Perturbation Methods

PhD in Enviromental Fluid Mechanics

Department of Civil, Environmental and Civil Engineering
University of Genoa, Italy

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Further sources of material have been taken from the following textbooks:

- Nayfeh (1973)

Introduction

Some Basic Tools

The Method of Multiple Scales

The Method of Strained Parameters/Coordinates

Matched and Composite Asymptotic Expansions

Introduction to Matched and Composite Asymptotic Expansions I

The method of **strained coordinates** is not capable of yielding uniformly valid expansions in cases in which sharp changes in dependent variables take place in some regions of the domain of the independent variables. In these cases straightforward expansions generally break down in these regions, and near-identity transformations of the independent variables (strained coordinates) cannot cope with such sharp changes. To obtain uniformly valid expansions, we must recognize and utilize the fact that the sharp changes are characterized by magnified scales which are different from the scale characterizing the behavior of the dependent variables outside the sharp-change regions.

One technique of dealing with this problem is to determine straightforward expansions (called outer expansions) using the original variables and to determine expansions (called inner expansions) describing the sharp changes using magnified scales. To relate these expansions a so-called matching procedure is used. This technique is called the method of inner and outer expansions or, after Bretherton (1962), the method of **Matched Asymptotic Expansions**.

A second technique for determining a uniformly valid expansion is to assume that each dependent variable is the sum of (i) a part characterized by the original independent variable and (ii) parts characterized by magnified independent variables, one for each sharp-change region. This is the method of **Composite Expansions** in its simplest form.

Prandtl's Technique I

Consider the simple boudary problem

$$\epsilon y'' + y' + y = 0 \tag{1}$$

with the following boundary conditions

$$y(0) = \alpha \quad , \quad y(1) = \beta \tag{2}$$

where ϵ is a parameter much smaller than unity, $\epsilon \ll 1$. The problem admits the exact solution

$$y = \frac{(\alpha e^{s_2} - \beta)\, e^{s_1 x} + (\beta - \alpha e^{s_1})\, e^{s_2 x}}{e^{s_2} - e^{s_1}} \quad \text{where} \quad s_{1,2} = \frac{-1 \pm \sqrt{1-4\epsilon}}{2\epsilon} \tag{3}$$

If we consider the case $\epsilon \longrightarrow 0$ then we obtain

$$s_1 = -1 - \epsilon \quad ; \quad s_2 = -\frac{1}{\epsilon} + 1 + \epsilon \qquad \Rightarrow \qquad y\,(x, \epsilon) = \beta e^{(1-x)} \tag{4}$$

This solution can not cope with the boundary condition $y(0) = \alpha$. The solution of the reduced equation is denoted by y^o and it is called the outer solution. For small ϵ the solution of the reduced equation is close to the exact solution except in a small interval at the end point $x = 0$ where the exact solution changes quickly in order to retrieve the boundary condition $y(0) = \alpha$ which is about to be lost. This small interval across which y changes very rapidly is called the boundary layer in fluid mechanics, the edge layer in solid mechanics, and the skin layer in electrodynamics.

It is possible to obtain the same result performing a straightforward expansion of the solution in the form $y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \ldots$.
Substituting this espression in the problem and equating likewise powers of ϵ, at the first order of approximation, we find that

$$\frac{dy_0}{dx} + y_0 = 0 \quad \text{with} \quad y_0(0) = \alpha \ , \quad y_0(1) = \beta \tag{5}$$

which can not be satisfied, since it is a first order equation with two boundary conditions. Being the general solution $y_0(x) = C_1 e^{-x}$, applying the boundary condition for $x = 1$ we obtain

$$y_0(1) = \beta = C_1 e^{-1} \quad \Rightarrow \quad C_1 = e\beta \quad \Rightarrow \quad y_0(x) = \beta e^{1-x} \tag{6}$$

The outer solution $y_0(x)$ does not satisfies the boundary condition for $x = 0$

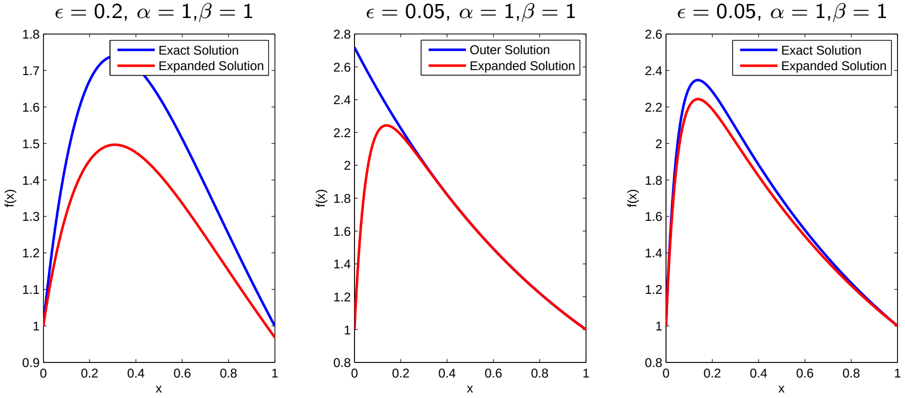
$$y_0(0) = \beta e^1 \neq \alpha \ \ !! \tag{7}$$

To understand what is happening close to $x = 0$, let us expand the exact solution close to the origin

$$y(x,\epsilon) = \frac{\left(\alpha e^{-\frac{1}{\epsilon}+1} - \beta\right) e^{-x} + \left(\beta - \alpha e^{-1}\right) e^{\left(-\frac{1}{\epsilon}+1\right)x}}{e^{-\frac{1}{\epsilon}+1} - e^{-1}} \tag{8}$$

which can be approximated for $\epsilon \longrightarrow 0$ with

$$y(x,\epsilon) = \beta e^{1-x} + (\alpha - \beta e)\, e^{\left(-\frac{1}{\epsilon}+1\right)x} + O(\epsilon) \tag{9}$$



Matching technique

One technique of dealing with this problem is to determine a straightforward expansion (called outer expansion) using the original variables and to determine an expansion (called inner expansion) describing the sharp changes close to $x = 0$ using a magnified scale

Outer Expansion

As presented in the previous slides, the outer expansion can be written as

$$y^o = y_0^o + \epsilon y_1^o + \epsilon^2 y_2^o + \dots \tag{10}$$

$$\frac{dy_0^o}{dx} + y_0^o = 0 \quad \text{with} \quad y_0^o(1) = \beta \tag{11}$$

Finally

$$y_0^o(x) = \beta e^{1-x} \tag{12}$$

Inner Expansion

To determine an expansion valid in the boundary layer (inner expansion), we magnify this layer using the stretching transformation which gives rise to the inner variable ζ

$$\zeta = \frac{x}{\epsilon} \quad \rightarrow \quad \frac{\partial}{\partial x} = \frac{1}{\epsilon}\frac{\partial}{\partial \zeta} \quad ; \quad y^i = y_0^i + \epsilon y_1^i + \epsilon^2 y_2^i + \dots \tag{13}$$

At the first order of approximation we obtain

$$\frac{d^2 y_0^i}{d\zeta^2} + \frac{dy_0^i}{d\zeta} = 0 \quad \text{with} \quad y_0^i(0) = \alpha \tag{14}$$

The solution is $y_0^i = C_1 + C_2 e^{-\zeta}$ and imposing the boundary condition $y_0^i(0) = C_1 + C_2 = \alpha$ we finally obtain

$$y_0^i(\zeta) = C_1 + (\alpha - C_1)\, e^{-\zeta} \tag{15}$$

The constant C_1 present in the inner solution can be determined imposing the matching between the inner and outer expansions

$$\lim_{x\to 0} y_0^o = \lim_{\zeta\to\infty} y_0^i \qquad \Rightarrow \qquad \beta e = C_1 \tag{16}$$

therefore

$$y_0^i(\zeta) = \beta e + (\alpha - \beta e)\, e^{-\zeta} \tag{17}$$

Matching principle

The matching principle $\lim_{x \longrightarrow 0} y_0^o(x,\epsilon) = \lim_{\zeta \longrightarrow \infty} y_0^i(\zeta,\epsilon)$ is equivalent to equating the inner limit of the outer solution $(y_0^o)^i$ to the outer limit of the inner solution $(y_0^i)^o$.

Composite Solution

To compute y as a function of all x, one must switch from one solution to the other as x increases at some small value of x such as the value where the solutions may intersect. This switching is not convenient, and we form from these solutions a single uniformly valid solution called the composite solution y^c

$$y^c = y^o + y^i - (y^o)^i = y^o + y^i - (y^i)^o \tag{18}$$

The composite solution is a uniform approximation over the whole interval of x including the gap between the outer and inner regions. The success of the matching may be due to the presence of an overlapping region in which both the outer and inner solutions are valid, hence there is no gap between the two regions.

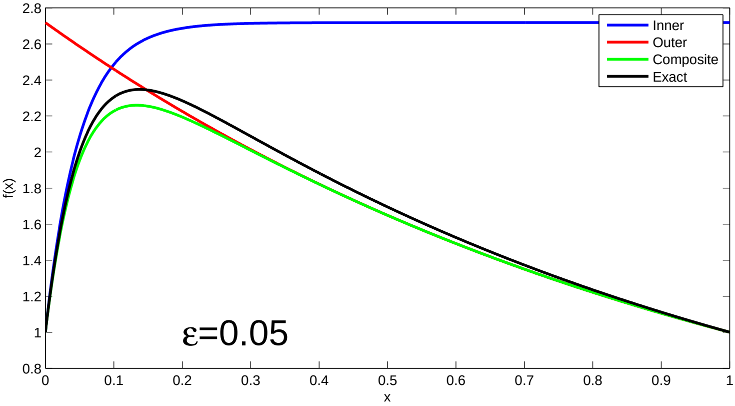
Note that $$((y^o)^i)^o = (y^o)^i = (y^i)^o = ((y^o)^i)^i$$

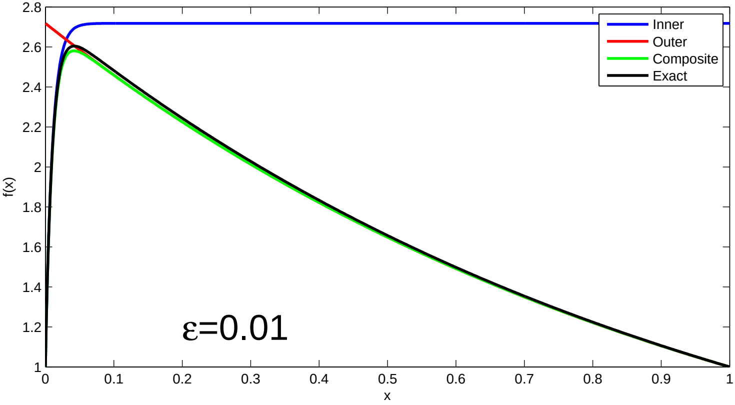
The composite solution reads

$$y^c = y^o + y^i - (y^o)^i = y^o + y^i - (y^i)^o =$$
$$= \beta e^{1-x} + \beta e + (\alpha - \beta e)\, e^{-\frac{x}{\epsilon}} - \beta e + O(\epsilon) \tag{19}$$

Finally

$$y^c = \beta e^{1-x} + (\alpha - \beta e)\, e^{-\frac{x}{\epsilon}} - \beta e + O(\epsilon) \tag{20}$$




Inner
Outer
Composite
Exact
ε=0.01
f(x)
x
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Higher Approximation and Refined Matching Procedures I

The procedure previously explained can be refined and the solution can be determined including higher order terms. We start by determining the boundary condition that must be dropped and determine the stretching transformation as a by-product. Then, we determine second-order inner and outer expansions and match them by using Van Dyke's principle. Finally, we form a uniformly valid composite expansion.

Which Boundary Contidion must be dropped?

Boundary Condition - $y(1) = \beta$

To determine if the boundary condition $y(1) = \beta$ must be dropped, we intoduce the stretching transformation

$$\zeta = (1 - x)\,\epsilon^{-\lambda} \quad ; \quad \lambda > 0 \tag{21}$$

hence we obtain

$$\frac{\partial}{\partial x} = -\epsilon^{-\lambda}\frac{\partial}{\partial \zeta} \tag{22}$$

and the original equation (1) becomes

$$\epsilon^{1-2\lambda}\frac{d^2 y}{d\zeta^2} - \epsilon^{-\lambda}\frac{dy}{d\zeta} + y = 0 \tag{23}$$

As $\epsilon \to 0$ three limiting forms arise depending on the value of λ

The Case $\lambda > 1$

Taking into accout that the first term is much larger than the second one, (23) becomes

$$\frac{d^2 y}{d\zeta^2} = 0 \qquad \Rightarrow \qquad y^i = A + B\zeta \tag{24}$$

Since the outer solution should be valid at $x = 0 \quad \Rightarrow y^o = \alpha e^{-x}$, the matching principle demads that

$$\lim_{\zeta \to \infty} (A + B\zeta) = \lim_{x \to 1} \alpha e^{-x} \qquad \Rightarrow \qquad B = 0 \quad \text{and} \quad A = \alpha e^{-1} \tag{25}$$

Hence $y^i = \alpha e^{-1}$. Since this solution is valid at $x = 1$, it should satisfy the boundary condition $y(x = 1) = \beta$, hence $\beta = \alpha e^{-1}$ which is not true in general.

The Case $\lambda < 1$

Taking into accout that the second term is much larger than the first term, (23) becomes

$$\frac{dy}{d\zeta} = 0 \qquad \Rightarrow \qquad y^i = A = \beta \tag{26}$$

This solution must be discarded because the matching principle demands that $\beta = \alpha e^{-1}$

The Case $\lambda = 1$

Equation (23) becomes

$$\frac{d^2 y}{d\zeta^2} - \frac{dy}{d\zeta} = 0 \qquad \Rightarrow \qquad y^i = A + Be^{\zeta} \tag{27}$$

Since the matching principle demands that

$$\lim_{\zeta \to \infty} \left(A + Be^{\zeta} \right) = \lim_{x \to 1} \alpha e^{-x} \qquad \Rightarrow \qquad B = 0 \quad \text{and} \quad A = \alpha e^{-1} \tag{28}$$

this case must be discarded also beacuse the application of the boundary condition $y(x = 1) = \beta$ demands $\beta = \alpha e^{-1}$.

> Therefore the boundary layer cannot exist at $x = 1$, hence the boundary condition $y(1) = \beta$ must not be dropped.

Boundary Condition - $y(0) = \alpha$

To determine if the boundary condition $y(0) = \alpha$ must be dropped, we intoduce the stretching transformation

$$\zeta = x\epsilon^{-\lambda} \quad ; \quad \lambda > 0 \tag{29}$$

hence we obtain

$$\frac{\partial}{\partial x} = \epsilon^{-\lambda} \frac{\partial}{\partial \zeta} \tag{30}$$

and the original equation (1) becomes

$$\epsilon^{1-2\lambda} \frac{d^2 y}{d\zeta^2} + \epsilon^{-\lambda} \frac{dy}{d\zeta} + y = 0 \tag{31}$$

In this case also three different limiting forms arise as $\epsilon \to 0$ depending on the value of λ

The Case $\lambda > 1$

$$\frac{\partial^2 y}{\partial \zeta^2} = 0 \quad \Rightarrow \quad y^i = A + B\zeta \tag{32}$$

The Case $\lambda < 1$

$$\frac{\partial y}{\partial \zeta} = 0 \quad \Rightarrow \quad y^i = A \tag{33}$$

The Case $\lambda = 1$

$$\frac{\partial^2 y}{\partial \zeta^2} + \frac{\partial y}{\partial \zeta} = 0 \quad \Rightarrow \quad y^i = A + Be^{-\zeta} \tag{34}$$

The first two cases should be discarded using arguments similar to those employed above. This leaves the third case which, by utilizing $y(0) = \alpha$, leads to the inner solution

$$y^i = A + (\alpha - A)\, e^{-\zeta} \tag{35}$$

Since the matching principle demands

$$\lim_{\zeta \to \infty} \left[A + (\alpha - A)\, e^{-\zeta} \right] = \lim_{x \to 0} \left(\beta e^{1-x} \right) \qquad \Rightarrow \qquad A = \beta e \tag{36}$$

then we can write

$$y^i = \beta e + (\alpha - \beta e)\, e^{-\zeta} \tag{37}$$

Therefore the boundary layer exists at the end $x = 0$ and the boundary condition $y(0) = \alpha$ cannot be imposed on the reduced equation $y' + y = 0$. Finally we have found as a by-product that the stretching transformation is

$$\zeta = x\epsilon^{-1} \tag{38}$$

hence the region of nonuniformity is $x = \mathcal{O}(\epsilon)$

Higher Order Expansion

Outer Expansion

We seek an outer expansion for y in the form

$$y^o(x, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n y_n^o(x) + \mathcal{O}(\epsilon^N) \tag{39}$$

By substituting in the equation we obtain

$$\frac{dy_0^o}{dx} + y_0^o = 0 \tag{40}$$

$$\frac{dy_n^o}{dx} + y_n^o = -\frac{d^2 y_{n-1}^o}{dx^2} \qquad \text{for} \qquad n \geq 1 \tag{41}$$

This outer solution is valid everywhere except in the region $x = \mathcal{O}(\epsilon)$, hence the boundary condition at $x = 1$ provides

$$y_0^o(1) = \beta \quad , \qquad y_n^o(1) = 0 \quad \text{for} \quad n \geq 1 \tag{42}$$

The solution of (40) subject to $y_0^o(1) = \beta$ is

$$y_0^o = \beta e^{1-x} \tag{43}$$

The solution of (41) for $n = 1$ subject to $y_1^o(1) = 0$ results

$$y_1^o = \beta\,(x)\,e^{1-x} \tag{44}$$

from

$$\frac{dy_1^o}{dx} + y_1^o = -\beta e^{1-x} \qquad \Rightarrow \qquad y_{1P}^o = C_1 x e^{1-x} \tag{45}$$

it follows that $C_1 = -\beta$. Then knowing that $y_1^o = C_2 e^{-x} - \beta x e^{1-x}$ we obtain $C_2 = \beta e$. Finally

$$y_1^o = \beta e e^{-x} - \beta x e^{1-x} = \beta e^{1-x}\,(1 - x) \tag{46}$$

Finally the outer expansion reads

$$y^o = \beta\,[1 + \epsilon\,(1 - x)]\,e^{1-x} + \mathcal{O}(\epsilon^2) \tag{47}$$

Inner Expansion

To determine an expansion valid near the origin we use the stretching transformation $\zeta = x\epsilon^{-1}$

$$\frac{d^2y^i}{d\zeta^2} + \frac{dy^i}{d\zeta} + \epsilon y^i = 0 \tag{48}$$

and then we seek an inner expansion of y in the form

$$y^i\,(x,\epsilon) = \sum_{n=0}^{N-1} \epsilon^n y_n^i(x) + \mathcal{O}(\epsilon^N) \tag{49}$$

By substituting in the equation we obtain

$$\frac{d^2y_0^i}{d\zeta^2} + \frac{dy_0^i}{d\zeta} = 0 \tag{50}$$

$$\frac{d^2y_n^i}{d\zeta^2} + \frac{dy_n^i}{d\zeta} = -y_{n-1}^i \qquad \text{for} \qquad n \geq 1 \tag{51}$$

While this inner expansion satisfies the boundary condition at $x = 0$, it is not expected to satisfy in general the boundary condition at $x = 1$. Since $x = 0$ corresponds to $\zeta = 0$, the boundary condition $y(x = 0) = \alpha$ together with the inner expansion (49) gives

$$y_0^i(0) = \alpha\;, \qquad y_n^i(0) = 0 \;\; \text{for} \qquad n \geq 1 \tag{52}$$

The solution of the problem posed by equation (50) reads

$$y_0^i = \alpha - A_0 \left(1 - e^{-\zeta}\right) \tag{53}$$

while that posed by eq. (51) reads

$$y_1^i = A_1 \left(1 - e^{-\zeta}\right) - \left[\alpha - A_0 \left(1 + e^{-\zeta}\right)\right] \zeta \tag{54}$$

Therefore, putting everything together we obtain

$$y^i = \alpha - A_0 \left(1 - e^{-\zeta}\right) + \epsilon \left\{A_1 \left(1 - e^{-\zeta}\right) - \left[\alpha - A_0 \left(1 + e^{-\zeta}\right)\right] \zeta\right\} + \mathcal{O}(\epsilon^2) \tag{55}$$

Finally the constants A_0 and A_1 should be evaluated from the matching between the inner and the outer expansions.

Refined Matching Procedures

The simplest possible form of matching the inner and outer expansions is that of Prandtl where

$$\lim_{x \to 0} y^o = \lim_{\zeta \to \infty} y^i \tag{56}$$

This condition leads to the matching of the first terms in both the inner and the outer expansions, giving $A_0 = \alpha - \beta e$. It can be easily seen that this matching principle cannot be used to match other than these first terms.

A more general form of the matching condition is

- The inner limit of (the outer limit) equals the outer limit of (the inner limit)

A still more general form of the matching condition is

- The inner expansion of (the outer expansion) equals the outer expansion of (the inner expansion)

Van Dyke (1964) proposed the following matching principle

Van Dyke's Matching Principle

The m-term inner expansion of (the n-term outer expansion) equals the n-term outer expansion of (the m-term inner expansion), where m and n may be taken to be any two integers which may be equal or unequal

To determine the m-term inner expansion of (the n-term outer expansion), we rewrite the first n terms of the outer expansion in terms of the inner variable, expand it for small ϵ keeping the inner variable fixed, and truncate the resulting expansion after m terms, and conversely for the right hand side.

Let fix $m = 2$ and $n = 2$.

- Outer Expansion

Two-term outer expansion	$y^o(x) \approx \beta\left[1+\epsilon\left(1-x\right)\right]e^{1-x}$
Rewritten in inner variable	$= \beta\left[1+\epsilon\left(1-\epsilon\zeta\right)\right]e^{1-\epsilon\zeta}$
Expanded for small ϵ	$= \beta\left(1+\epsilon-\epsilon\zeta+\mathcal{O}(\epsilon^2)\right)$ [1]
Two-term expansion reads	$= \beta e\left(1+\epsilon-\epsilon\zeta\right)$

- Inner Expansion

Two-term inner expansion	$y^i(\zeta) \approx \alpha - A_0\left(1-e^{-\zeta}\right) + \epsilon\left\{A_1\left(1-e^{-\zeta}\right) - \left[\alpha - A_0\left(1+e^{-\zeta}\right)\right]\zeta\right\}$
Rewritten in outer variable	$= \alpha - A_0\left(1-e^{-x/\epsilon}\right) + \epsilon\left\{A_1\left(1-e^{-x/\epsilon}\right) - \left[\alpha - A_0\left(1+e^{-x/\epsilon}\right)\right]x/\epsilon\right\}$
Expanded for small ϵ	$= \left(\alpha - A_0\right)\left(1-x\right) + \epsilon A_1$
Two-term expansion reads	$= \left(\alpha - A_0\right)\left(1-x\right) + \epsilon A_1$

[1] Note that $e^{1-\epsilon\zeta} = e - e\zeta\epsilon + \mathcal{O}(\epsilon^2)$

Equating the two term outer expansion $y^o \approx \beta e \left(1 + \epsilon - \epsilon\zeta\right)$ and the two term inner expansion $y^i \approx \left(\alpha - A_0\right)\left(1 - x\right) + \epsilon A_1$, according to the matching principle, we obtain

$$A_0 = \alpha - \beta e \qquad\qquad A_1 = \beta e \tag{57}$$

Hence the expansions reads

$$y^i\left(\zeta\right) = \beta e + \left(\alpha - \beta e\right) e^{-\zeta} + \epsilon \left\{\beta e \left(1 - e^{-\zeta}\right) - \left[\beta e - \left(\alpha - \beta e\right) e^{-\zeta}\right]\zeta\right\} + \mathcal{O}(\epsilon^2) \tag{58}$$

Composite Expansion

As discussed previously, the outer expansion is not valid near the origin, while the inner expansion is not valid in general away from the region $x = \mathcal{O}(\epsilon)$. To determine an expansion valid over the whole interval, we form a composite expansion y^c

$$y^c = y^o + y^i - (y^o)^i = y^o + y^i - \left(y^i\right)^o \tag{59}$$

Since $(y^o)^i$ is given by either $\beta e\,(1 + \epsilon - \epsilon\zeta)$ or $(\alpha - A_0)\,(1 - x) + \epsilon A_1$, a composite expansion can be formed by adding the outer expansion (47)

$$y^o = \beta\,[1 + \epsilon\,(1 - x)]\,e^{1-x} + \mathcal{O}(\epsilon^2) \tag{60}$$

and the inner expansion (58)

$$y^i\,(\zeta) = \beta e + (\alpha - \beta e)\,e^{-\zeta} + \epsilon\left\{\beta e\left(1 - e^{-\zeta}\right) - \left[\beta e - (\alpha - \beta e)\,e^{-\zeta}\right]\zeta\right\} + \mathcal{O}(\epsilon^2) \tag{61}$$

and subtracting the inner expansion of the outer expansion $(\alpha - A_0)\,(1 - x) + \epsilon A_1$, this results in

$$y^c = \beta\,[1 + \epsilon\,(1 - x)]\,e^{1-x} + [(\alpha - \beta e)\,(1 + x) - \epsilon\beta e]\,e^{-x/\epsilon} + \mathcal{O}(\epsilon^2) \tag{62}$$

An application of Matched Asymptotic Expansions

Consider the simple boundary problem:

$$\epsilon \frac{dy}{dx} + y = x \qquad y(0) = 1 \tag{63}$$

where the parameter ϵ is much smaller than one. The correct solution is

$$y(x) = (1+\epsilon)\, e^{-x/\epsilon} + x - \epsilon \tag{64}$$

Let us see how it is possible to obtain this solution with an asymptotic approach. When x is of $\mathcal{O}(1)$, the strightforward expansion in the *outer region* provides

$$y^o = y_0^o + \epsilon y_1^o + \ldots \tag{65}$$

where

$$y_0^o = x \qquad y_1^o = -\frac{dy_0^o}{dx} = -1 + \ldots \tag{66}$$

Hence

$$y^o = x - \epsilon + \ldots \tag{67}$$

which does not satisfy the boundary condition because close to $x = 0$ the derivative becomes very large and the first term cannot be neglected.

We need to provide a stretching close to $x = 0$, so let us assume

$$\zeta = x\epsilon^{-\lambda} \qquad \lambda > 0 \qquad \Rightarrow \qquad \frac{\partial}{\partial x} = \epsilon^{-\lambda}\frac{\partial}{\partial \zeta} \tag{68}$$

Hence the boundary problem reads

$$\epsilon^{1-\lambda}\frac{dy}{d\zeta} + y = \epsilon^{\lambda}\zeta \tag{69}$$

Let is analyse what happen for different values of λ

- The case $\lambda > 1$

  $$\frac{dy}{d\zeta} = 0 \qquad \rightarrow \qquad y = Constant, \quad \text{the boundary condition forces} \quad y = 1$$

- The case $\lambda < 1$

  $$y = 0 \qquad \text{which is not acceptable}$$

- The case $\lambda = 1$

  Let us analyse this case in detail

Let us introduce the following stretching variable

$$\zeta = \frac{x}{\epsilon} \qquad \rightarrow \qquad \frac{d}{dx} = \frac{1}{\epsilon}\frac{d}{d\zeta} \tag{70}$$

and expand the solution in the region where ζ is of order one (i.e. the inner region) in the form

$$y^i = y_0^i + \epsilon y_1^i + \ldots \tag{71}$$

Substituting at the leading order of approximation, we obtain

$$\frac{dy_0^i}{d\zeta} + y_0^i = 0 \qquad \rightarrow \qquad y_0^i = C_1 e^{-\zeta} \tag{72}$$

$$\frac{dy_1^i}{d\zeta} + y_1^i = \zeta \qquad \rightarrow \qquad y_1^i = C_2 e^{-\zeta} + \zeta - 1 \tag{73}$$

The boundary condition $y(0) = 1$ suggests that

$$y_0^i = C_1 = 1 \qquad y_1^i = C_2 - 1 = 0 \qquad \Rightarrow \qquad C_1 = C_2 = 1 \tag{74}$$

Finally

$$y^i = e^{-\zeta} + \epsilon \left[e^{-\zeta} + \zeta - 1 \right] \tag{75}$$

It is possible to verify, operating the limits of the outer an inner expansions, that

$$\lim_{x \longrightarrow 0} y^o \approx x - \epsilon \qquad\qquad \lim_{\zeta \longrightarrow \infty} y^i \approx \epsilon\,[\zeta - 1] \tag{76}$$

hence the two functions overlap because $\lim_{x \longrightarrow 0} y^o = \lim_{\zeta \longrightarrow \infty} y^i$

Finally we can build a composite solution valid both in the inner region and in the outer region in the following way

$$y = y^o + y^i - \lim_{x \longrightarrow 0} y^o \qquad \text{or} \qquad y = y^o + y^i - \lim_{\zeta \longrightarrow \infty} y^i \tag{77}$$

In both cases the final solution results in

$$y = (1 + \epsilon)\, e^{-x/\epsilon} + x - \epsilon \tag{78}$$

The Method of Composite Expansions I

The composite expansion obtained in the previous sections are generalized expansions having the form

$$y\left(x,\epsilon\right)=y^{o}\left(x,\epsilon\right)+y^{i}\left(x,\epsilon\right)-\left(y^{o}\right)^{i}=y^{o}+y^{i}-\left(y^{i}\right)^{o} \tag{79}$$

where y is the dependent variable, ϵ is the small parameter, x is the outer variable, and ζ is the inner variable. This composite expansion can be viewed as the sum of two terms $F(x,\epsilon)=y^{o}$ and $G(\zeta,\epsilon)=y^{i}-(y^{o})^{i}$ that is

$$y(x,\epsilon)=F(x,\epsilon)+G(\zeta,\epsilon) \tag{80}$$

Rather than determining outer and inner expansions, matching them, and then forming a composite expansion Bromberg (1956) and Visik and Lyusternik (1957) assumed the solution to have the form (80) which is valid everywhere hence it satisfies all the boundary conditions. To determine an approximate solution, F and G are expanded in terms of ϵ, and equations and boundary conditions are derived for each level of approximation.

Another method has been developed by Latta (1951), assuming that the the function G is also a function of the outer variable and the inner variable can be generalized as $g(x)/\delta(\epsilon)$ with g to be determined from the analysis.

Let us consider the simple boundary problem

$$\epsilon\frac{d^{2}y}{dx^{2}}+\frac{dy}{dx}+y=0 \tag{81}$$

with the following boundary conditions

$$y(0)=\alpha \qquad y(1)=\beta \tag{82}$$

As discussed before, it is well known that the straightforward expansion breaks down near $x = 0$, and an inner expansion using the stretching transformation $\zeta = x\epsilon^{-l}$ was introduced to describe y in the region of nonuniformity. The inner expansion was shown to involve the function $e^{-\zeta} = e^{-x/\epsilon}$. Since upon differentiation $e^{-x/\epsilon}$ reproduces itself, no other special functions are needed to represent the composite expansion.
Latta (1951) assumed that y has uniformly valid expansion of the form

$$y = \sum_{n=0}^{\infty} \epsilon^n f_n(x) + e^{-\frac{x}{\epsilon}} \sum_{n=0}^{\infty} \epsilon^n h_n(x) \tag{83}$$

where the term $e^{-x/\epsilon}$ has been introduced to enlarge the *boundary layer*.

Substituting expansion (83) in the boundary problem and equating the coefficients of ϵ, ϵ^2, ..., $\epsilon e^{-x/\epsilon}$, $\epsilon^2 e^{-x/\epsilon}$, ... we obtain the following equations for f_n and h_n

$$\begin{gathered}\epsilon \left\{ \sum_{n=0}^{\infty} \epsilon^n f_n'' + \frac{1}{\epsilon^2} e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n - \frac{2}{\epsilon} e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n' + e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n'' \right\} + \\ \sum_{n=0}^{\infty} \epsilon^n f_n' - \frac{1}{\epsilon} e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n + e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n' + \sum_{n=0}^{\infty} \epsilon^n f_n + e^{-x/\epsilon} \sum_{n=0}^{\infty} \epsilon^n h_n = 0\end{gathered} \tag{84}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \epsilon^{n+1} f_n'' + \sum_{n=0}^{\infty} \epsilon^n f_n' + \sum_{n=0}^{\infty} \epsilon^n f_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \epsilon^{n+1} h_n'' - \sum_{n=0}^{\infty} \epsilon^n h_n' + \sum_{n=0}^{\infty} \epsilon^n h_n = 0 \tag{85}$$

Finally, at the different orders of approximation we obtain

$$f_0' + f_0 = 0 \qquad h_0' - h_0 = 0 \tag{86}$$

$$f_1' + f_1 = -f_0'' \qquad h_1' - h_1 = h_0'' \tag{87}$$

$$f_2' + f_2 = -f_1'' \qquad h_2' - h_2 = h_1'' \tag{88}$$

and the boundary conditions leads to

$$f_0(1) = \beta \qquad f_0(0) + h_0(0) = \alpha \tag{89}$$

$$f_n(1) = 0 \qquad f_n(0) + h_n(0) = 0 \quad \text{for} \quad n \geq 1 \tag{90}$$

where the eponentially small terms $e^{-1/\epsilon} h_n(1)$ are neglected.
Solutions of equations (86)-(88) can be otbained straightforward

$$f_0 = \beta e^{1-x} \qquad h_0 = (\alpha - \beta e)\, e^x \tag{91}$$

$$f_1 = \beta\,(1-x)\, e^{1-x} \qquad h_1 = [-\beta e + (\alpha - \beta e)\, x]\, e^x \tag{92}$$

$$f_2 = \frac{1}{2}\beta\,(1-x)\,(5-x)\, e^{1-x} \qquad h_2 = \left[-\frac{5}{2}\beta e + (2\alpha - 3\beta e)\, x + \frac{1}{2}\,(\alpha - \beta e)\, x^2\right] e^x \tag{93}$$

With the above solutions the expansion (83) becomes

$$y = \beta e^{1-x} + (\alpha - \beta e)\, e^{x - x/\epsilon} + \epsilon \left\{ \beta\,(1-x)\, e^{1-x} \; + \left[-\beta e + (\alpha - \beta e)\, x\right] e^{x-x/\epsilon} \right\} +$$
$$+\epsilon^2 \left\{ \frac{1}{2}\beta\,(1-x)\,(5-x)\, e^{1-x} + \left[-\frac{5}{2}\beta e + (2\alpha - 3\beta e)\, x + \frac{1}{2}\,(\alpha - \beta e)\, x^2 \right] e^{x-x/\epsilon} \right\} + \mathcal{O}(\epsilon^3) \quad (94)$$

It can be easily verified that the outer expansion ($\lim \epsilon \rightarrow 0$ with x kept fixed) of the first two terms of this expansion is given by (47), while the inner expansion ($\epsilon \rightarrow 0$ with $\zeta = x/\epsilon$ kept fixed) is given by (58). Therefore the method of composite expansions gives a uniformly valid expansion directly without the need to determine an outer expansion and an inner expansion, to match them, and then to form a composite expansion.

An application of Composite Asymptotic Expansions

Let us consider the following boundary value problem

$$\epsilon \frac{d^2 y}{dx^2} + (2x+1)\frac{dy}{dx} + 2y = 0 \tag{95}$$

with the following boundary conditions

$$y(0) = \alpha \qquad y(1) = \beta \tag{96}$$

where $\epsilon \ll 1$.
Since the coefficient of y' is positive, the nonuniformity occurs near $x = 0$. To describe y in the region of nonuniformity, we need a stretching transformation $\zeta = x\epsilon^{-l}$, and the inner expansion is described in terms of the special function $e^{-\zeta} = e^{-x/\epsilon}$.
Since the problem has variable coefficients, we assume that y possesses a uniformly valid expansion of the form

$$y = \sum_{n=0}^{\infty} \epsilon^n f_n(x) + e^{\frac{-g(x)}{\epsilon}} \sum_{n=0}^{\infty} \epsilon^n h_n(x) \tag{97}$$

where the function $g(x)$, which is determined from the analysis, tends to x as x tends to zero.
Substituting the expansion in the original problem and equating the coefficients of ϵ^n and $\epsilon^n e^{-g(x)/\epsilon}$ we obtain different equations to determine g, f_n and h_n

After some manipulation

$$\epsilon^{n+1} f_n'' - g'' e^{-g(x)/\epsilon} \epsilon^n h_n + g'^2 e^{-g(x)/\epsilon} \epsilon^{n-1} h_n - 2g' e^{-g(x)/\epsilon} \epsilon^n h_n' + \\ + e^{-g(x)/\epsilon} \epsilon^{n+1} h_n'' + (2x+1) \left[\epsilon^n f_n' - g' e^{-g(x)/\epsilon} \epsilon^{n-1} h_n + e^{-g(x)/\epsilon} \epsilon^n h_n' \right] + 2\epsilon^n f_n + \\ + 2e^{-g(x)/\epsilon} \epsilon^n h_n = 0 \tag{98}$$

Equating the coefficients of ϵ^n and $\epsilon^n e^{-g(x)/\epsilon}$ we obtain

$$2f_0 + 2(x+1) f_0' = 0 \tag{99}$$

$$h_0' \left(2x + 1 - 2g'\right) + h_0 \left(2 - g''\right) = 0 \tag{100}$$

$$h_0 g' \left[g' - 2(x+1)\right] = 0 \tag{101}$$

with the boundary conditions

$$f(1) = \beta \qquad\qquad f_0(0) + h_0(0) = \alpha \tag{102}$$

To determine a non trivial solution for h_0 it is required that

$$g'(x) = 0 \qquad \text{or} \qquad g'(x) = 2x + 1 \tag{103}$$

The first case yields $g = constant$, which must rejected because $g(x) \to x$ as x tends to zero. Hence

$$g(x) = x^2 + x \tag{104}$$

Solving for $f_0(x)$ and applying the boundary condition provides

$$f_0(x) = \frac{3\beta}{2x+1} \tag{105}$$

Solving for $h_0(x)$ and applying the boundary condition provides

$$h_0(x) = \alpha - 3\beta \tag{106}$$

Therefore we can write the solution as

$$y = \frac{3\beta}{2x+1} + (\alpha - 3\beta)\, e^{-\frac{x^2+x}{\epsilon}} + \mathcal{O}(\epsilon) \tag{107}$$

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