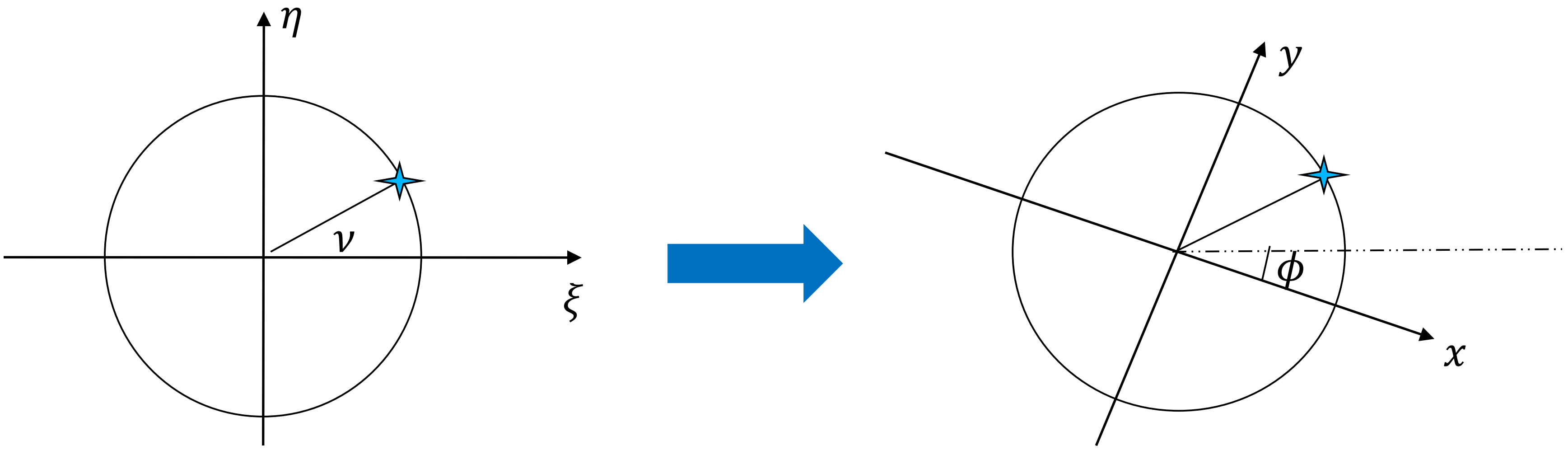




circle of radius $1/a$ ➡ circle of unit radius

Some elementary mappings $z = f(\zeta)$

Rotation: $z = e^{i\phi}\,\zeta \quad \rightarrow \quad re^{i\theta} = \rho e^{i(\nu+\phi)}$



the circle is mapped onto itself, with the complex z plane rotating clockwise around the origin by the (real) angle ϕ

Some elementary mappings $z = f(\zeta)$

Inversion: $z = f(\zeta) = \frac{1}{\zeta}$

This is a one-to-one analytic mapping everywhere except at the origin of the $\mathcal{D}$ plane ($\zeta = 0$).

$$\zeta = f^{-1}(z) = g(z) = \frac{1}{z} \qquad \frac{\mathrm{d}}{\mathrm{d}z}[f^{-1}(z)] = -\frac{1}{z^2} = \frac{1}{f'(\zeta)} = -\zeta^{\,2}$$

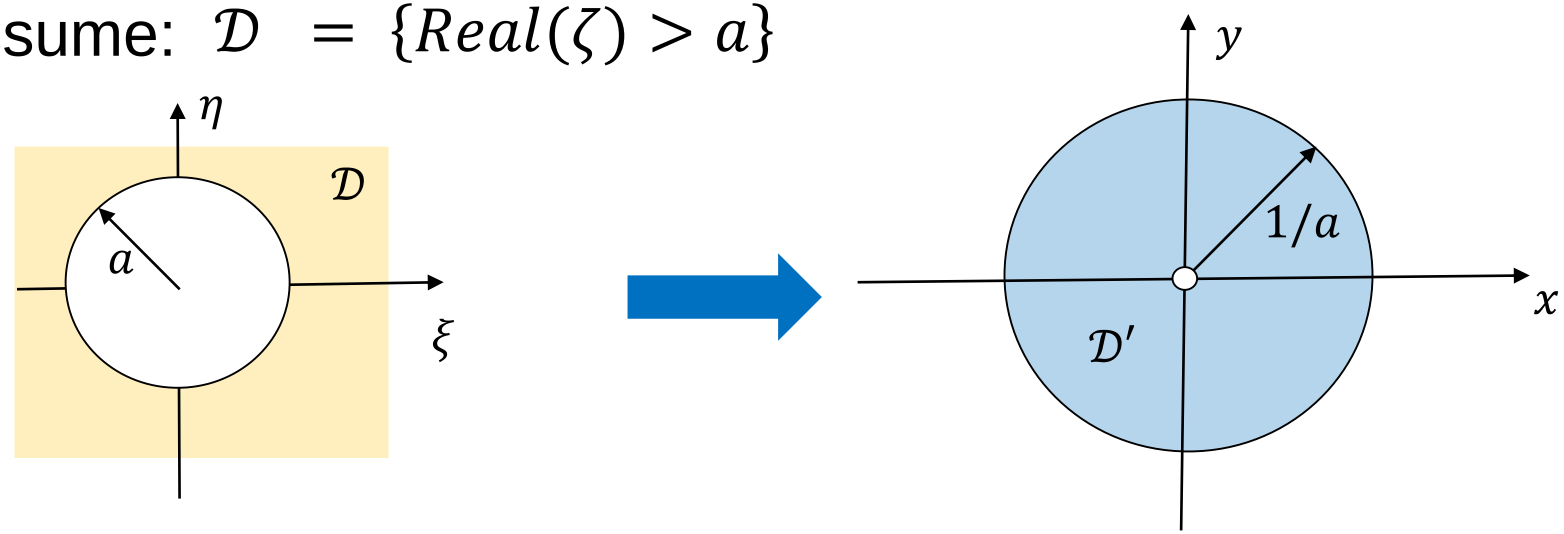
Critical points: $f'(\zeta) = 0$. Since $f'(\zeta) = -z^2$, the critical point is $z = 0$. On $z = 0$ the transformation is non-conformal.

$$\boxed{r = |z| = \frac{1}{|\zeta|} = \frac{1}{\rho} \quad \text{and} \quad \theta = \arg(z) = -\nu = -\arg(\zeta)}$$

Some elementary mappings $z = f(\zeta)$

Inversion: $z = f(\zeta) = \frac{1}{\zeta}$

Assume: $\mathcal{D} = \{Real(\zeta) > a\}$

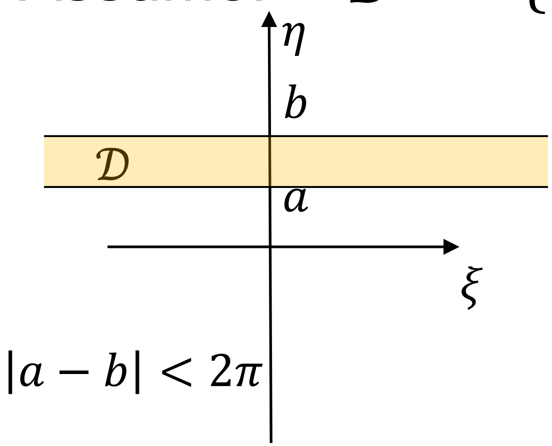


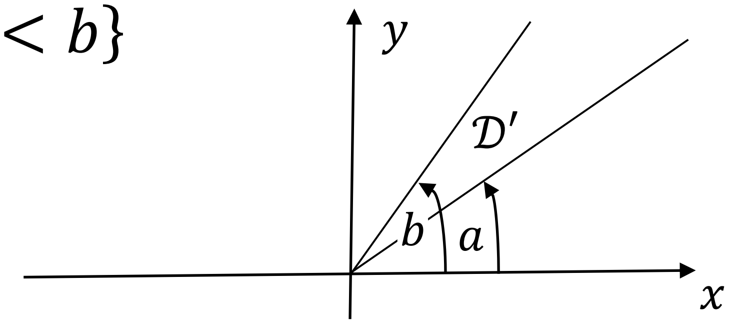
$\mathcal{D}$ is the *exterior* of the circle of radius a; it is mapped onto the *punctured* disk $\mathcal{D}' = \{0 < |z| < 1/a\}$

Some elementary mappings $z = f(\zeta)$

The exponential mapping: $z = f(\zeta) = e^{\zeta}$ *(single-valued ?? … $e^{\zeta} = e^{\zeta + 2\pi i}$)*

Assume: $\mathcal{D} = \{a < Imag(\zeta) < b\}$





$$re^{i\theta} = e^{\xi + i\eta}$$

The horizontal strip $\mathcal{D}$ is mapped onto the wedge-shaped domain $\mathcal{D}' = \{a < \theta = \arg(z) < b\}$

The Joukowski mapping

The most well-known mapping to go from the flow past a circle (in the ζ −plane) to the flow around airfoils (in the physical or z −plane). The J transformation must be used together with a condition (*Kutta condition*) which loosely states that the flow must exit from the trailing edge of the airfoil smoothly, or the rear stagnation point on the circle in the ζ-plane must map on the TE (which is a **cusp** for the J airfoil) in the z −plane. The Kutta condition permits to set the circulation Γ around the circle (and around the airfoil).

$$z = \zeta + \frac{\lambda^2}{\zeta} \qquad \lambda^2 \in \mathcal{R}$$

The Joukowski mapping

$$z = \zeta + \frac{\lambda^2}{\zeta}$$

Observations:

- $\zeta = 0$ is a singularity of the function $f(\zeta)$
- when $|\zeta| \to \infty$ we have that $z \to \zeta$, i.e. far from the origin we have the *identity mapping*, so that $F(z) = \tilde{F}(\zeta)$ and $W(z) = \widetilde{W}(\zeta)$. In other words, the complex velocity in the two planes is the same far away from the axes' origins
- $\frac{\mathrm{d}z}{\mathrm{d}\zeta} = 1 - \frac{\lambda^2}{\zeta^2} = 0$ for $\zeta = \pm\lambda$. These are the **critical points** of the J transformation. They are stagnation points on the ζ −plane (cf. slide 54) and for these pts angles between corresponding elements are not conserved (cf. slide 53)

The Joukowski mapping

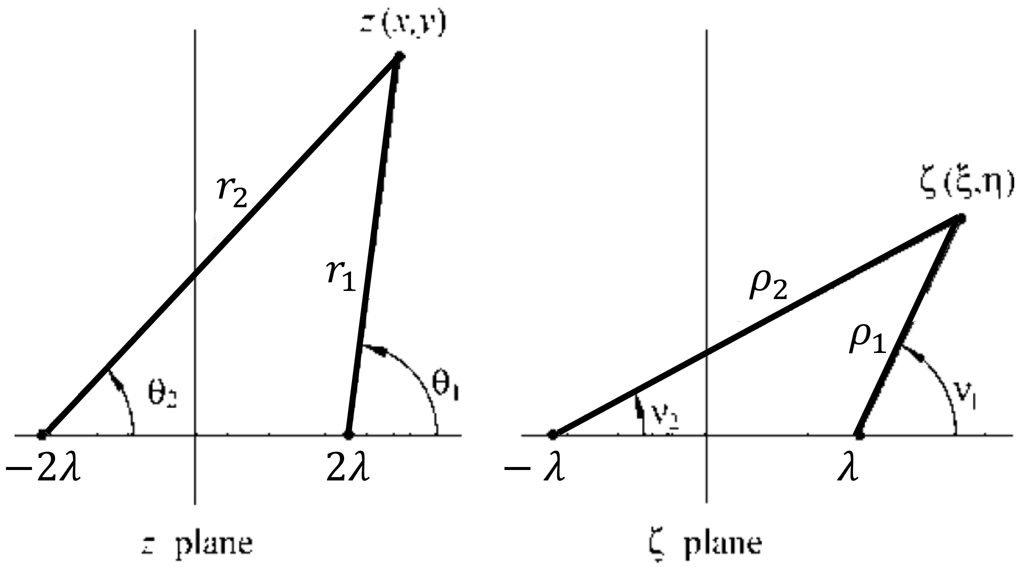
$$z = \zeta + \frac{\lambda^2}{\zeta}$$

The last observation amounts to stating that the J mapping is non-conformal on the critical points $\zeta = \pm\lambda$ (which map onto $z = \pm 2\lambda$ in the z –plane).

Let us write the J mapping as: $z \pm 2\lambda = \frac{(\zeta \pm \lambda)^2}{\zeta}$ so that

$$\frac{z - 2\lambda}{z + 2\lambda} = \left(\frac{\zeta - \lambda}{\zeta + \lambda}\right)^2$$

The Joukowski mapping



$$\frac{z-2\lambda}{z+2\lambda}=\left(\frac{\zeta-\lambda}{\zeta+\lambda}\right)^2$$

$$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}=\left(\frac{\rho_1e^{i\nu_1}}{\rho_2e^{i\nu_2}}\right)^2$$

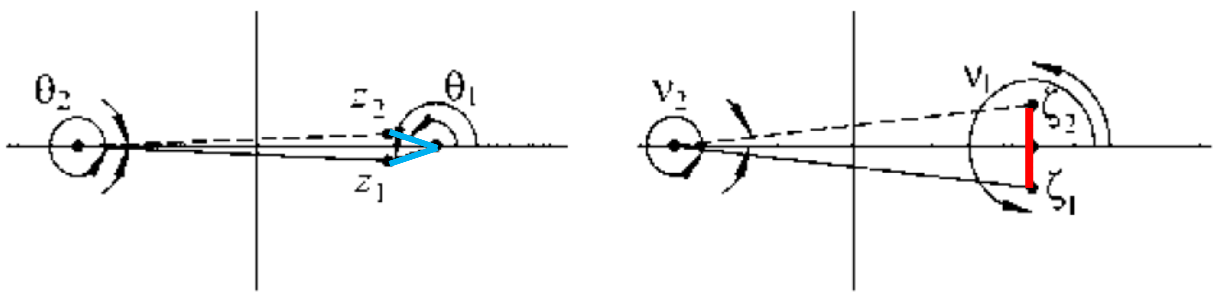
$$\frac{r_1}{r_2}=\left(\frac{\rho_1}{\rho_2}\right)^2, \qquad \theta_1-\theta_2=2(\nu_1-\nu_2)$$

The Joukowski mapping

Let us now consider the smooth curve about $\zeta = \lambda$, with two points very close to one another, ζ_1 and ζ_2. The corresponding curve in the z–plane forms a knife-edge or **cusp**.

Angles variations as we move along the curve from ζ_1 to ζ_2:

ν_1 goes from 3π/2 to π/2, ν_2 goes from 2π to 0
$\theta_1 = \pi$ both before and after, θ_2 goes from 2π to 0



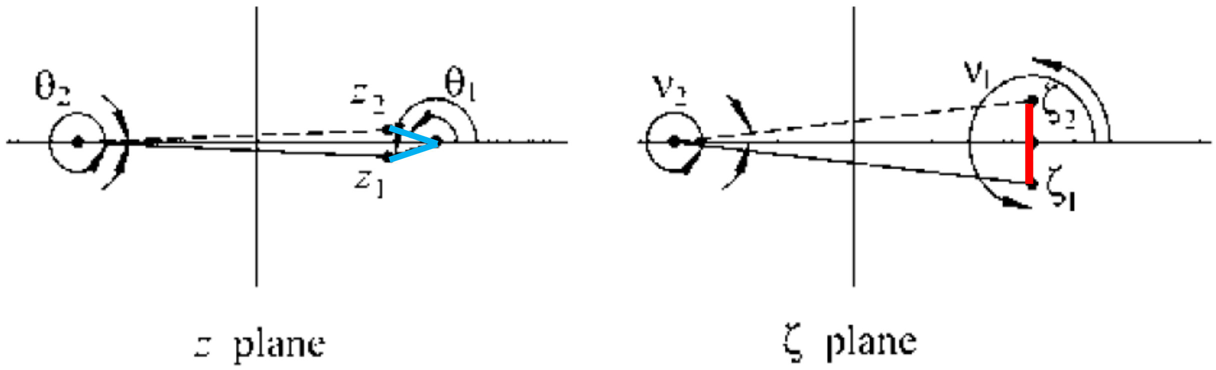
z plane $\qquad$ ζ plane

The Joukowski mapping

$(\nu_1 - \nu_2)$ varies from $-\pi/2$ to $\pi/2$ and $(\theta_1 - \theta_2)$ from $-\pi$ to π

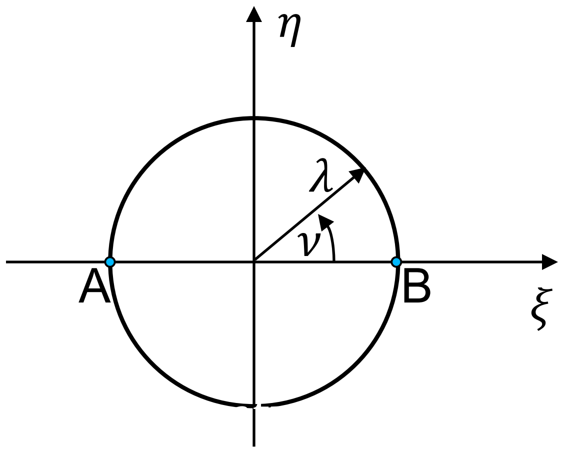
(i.e. letting a line cross the critical point in $\boldsymbol{\zeta = \lambda}$ a **cusp** is created in $\boldsymbol{z = 2\lambda}$; on this pt we have $W(z) \to \infty$, cf. slide 14)

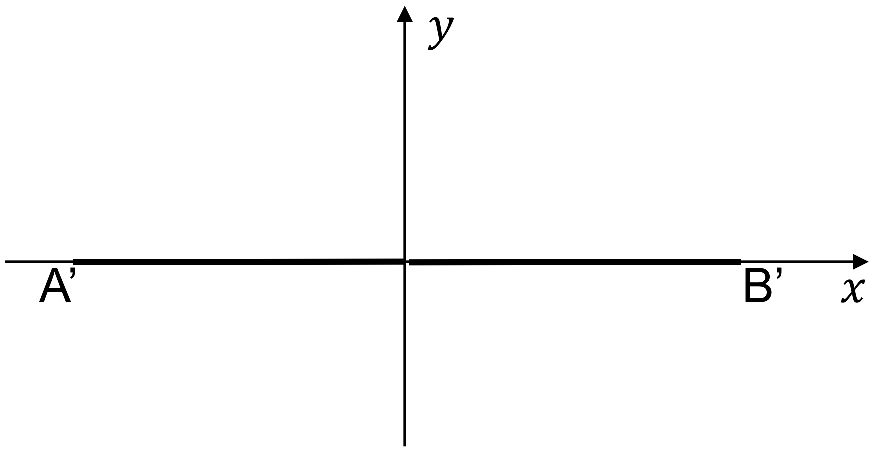
Remember: *a smooth curve through either one of the critical points in $\zeta = \pm\lambda$ forms a cusp in the z –plane in $z = \pm 2\lambda$*



The Joukowski mapping

Case 1: circle of radius λ centered on the origin





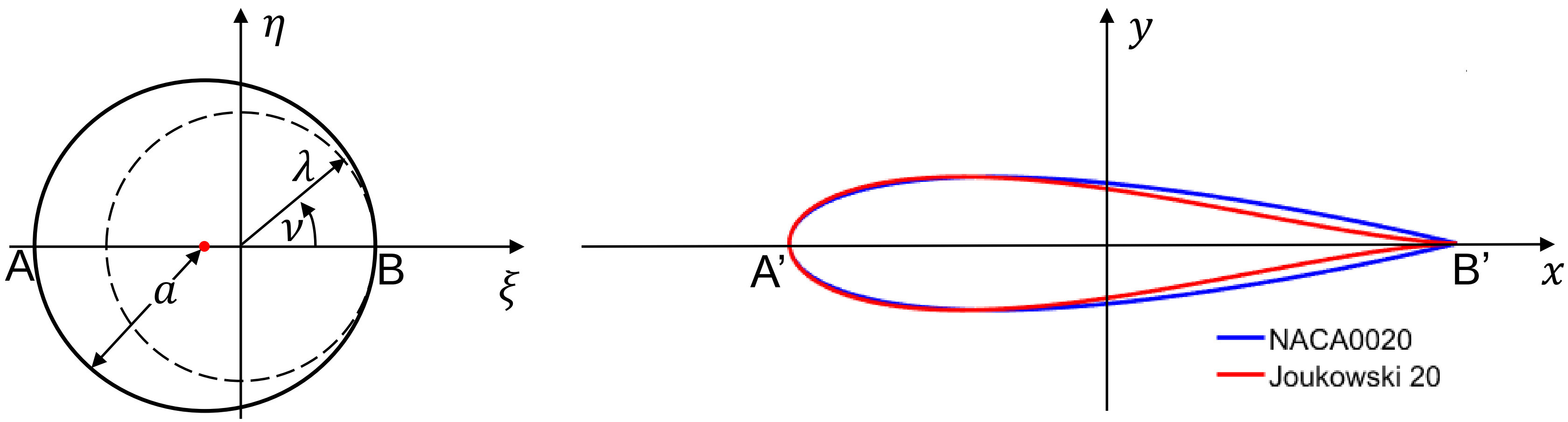
$$\zeta = \lambda\, e^{i\nu}$$

$$z = \lambda\, e^{i\nu} + \lambda\, e^{-i\nu} = 2\,\lambda \cos(\nu)$$

The critical points A and B are mapped onto A' and B' (cusps). The circle of radius λ maps onto a segment in the real plane (a *flat plate airfoil*) of length (or chord) $c = 4\lambda$

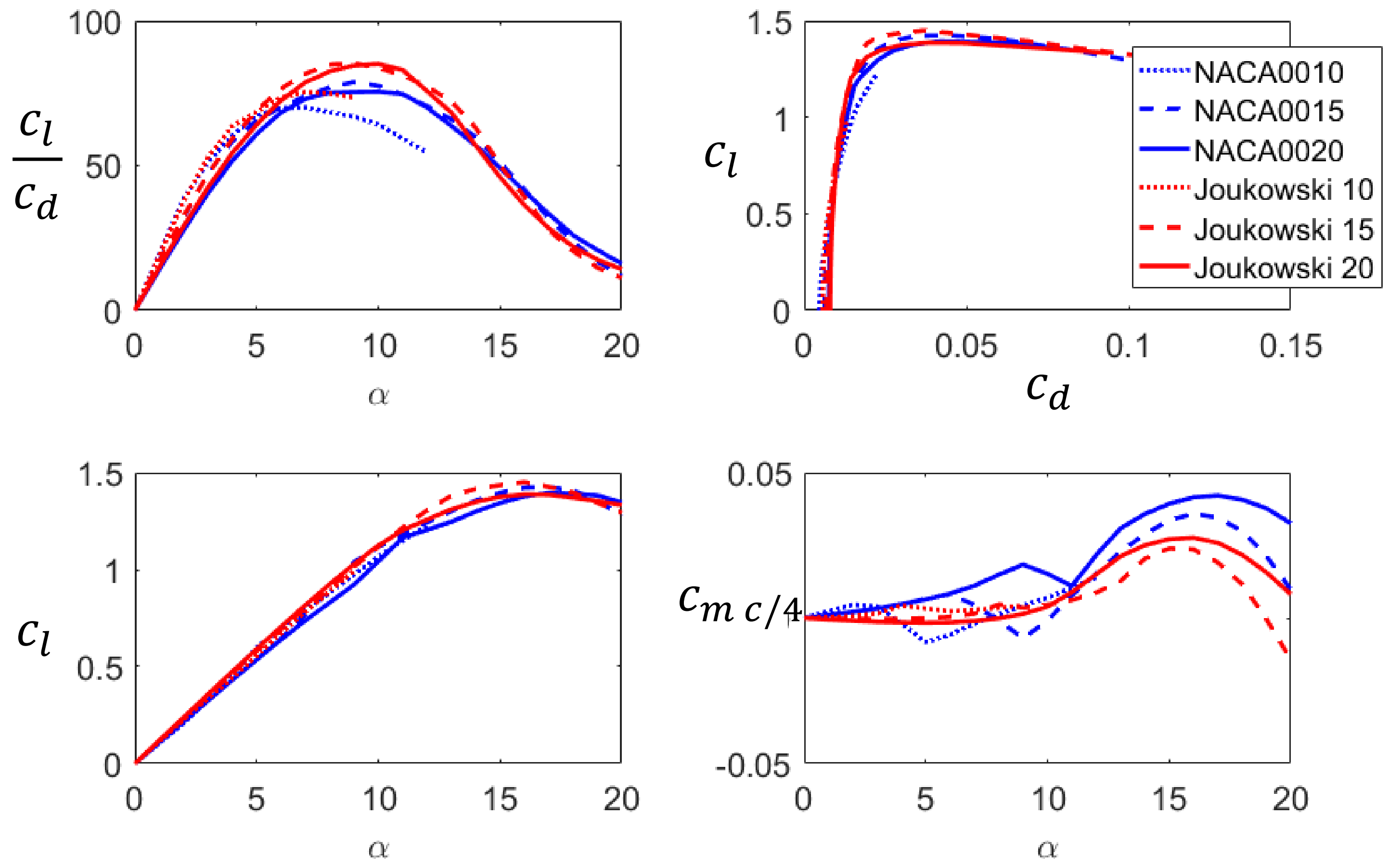
The Joukowski mapping

Case 2: circle of radius $a > \lambda$ centered on the real axis



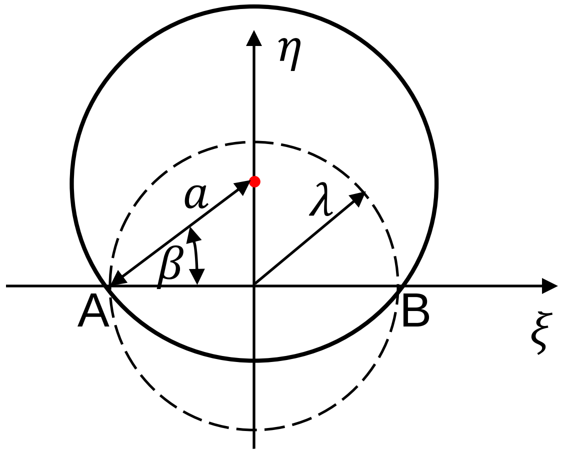
The critical point B is mapped onto a cusp in B' (cusps). If $\varepsilon = a - \lambda \ll \lambda$ the symmetric airfoil which is generated has max thickness equal to approximately $3\sqrt{3}\,\varepsilon$ and this max thickness occurs at a position distant about ${}^{c}/_{4} \approx \lambda$ from A'.

The symmetric J versus NACA airfoil

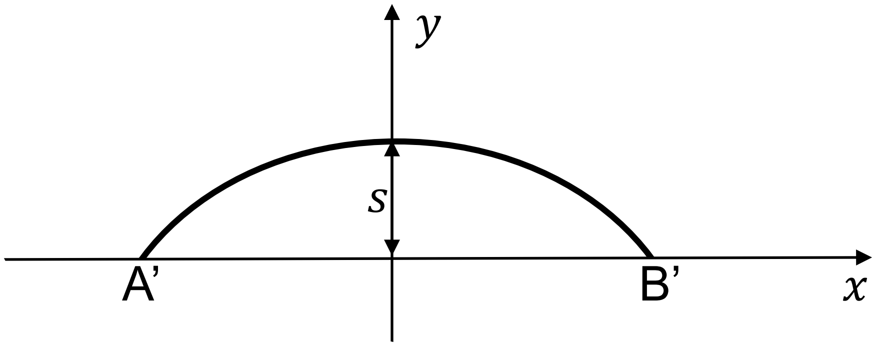


The Joukowski mapping

Case 3: circle of radius $a > \lambda$ centered on the imaginary axis



$\lambda = a\cos\beta$



circular arc airfoil of chord $c = 4\lambda$
with cusps in A' and B'

$\varepsilon = a\sin\beta$ = distance of center of circle from origin

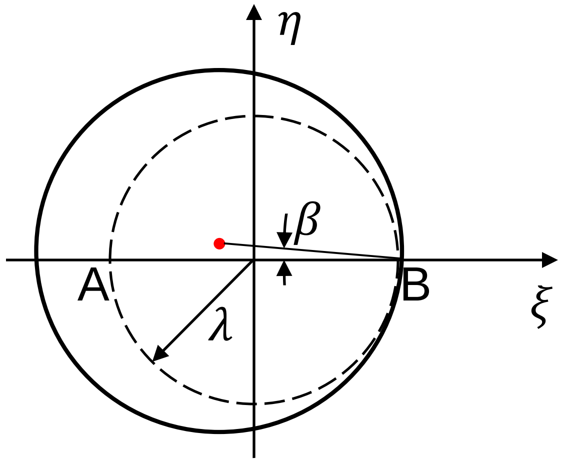
max camber height: $s = 2a\sin\beta$

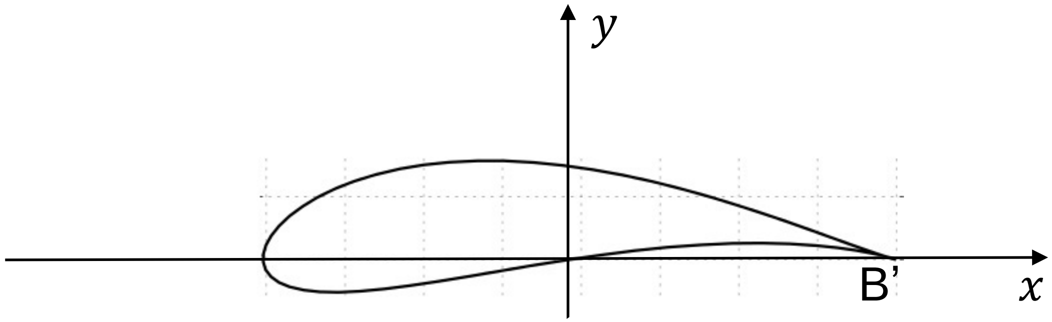
airfoil's equation (for $\varepsilon \ll \lambda$):

$$x^2 + \left(y + \frac{\lambda^2}{\varepsilon}\right)^2 \approx \lambda^2\left(4 + \frac{\lambda^2}{\varepsilon^2}\right)$$

The Joukowski mapping

Case 4: circle of radius $a > \lambda$ centered in the complex plane





Joukowski airfoil
with cusp in B'

$\frac{a-\lambda}{\lambda}$ controls the J airfoil's thickness

β controls the camber of the airfoil

By increasing the thickness, circulation, and thus lift, around the airfoil increase; however, large thickness means large D' …

Joukowski transformation

https://demonstrations.wolfram.com/TheJoukowskiMappingAirfoilsFromCircles/

http://www.dicat.unige.it/~irro/

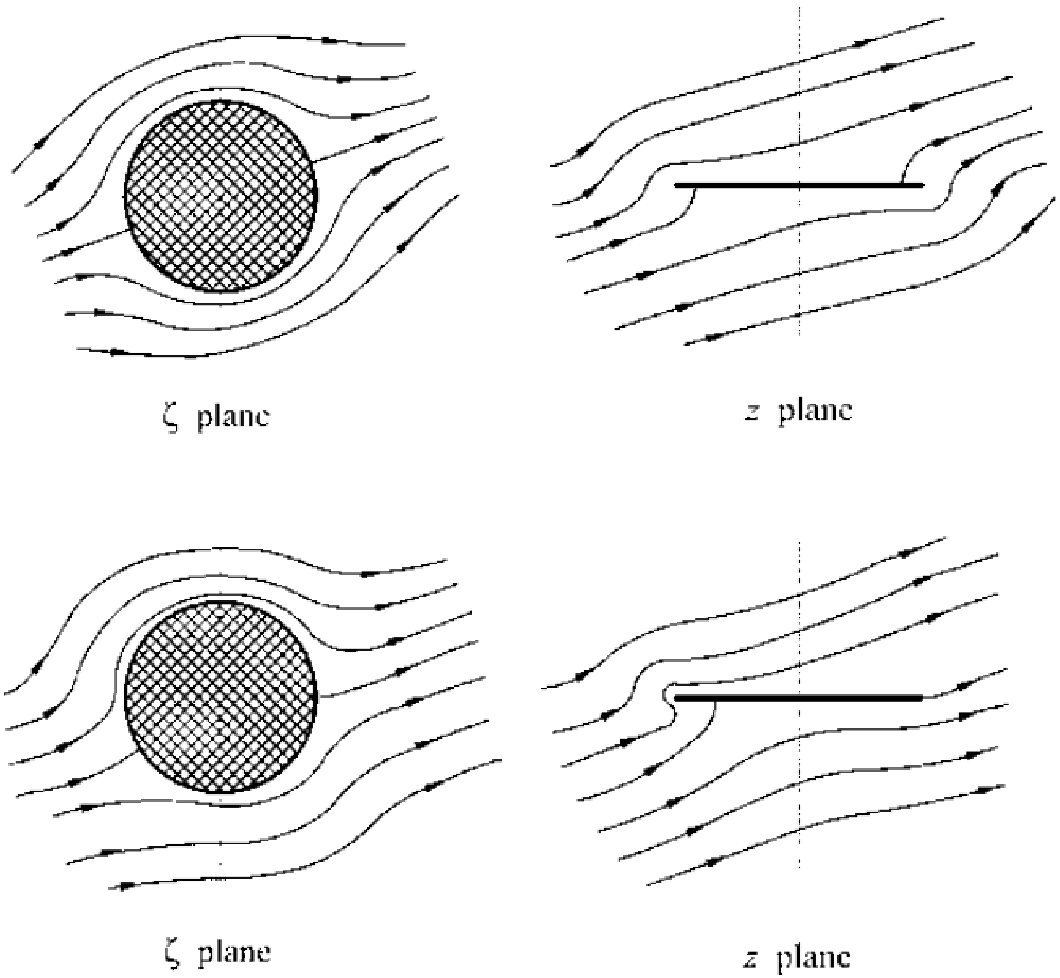
Kutta condition

The *Kutta condition* (slide 65) imposes that the rear stagn pt in the circle in the ζ $-$plane must map onto a cusp in z.

This condition mimics the effect of viscosity, i.e. the presence of a thin boundary layer around the airfoil: to let the flow out smoothly at the trailing edge we must add circulation to our potential flow solution. In physical reality this circulation is provided by the vorticity within the boundary layer.

http://dimanov.com/airfoil/feature.html

The kinematic problem for the flat plate



Flow past a cylinder with a small angle of attack

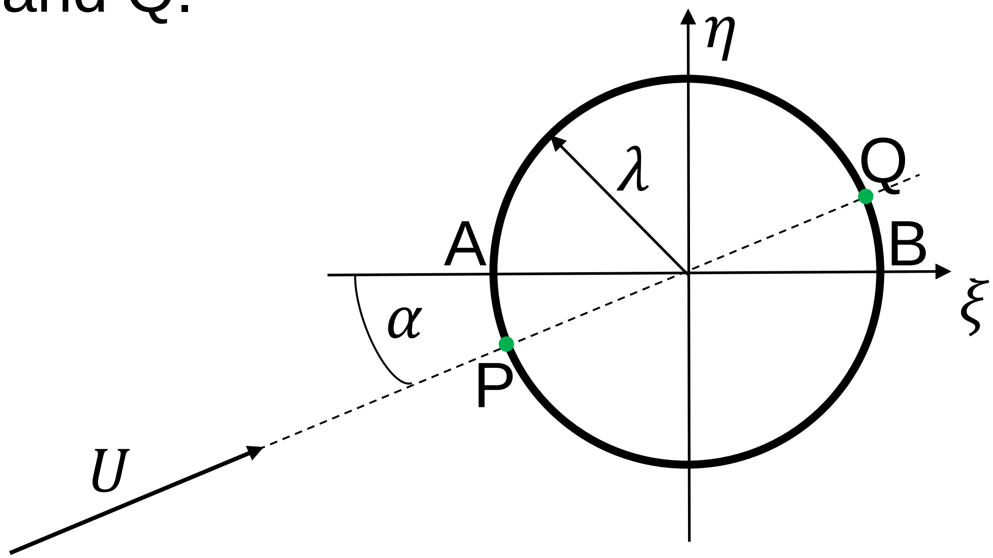
$\Gamma = 0$

$\Gamma \neq 0$

the *correct amount* of Γ is supplied to let the flow go out smoothly at the TE

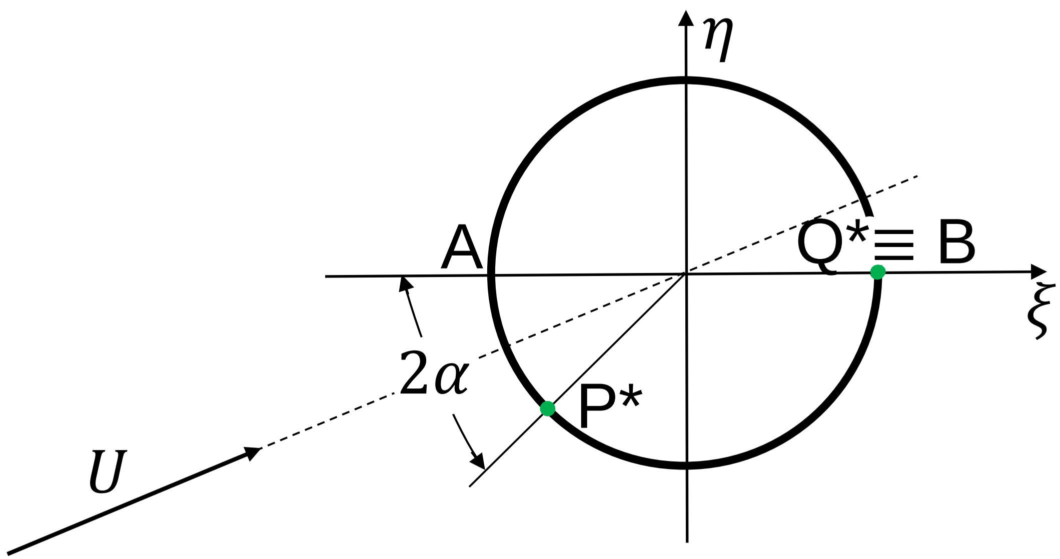
The kinematic problem for the flat plate

What is the *correct amount* of Γ? Assume that the uniform incoming flow, of speed U, has an angle of attack α with respect to the AB segment. For $\Gamma = 0$ stagnation points are P and Q.



The kinematic problem for the flat plate

To satisfy the Kutta condition a *clockwise* vortex must be added so that the rear stagnation point is moved from Q to Q* (to coincide with B), while at the same time P moves to P*.

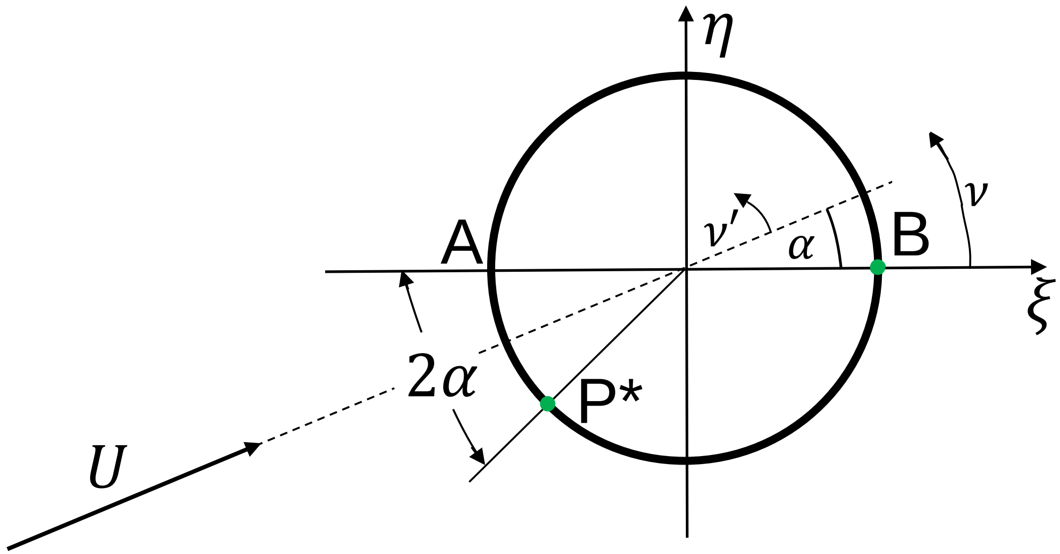


The kinematic problem for the flat plate

We know that on the circle of radius λ we have (slide 23):

$$v_r(\lambda) = 0,\ \ v_\theta(\lambda) = -2U\sin\nu' + \frac{\Gamma}{2\pi\lambda}$$

On B: $\quad \nu' = -\alpha$

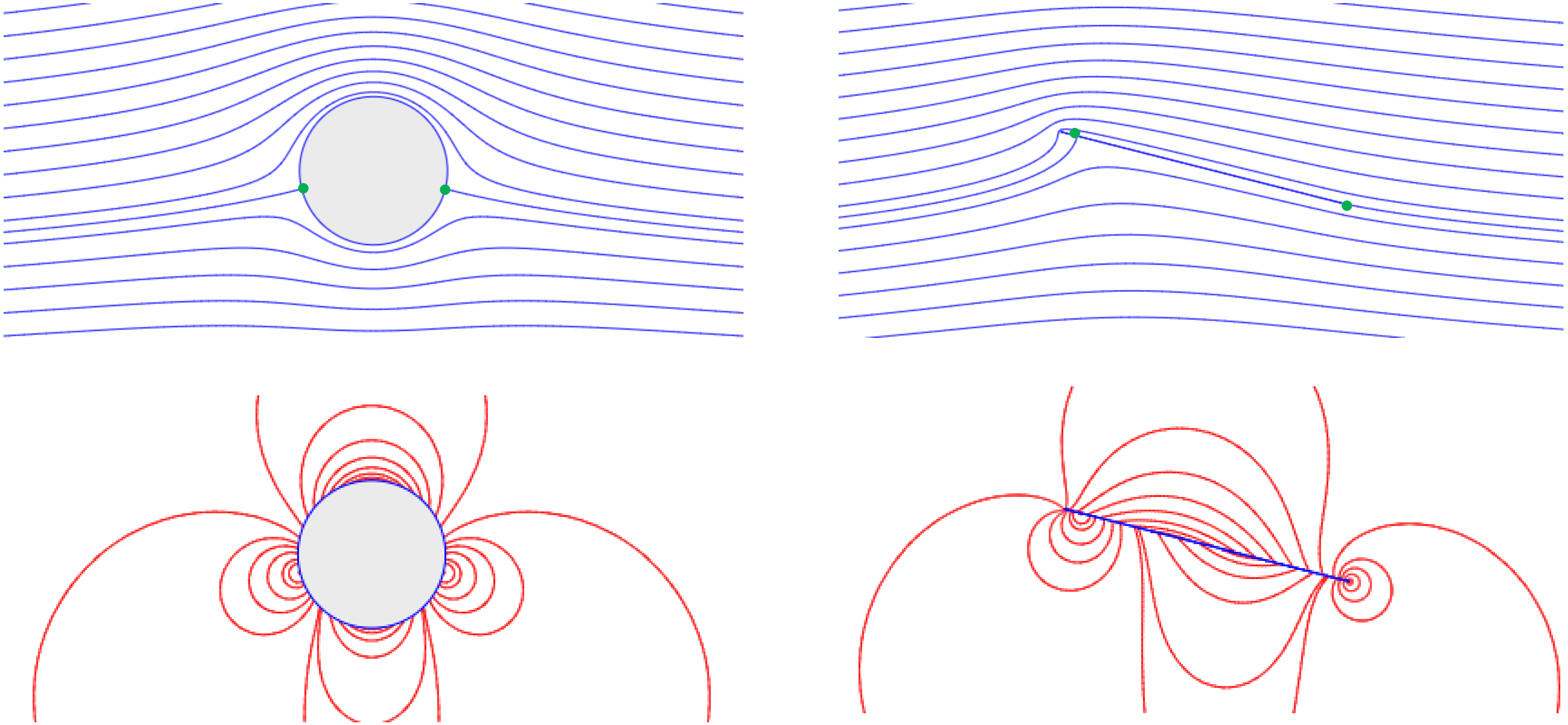


It must thus be:

$$\Gamma = -4\,\pi\,\lambda\,U\sin\alpha$$

This same Γ is also the circulation about the flat plate (slide 58).

The kinematic problem for the flat plate



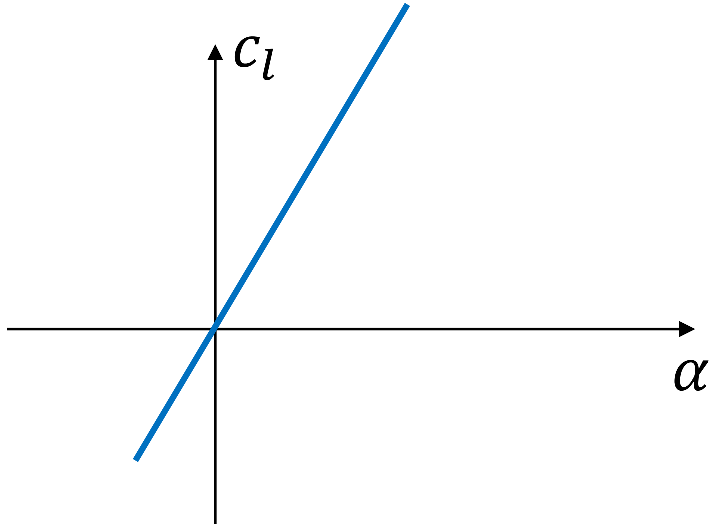
The kinematic problem for the flat plate

The lift force on the flat plate is (from KJ theorem):

$$L' = -\rho\, U\, \Gamma = 4\,\pi\,\rho\, U^2 \lambda \sin\alpha = \pi\,\rho\, U^2 c \sin\alpha$$

The lift coefficient is: $c_l = \dfrac{\pi\,\rho\, U^2 c \sin\alpha}{\frac{1}{2}\rho\, U^2 c} = 2\,\pi \sin\alpha \approx 2\,\pi\,\alpha$

The last relation is ok for small angles of incidence α; the flat plate displays a linear behavior of c_l with α (measured in radians)



The kinematic problem for the flat plate

Bernoulli: $p_\infty + \frac{1}{2}\rho\, U^2 = p + \frac{1}{2}\rho\, \mathbf{v}\cdot\mathbf{v}$ (p and $\mathbf{v}$ on the surface of the flat plate)

$$c_p = 1 - \left(\frac{W\bar{W}}{U^2}\right)$$

We thus need the complex velocity $W(z)$ on the flat plate surface. From slide 54 we know that

$$|W(z)| = \frac{|\widetilde{W}(\zeta)|}{|f'(\zeta)|} = \frac{|\widetilde{W}(\zeta)|}{\left|1 - \frac{\lambda^2}{\zeta^2}\right|} = \frac{|\widetilde{W}(\zeta)|}{|1 - e^{-2i\nu}|}$$

since $\zeta = \lambda\, e^{i\nu}$ on the cylinder

The kinematic problem for the flat plate

$$|W(z)| = \frac{|\widetilde{W}(\zeta)|}{|1-\cos(2\nu)+i\sin(2\nu)|} = \frac{|\widetilde{W}(\zeta)|}{\sqrt{[1-\cos(2\nu)]^2+\ \sin^2(2\nu)}}$$

$$= \frac{|\widetilde{W}(\zeta)|}{\sqrt{2\,[1-\cos(2\nu)]}} = \frac{|\widetilde{W}(\zeta)|}{|2\sin\nu|} = \frac{|\widetilde{W}(\zeta)|}{|2\sin(\nu'+\alpha)|}$$

Furthemore, from slide 81 we know that $\Gamma = -4\,\pi\,\lambda\,U\sin\alpha$ and

$$v_\theta(\lambda) = -2U\sin\nu' + \frac{\Gamma}{2\pi\lambda} \;\rightarrow\; v_\theta(\lambda) = -2U\,(\sin\nu' + \sin\alpha)$$

$$|W(z)| = \frac{|U\,(\sin\nu' + \sin\alpha)\,|}{|\sin\nu'\cos\alpha + \cos\nu'\sin\alpha|}$$

The kinematic problem for the flat plate

$$|W(z)| = \frac{|U\,(\sin\nu' + \sin\alpha)\,|}{|\sin\nu' \cos\alpha + \cos\nu' \sin\alpha|}$$

on A (LE of flat plate, A'): $\nu' \to \pi - \alpha$

on B (TE of flat plate, B'): $\nu' \to -\alpha$

on LE: $|W(z)| \to \dfrac{|2\,U\sin\alpha\,|}{|\sin\alpha\cos\alpha - \cos\alpha\sin\alpha|} \to \infty$

on TE: $|W(z)| \to \dfrac{|U\,(-\sin\alpha + \sin\alpha)|}{|-\sin\alpha\cos\alpha + \cos\alpha\sin\alpha|} \to\ ?$

The kinematic problem for the flat plate

$$|W(z)| = \frac{|U\,(\sin\nu' + \sin\alpha)\,|}{|\sin\nu'\cos\alpha + \cos\nu'\sin\alpha|}$$

on B/B’: $\lim_{\nu' \to -\alpha} |W(z)|$ (using l'Hôpital's rule)

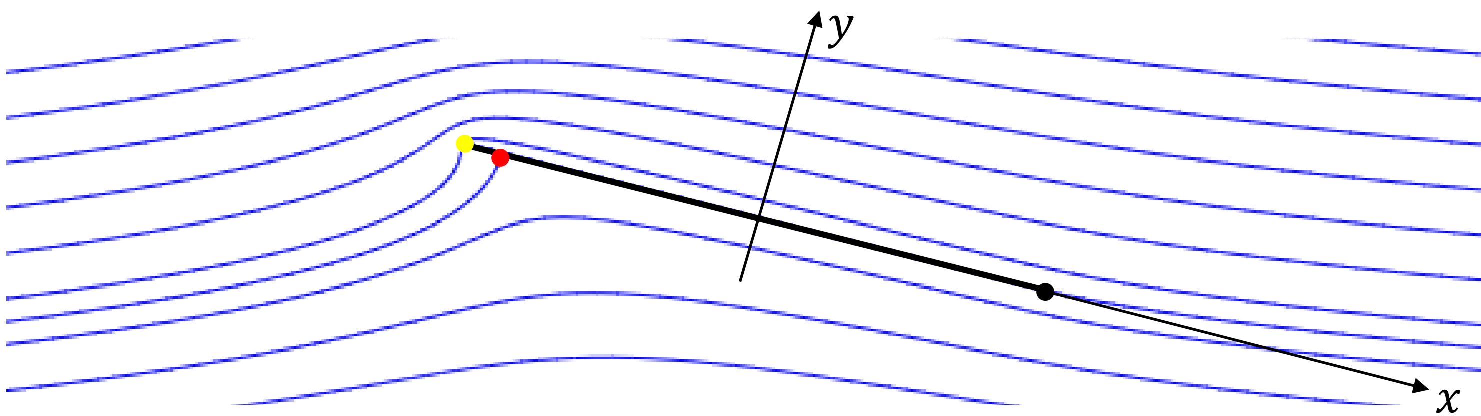
$$= \frac{|U\cos\nu'|}{|\cos\nu'\cos\alpha - \sin\nu'\sin\alpha|} = \frac{|U\cos\alpha|}{|\cos^2\alpha + \sin^2\alpha|} = |U\cos\alpha|$$

and for small angles of incidence, α, the velocity in B’, TE of the flat plate, has modulus equal to that of the free stream speed.

Notice: neither A’ nor B’ are stagnation points (cf. slide 54)

The kinematic problem for the flat plate

The only stagnation point on the flat plate is in P*'



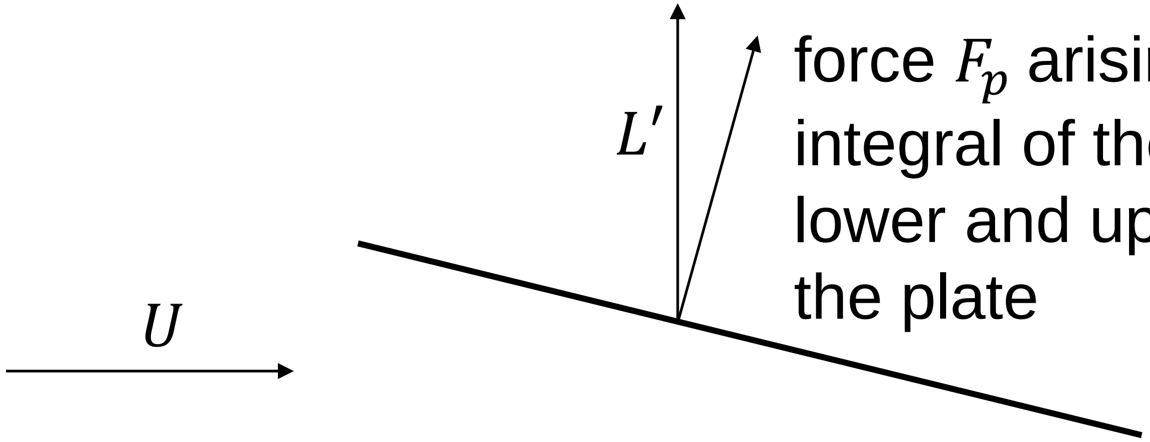
- A'
- P*'
- B'

To find the position of P*' we note that P* is in $\nu' = \pi + \alpha$ (or $\nu = \pi + 2\alpha$); since the plate has eq: $z = 2\,\lambda\cos(\nu)$ (slide 71) we finally have

$$z_{\mathrm{P}*'} = -2\,\lambda\cos(2\alpha)$$

The kinematic problem for the flat plate

Kutta-Joukowski theorem states that lift is always perpendicular to U (slide 28). However, since pressure acts always normal to the flat plate, we seem to have a problem …

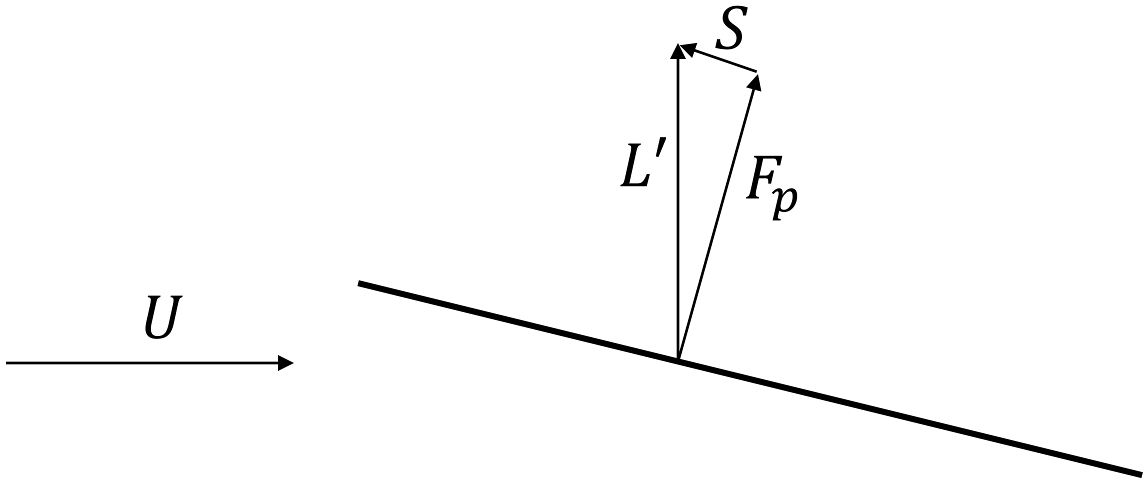


force F_p arising from the integral of the pressure on lower and upper sides of the plate

(go back and check the heuristic argument of slide 29)

The kinematic problem for the flat plate

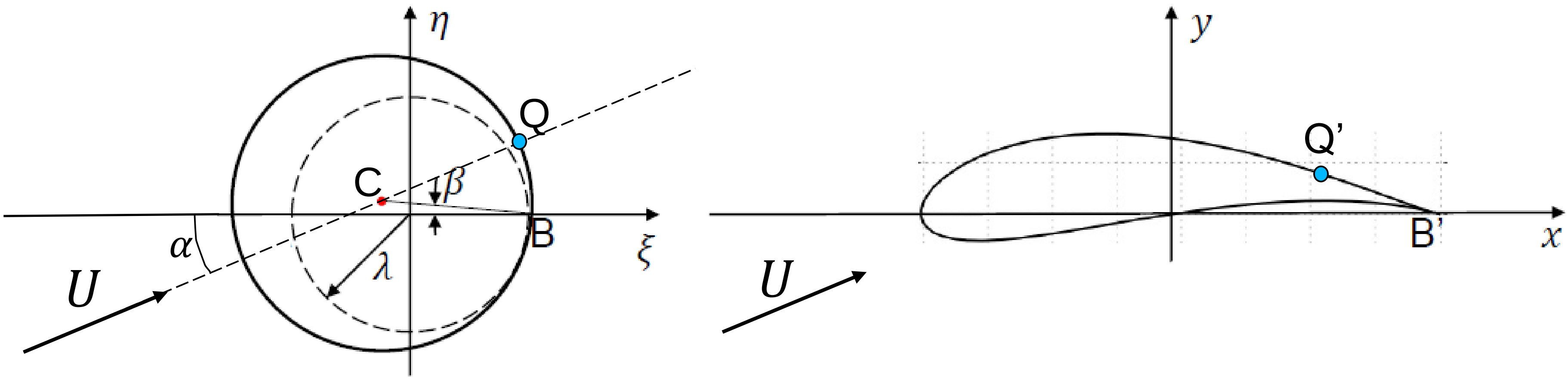
Apparent paradox … the velocity at the LE tends to ∞ and the pressure thus tends to $-\infty$ (Bernoulli). This p_{LE} produces a finite suction force S at the LE, the product of a "very large" (in modulus) pressure and a "very small" LE area.



D'Alembert paradox stands! There is no drag force $/\!/\ U$.

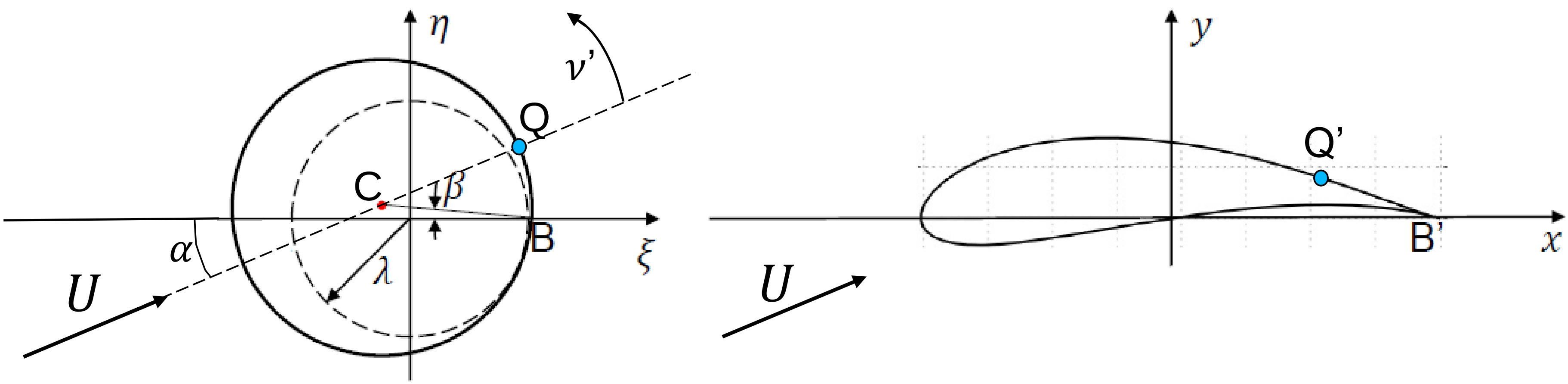
The kinematic problem for the J airfoil

J airfoil with α angle of incidence



The point Q on the circle must rotate (clockwise) by an angle $\alpha + \beta$ for the fluid to flow smoothly out of the TE in B'. Clearly, also the front stagnation point on the circle rotates (counter-clockwise) so that the stagnation point on the physical plane moves on the lower side of the airfoil.

The kinematic problem for the J airfoil



The tangential velocity on the point B of the circle centered in C and of radius a is $v_\theta = -2U\sin\nu' + \frac{\Gamma}{2\pi a} = -2U\sin(-\alpha-\beta) + \frac{\Gamma}{2\pi a} =$ $= 2U\sin(\alpha+\beta) + \frac{\Gamma}{2\pi a}$. Thus, B is stagnation point iff

$$\Gamma = -4\,\pi\,a\,U\sin(\alpha+\beta)$$

The kinematic problem for the J airfoil

The *same* clockwise circulation $\Gamma = -4\,\pi\, a\, U\, \sin(\alpha+\beta)$ is applied on the physical plane.

Lift on the airfoil is $L' = -\rho\, U\, \Gamma = 4\,\pi\,\rho\, a\, U^2 \sin(\alpha+\beta)$ and the lift coefficient is

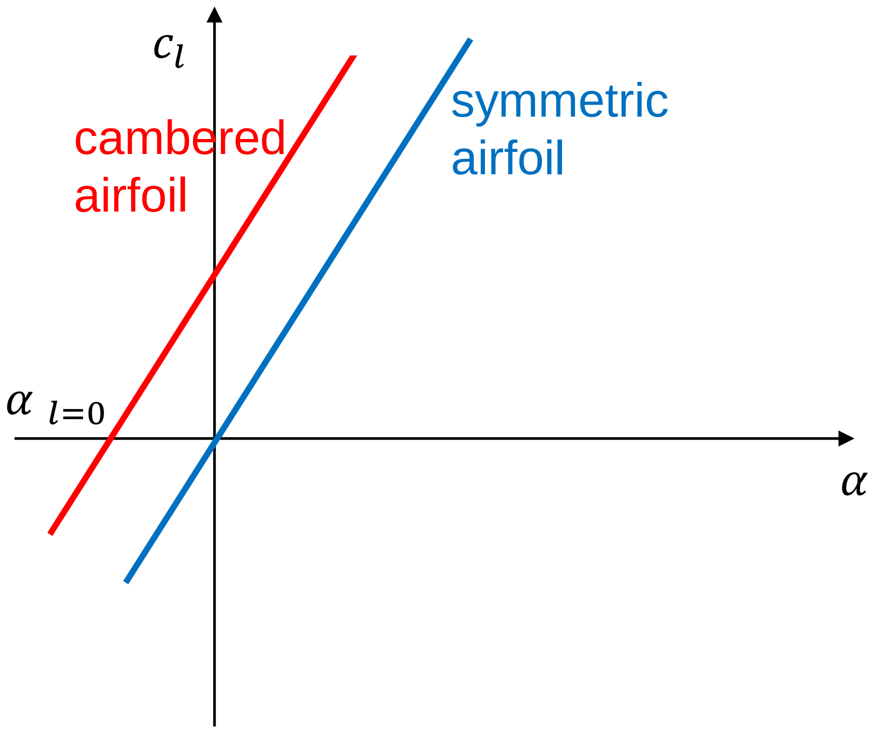
$$c_l = \frac{L'}{\frac{1}{2}\rho U^2 c} = \frac{8\,\pi\, a\, \sin(\alpha+\beta)}{c}$$

if the point C is not too far from the origin of the ζ −plane, the chord of the airfoil is $c \approx 4\lambda \approx 4a$ and the lift coefficient reads:

$$c_l \approx 2\,\pi\, \sin(\alpha+\beta) \approx 2\pi\,(\alpha+\beta)$$ (for α and β small)

The kinematic problem for the J airfoil

$c_l \approx 2\pi\,(\alpha + \beta) = 2\,\pi\,\alpha'$ with α' the *effective angle of attack*, which accounts for the camber of the airfoil (through β).



symmetric airfoil

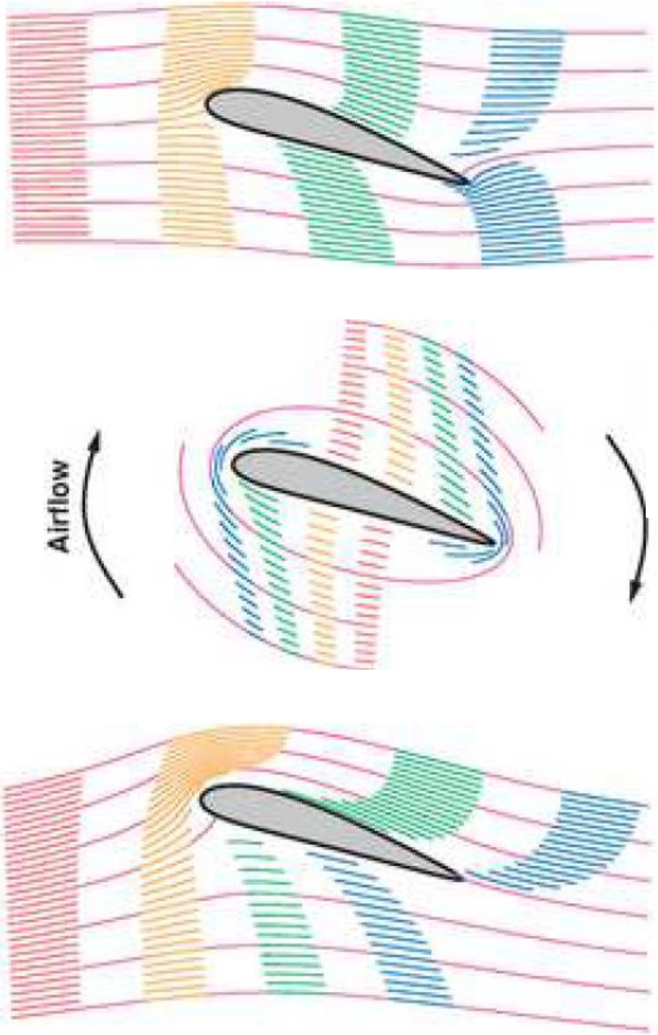
$c_l = 0$ for $\alpha = 0$

cambered airfoil

$c_l = 0$ for $\alpha = -\beta = \alpha_{l=0}$

when the geometric angle of attack vanishes there is still some lift: $c_l \approx 2\,\pi\,\beta$

To Kutta or not to Kutta

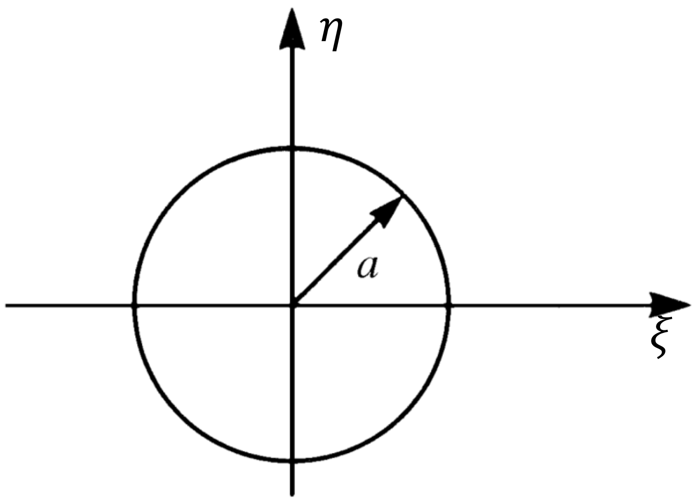


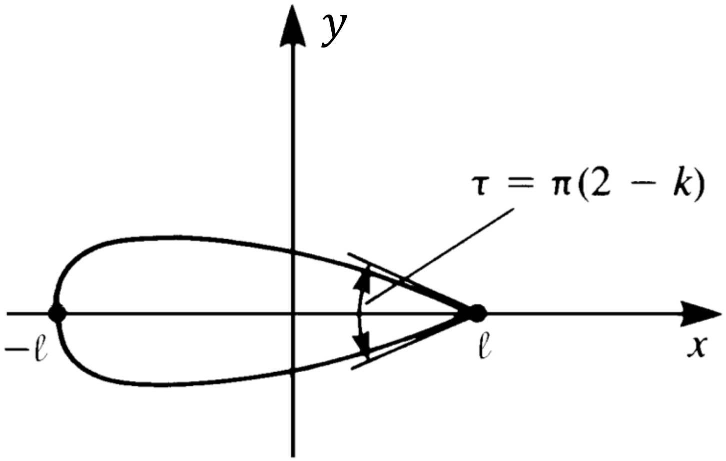
Another conformal mapping

Mapping by van de Vooren and de Jong

$$z = \frac{(\zeta - a)^k}{(\zeta - \epsilon a)^{k-1}} + \ell$$

with a *finite TE angle*

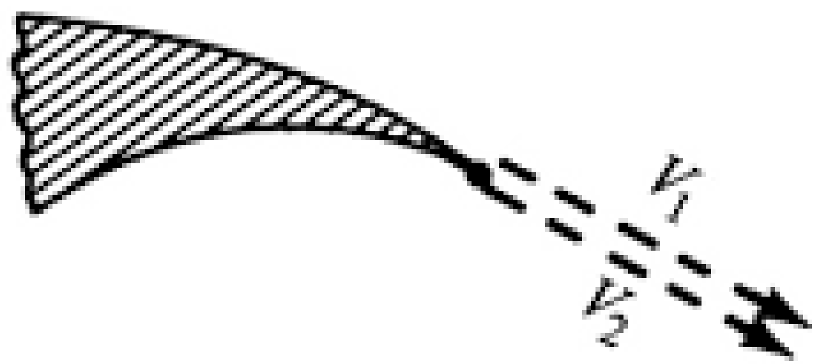




TE angle

finite angle

cusp



at TE: $V_1 = V_2 = 0$

at TE: $V_1 = V_2 \neq 0$

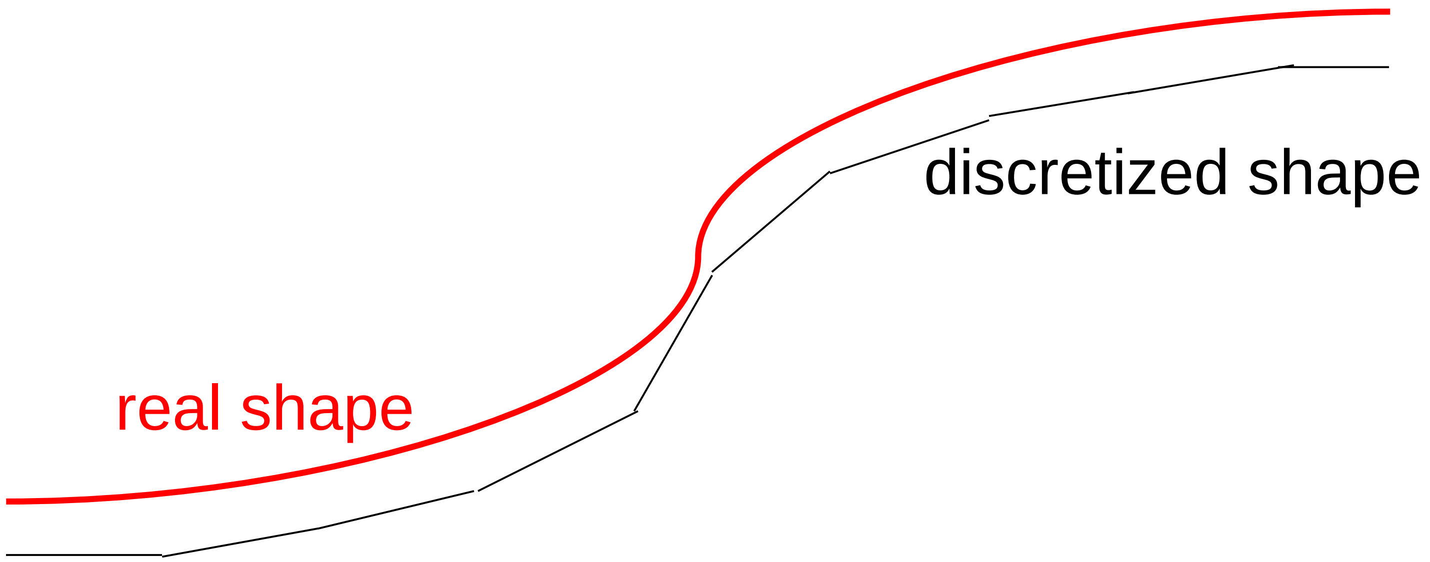
Conformal mappings

Exercises (third set)

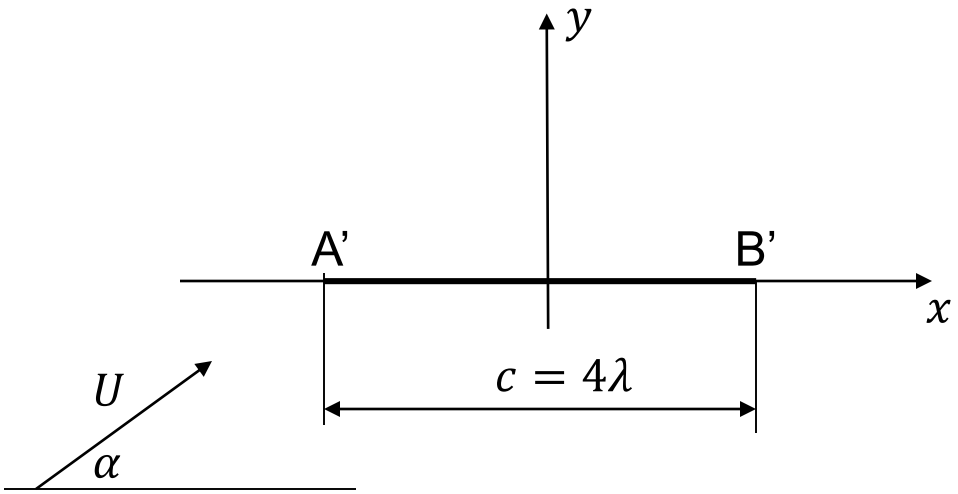
1. For the symmetric Joukowski airfoil (slide 72) find the coordinate $z_{\mathrm{A}'}$ of the leading edge.
2. For the circular arc airfoil (slide 74) show that the maximum camber height is $s = 2a\,\sin\beta$.
3. Consider the cambered Joukowski airfoil (slides 75 and 92). Where does the center C of the circle in the ζ −plane go in the z −plane?
4. Show that the conformal mapping $z = \zeta + \frac{a^2-b^2}{4\,\zeta}$ maps a circles of radius $\frac{a+b}{2}$ in the ζ −plane to an ellipse of semi-axis a and b onto the z −plane.

Using J mapping for arbitrary shapes

Let us now see how the J mapping for a flat plate of length c can be used to model arbitrarily shaped 2D bodies, formed by many finite-length segments



Using J mapping for arbitrary shapes



The fluid exits smoothly at B' iff

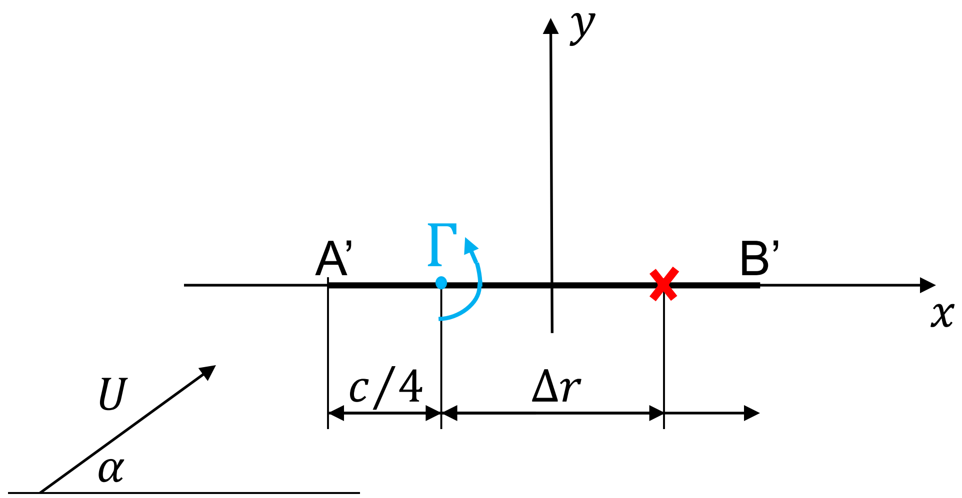
$$\Gamma = -\pi\, c\, U \sin\alpha$$

(cf. slide 81)

Let us imagine to replace the flat plate with a **potential vortex** of circulation Γ, **positioned in the AC** (which is $c/4$ from the LE of the plate) plus a **collocation point**, which is a point on the plate where we impose the flow to be **tangent** to the plate.

Recall: **AC** is defined such that $\frac{dm_{AC}}{d\alpha} = 0$. For symmetric airfoils, e.g. the flat plate, it will also be shown that $m_{AC} = 0$ in the potential flow case.

Using J mapping for arbitrary shapes



$\Gamma = -\pi\, c\, U \sin\alpha$

Distance vortex-collocation point $= \Delta r$

The azimuthal velocity induced by the vortex on the collocation point is $\Gamma/(2\pi\Delta r)$: this velocity is perpendicular to the plate and downwards (because circulation is negative). The total vertical velocity on the collocation pt vanishes if

$U \sin\alpha = c\, U \sin\alpha\, /(2\, \Delta r) \quad \rightarrow \quad \Delta r = c/2$

The lumped vortex element method

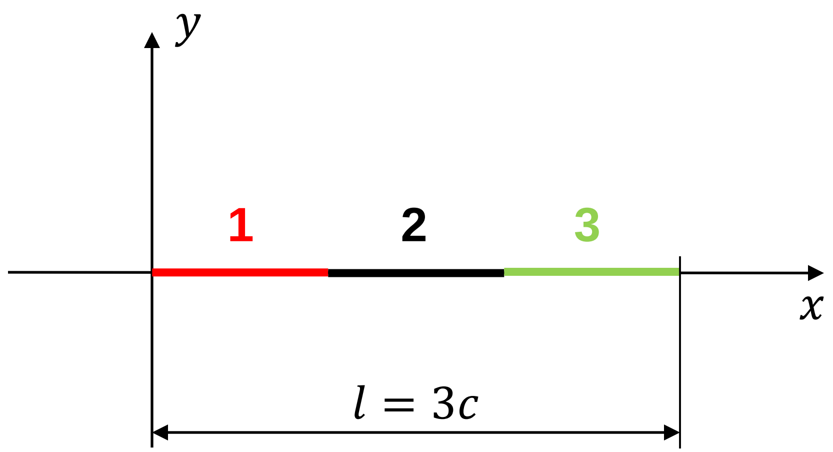
All individual segments of length c which, taken one after the other, make up a complex 2D shape can be represented as a series of vortices positioned on the AC of the segments, plus a series of collocation points, positioned $c/2$ downstream of the vortices.

This is called the **lumped vortex element method**.

Note: for symmetric airfoils, and thus for the flat plate as well, the AC coincides with the CP (both of them are in $c/4$)

The lumped vortex element method

Example: let us use the lumped vortex element method to represent a flat plate of length l made up by three equal segments of length $c = l/3$

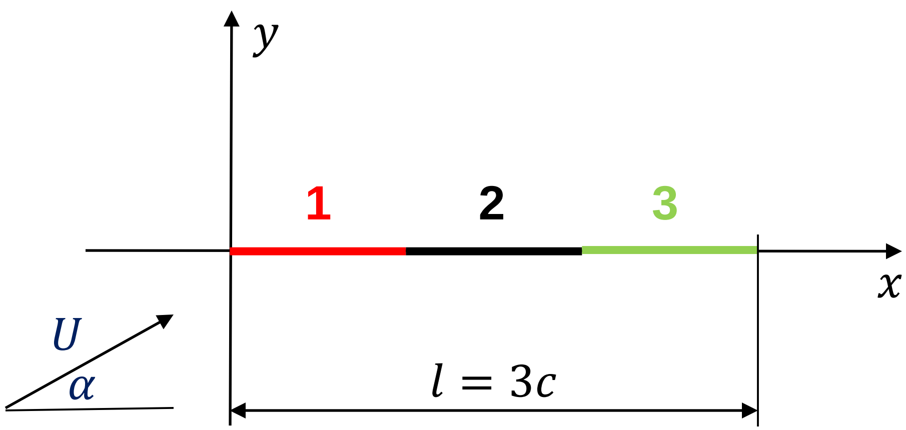


element	distance from origin
V1	$c/4$
C1	$3c/4$
V2	$c+c/4 = 5c/4$
C2	$c+3c/4 = 7c/4$
V3	$2c+c/4 = 9c/4$
C3	$2c+3c/4 = 11c/4$

The lumped vortex element method

Vertical velocity on **C1** induced by the 3 vortices:

$$\frac{\Gamma_1}{2\pi\left(x_{C_1} - x_{V_1}\right)} - \frac{\Gamma_2}{2\pi\left(x_{V_2} - x_{C_1}\right)} - \frac{\Gamma_3}{2\pi\left(x_{V_3} - x_{C_1}\right)}$$



V1	$c/4$
C1	$3c/4$
V2	$c+c/4 = 5c/4$
C2	$c+3c/4 = 7c/4$
V3	$2c+c/4 = 9c/4$
C3	$2c+3c/4 = 11c/4$

$$v_{C_1} = U\sin\alpha + \sum_{i=1}^{3} \frac{\Gamma_i}{2\pi\left(x_{C_1} - x_{V_i}\right)}$$

The lumped vortex element method

$$v_{c_j} = U \sin\alpha + \sum_{i=1}^{3} \frac{\Gamma_i}{2\pi\left(x_{c_j} - x_{v_i}\right)}$$

The vertical velocity components on the 3 collocation points must vanish:

C_1: $$2\,\pi\, U \sin\alpha + \frac{\Gamma_1}{l/6} - \frac{\Gamma_2}{l/6} - \frac{\Gamma_3}{l/2} = 0$$

C_2: $$2\,\pi\, U \sin\alpha + \frac{\Gamma_1}{l/2} + \frac{\Gamma_2}{l/6} - \frac{\Gamma_3}{l/6} = 0$$

C_3: $$2\,\pi\, U \sin\alpha + \frac{\Gamma_1}{5l/6} + \frac{\Gamma_2}{l/2} + \frac{\Gamma_3}{l/6} = 0$$

The lumped vortex element method

The system of three equations in three unknowns yields:

$\Gamma_1 = -\frac{5}{8}\,\pi\,U\,l\sin\alpha$ (clockwise circulation)

$\Gamma_2 = -\frac{1}{4}\,\pi\,U\,l\sin\alpha$ (clockwise circulation)

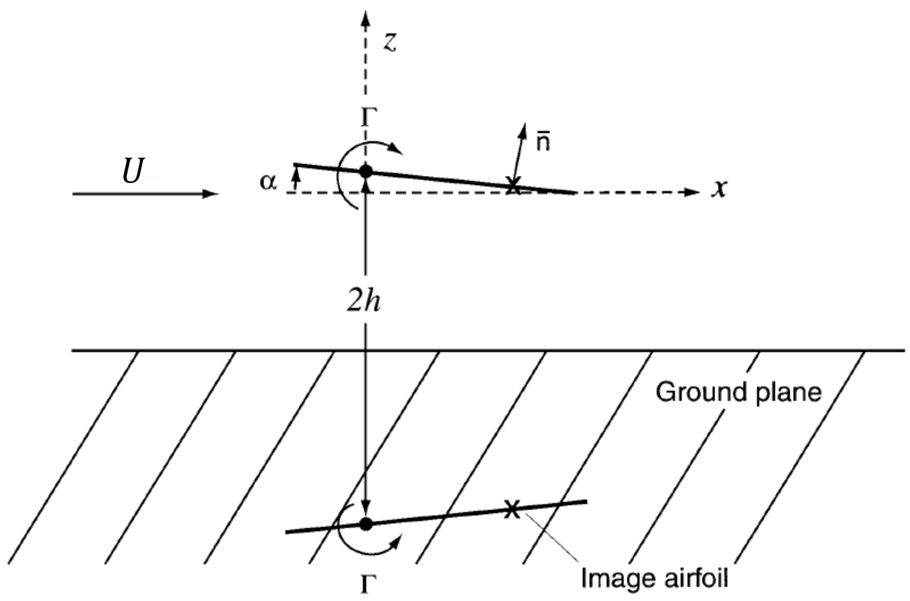
$\Gamma_3 = -\frac{1}{8}\,\pi\,U\,l\sin\alpha$ (clockwise circulation)

The total circulation around the flat plate is $\Gamma = \sum_{i=1}^{3}\Gamma_i$

$\Gamma = -\,\pi\,U\,l\sin\alpha$ (cf. slide 81!)

The mirror image method

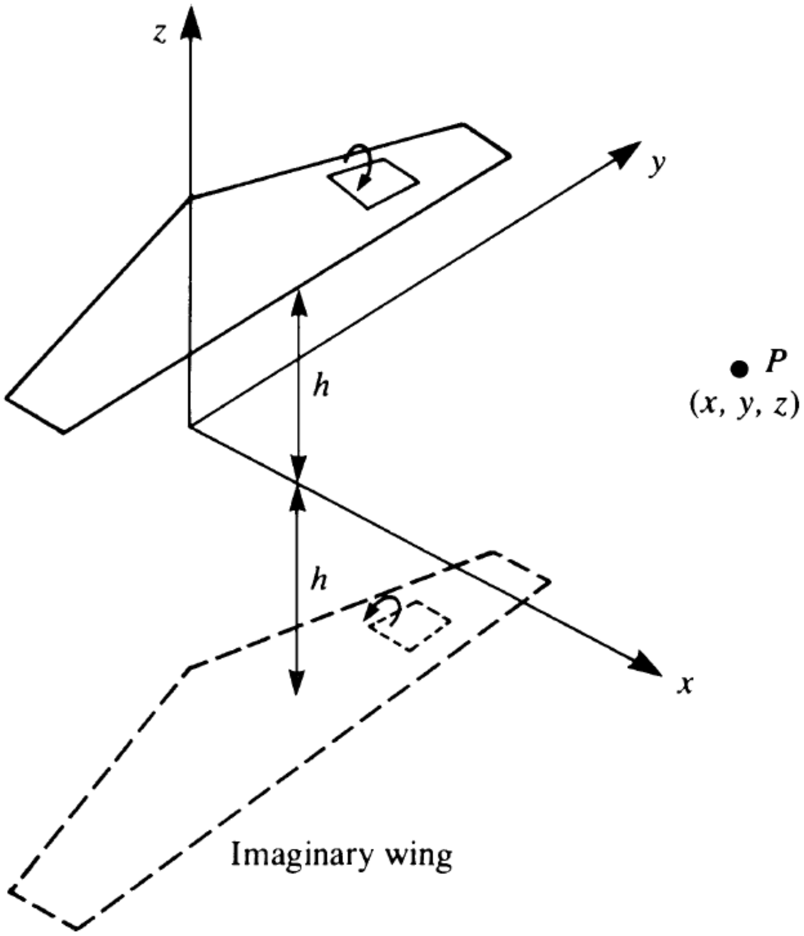
When in the vicinity of the ground (landing or takeoff) the behavior of a wing is modified from that observed in an unrestricted freestream. This is called *ground effect.* If the wing is modelled by *lumped vortex elements*, the presence of the ground can be modelled by the **method of images**, for the ground to become a streamline.



A similar strategy can be adopted, for example, to model wind tunnel walls.

The mirror image method

The same technique can be used in 3D when using distributions of surface singularities (**3D panel method**)



Lumped vortex element method

Exercises (fourth set)

1. Let us consider a tandem of flat plate airfoils, as in the figure, in a uniform stream of velocity U and angle of attack α. Compute the circulation of both plates.

2. Let us model a symmetric airfoil as an equilateral triangle, formed by three panels of length l, potential vortices on the vertices of the triangle and collocation points centered along the sides. Compute the circulation of each potential vortex, when the profile is in a stream of constant velocity U.