Viscous flow at low Reynolds numbers

PhD Course on Fluid Mechanics

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Navier-Stokes Equations

$$\frac{\partial \mathbf{U}^*}{\partial t^*} + \left(\mathbf{U}^* \cdot \nabla^*\right)\mathbf{U}^* = -\frac{1}{\rho}\nabla^* P^* + \nu \nabla^{*2}\mathbf{U}^* \qquad (1)$$

$$\nabla^* \cdot \mathbf{U}^* = 0 \qquad (2)$$

Hypotesis:

- incompressible flow
- dynamic pression $\Rightarrow P^* = p^* - p_0 - \rho \mathbf{g}\mathbf{x}$

In order to make dimensionless the equations, two scales are chosen: U^* for the velocity and L^* for the length

Navier-Stokes Equations

Dimensionless Variables

$$\mathbf{x} = \frac{\mathbf{x}^*}{L^*} \qquad \mathbf{U} = \frac{\mathbf{U}^*}{U^*} \qquad P = \frac{P^*}{\rho {U^*}^2} \qquad \nabla = L^* \nabla^* \qquad t = \frac{t^* U^*}{L^*} \tag{3}$$

Plugging (3) in (1) and (2) we obtain

$$\frac{\partial \mathbf{U}}{\partial t} \frac{{U^*}^2}{L^*} + (\mathbf{U} \cdot \nabla) \mathbf{U} \frac{{U^*}^2}{L^*} = -\nabla P \frac{{U^*}^2}{L^*} + \nu \nabla^2 \mathbf{U} \frac{U^*}{{L^*}^2} \tag{4}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{5}$$

Inertial terms $\mathcal{O}\left(\frac{{U^*}^2}{L^*}\right)$ Viscous terms $\mathcal{O}\left(\nu \frac{U^*}{{L^*}^2}\right)$

Navier-Stokes Equations

The ratio between inertial terms and viscous terms reads

$$\left(\frac{U^{*^2}}{L^*}\right) / \left(\nu \frac{U^*}{L^{*^2}}\right) = \frac{U^* L^*}{\nu} = Re \qquad \text{Reynolds Number} \tag{6}$$

Dimensionless form of eqs (4)-(5) read

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla\right) \mathbf{U} = -\nabla P + \frac{1}{Re} \nabla^2 \mathbf{U} \tag{7}$$

$$\nabla \cdot \mathbf{U} = 0 \tag{8}$$

The system of eqs (7)-(8) is nonlinear and exact solutions are rare in any branch of fluid mechanics. Solution existence and unicity has not been demonstrated for the full problem, while it is possible to obtain exact solution when one or more of the parameter or variables in the problem is small (or large)

Navier-Stokes Equations

Assuming the Reynolds number *Re* as the parameter, we can define two different cases

- $Re \gg 1 \Rightarrow$ Boundary Layer Theory
- $Re \ll 1 \Rightarrow$ Creeping Flows, Sedimentation

Low Reynolds Number $Re \ll 1$

- *Re* is small because velocity is small. It seems licit to neglect nonlinear terms of the velocity in Navier-Stokes equations $\Rightarrow$ Stokes Flow
- *Re* is small because the length scale is small. Body dimensions could not influence the surrounding flow: it seems licit to express inertial terms employing as convection velocity that of undisturbed flow $\Rightarrow$ Oseen Flow

Flow at low Reynolds number

Applying the curl $(\nabla\times)$ to (8) we obtain

$$\frac{\partial \omega}{\partial t} + (\mathbf{U}\cdot\nabla)\,\omega - (\omega\cdot\nabla)\,\mathbf{U} = \frac{1}{Re}\nabla^2\omega \tag{9}$$

Assuming plane or axialsymmetric flows $(\partial/\partial z = 0\ ;\ \ \partial/\partial\varphi = 0)$ it is possible to say that

- vorticity has just one component along the z or φ direction
- continuity equation implies the existence of a stream function such that $\omega = -[\mathfrak{L}_1^2]\psi$

$$\text{Cartesian coord.}\quad [\mathfrak{L}_1^2] = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{10}$$

$$\text{Spherical coord.}\quad [\mathfrak{L}_1^2] = \frac{1}{r\sin(\vartheta)}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} - \frac{1}{r^2}\frac{\cos(\vartheta)}{\sin(\vartheta)}\frac{\partial}{\partial\vartheta}\right] \tag{11}$$

$$\text{Cylindrical coord.}\quad [\mathfrak{L}_1^2] = \nabla^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\right] \tag{12}$$

Flow at low Reynolds number

- $(\omega \cdot \nabla)\,\mathbf{U} = 0$ Three-dimensional term (stretching)

$$\omega \cdot \nabla = \begin{cases} \omega_z \frac{\partial}{\partial z} & \text{Cartesian, Cylindrical} \\ \omega_\varphi \frac{1}{r\sin(\varphi)} \frac{\partial}{\partial \varphi} & \text{Spherical} \end{cases} \tag{13}$$

- $\nabla^2 \omega = [\mathfrak{L}_2^2]\omega$

$$[\mathfrak{L}_2^2] = \begin{cases} \nabla^2 & \text{Cartesian} \\ \nabla^2 & \text{Cylindrical} \\ \nabla^2 - \frac{1}{r^2 \sin^2(\vartheta)} & \text{Spherical} \end{cases} \tag{14}$$

Flow at low Reynolds number

- Taking into account steady flows, i.e. $\frac{\partial}{\partial t} = 0$ eq (9) becomes

$$(\mathbf{U} \cdot \nabla)\,[\mathfrak{L}_1^2]\psi = \frac{1}{Re}[\mathfrak{L}_2^2][\mathfrak{L}_1^2]\psi \qquad (15)$$

or

$$(\mathbf{U} \cdot \nabla)\,\omega = \frac{1}{Re}[\mathfrak{L}_2^2]\omega \qquad (16)$$

Remember that $\omega = -[\mathfrak{L}_1^2]\psi$

Flow at low Reynolds number

Problem could be solved as a power series of Re (Direct Perturbation)

$$\psi = \psi_0 Re^0 + \psi_1 Re^m + \psi_2 Re^{2m} + \ldots \tag{17}$$

If $Re \ll 1$ then viscous effects are dominant $\Rightarrow m > 0$

Solution at the first order for a cylinder and a sphere (Stokes)
Paradoxes

- Stokes: cylinder at the first order
- Whitehead: sphere at second order

The presence of paradoxes shows that inertial effects are not negligible EVERYWHERE with respect to viscous effects
Direct solution is not valid far from the body

Stokes solution for sphere

The problem, in Cartesian Coordinates, is posed by

$$\nabla^2 \omega|_{Cart} = [\mathfrak{L}_2^2]\omega = 0 \quad \text{where} \quad [\mathfrak{L}_2^2] = \nabla^2 - \frac{1}{r^2 \sin^2(\vartheta)} \tag{18}$$

with the following boundary conditions

$$\mathbf{v}^* \to \mathbf{U}^* \quad \mathbf{x}^* \to \infty \;\; ; \quad P^* \to P_0^* \quad \mathbf{x}^* \to \infty \tag{19}$$

We solve the problem in dimensionless form using the following dimensionless quantities

$$Re = \frac{a^* U^*}{\nu} \qquad r = \frac{r^*}{a^*} \qquad \mathbf{U} = \frac{\mathbf{U}^*}{U^*}$$

- Axial symmetry $\Rightarrow \frac{\partial}{\partial \varphi} = 0 \;\; ; \quad U_\varphi = 0 \quad \Rightarrow \quad \omega = (0, 0, \omega_\varphi)$

Stokes solution for sphere

- We define the stream function $\psi = \psi\,(r, \vartheta)$ such that

$$U_\varphi = 0 \;; \qquad U_r = \frac{1}{r^2 \sin(\vartheta)} \frac{\partial \psi}{\partial \vartheta} \;; \qquad U_\vartheta = -\frac{1}{r \sin(\vartheta)} \frac{\partial \psi}{\partial r} \tag{20}$$

Stream function ψ verifies directly continuity equation:

$$\nabla \cdot \mathbf{U} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 U_r \right) + \frac{1}{r \sin(\vartheta)} \frac{\partial}{\partial \vartheta} \left(\sin(\vartheta) U_\vartheta \right) = \ldots = 0 \tag{21}$$

Stokes solution for sphere

- Link between vorticity and stream function

$$\omega = (\nabla \times \mathbf{U})_{\varphi} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rU_{\vartheta}) - \frac{\partial}{\partial \vartheta}(U_r)\right] =$$
$$= \frac{1}{r}\left[-\frac{1}{\sin(\vartheta)}\frac{\partial}{\partial r}(\psi_{,r}) - \frac{1}{r^2}\frac{\partial}{\partial \vartheta}\left(\frac{\psi_{,\vartheta}}{\sin(\vartheta)}\right)\right] =$$
$$= \frac{1}{r\sin(\vartheta)}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \vartheta^2} - \frac{\cos(\vartheta)}{\sin(\vartheta)}\frac{\partial}{\partial \vartheta}\right)\right]\psi =$$
$$= -\frac{1}{r\sin(\vartheta)}[D^2]\psi \qquad (22)$$

where

$$[D^2] = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \vartheta^2} - \frac{\cos(\vartheta)}{\sin(\vartheta)}\frac{\partial}{\partial \vartheta}\right)\right] \qquad (23)$$

Stokes solution for sphere

$$\omega_{,r} = \left[\frac{1}{r^2\sin(\vartheta)}[D^2] - \frac{1}{r\sin(\vartheta)}[D^2]_{,r}\right]\psi = \frac{1}{r\sin(\vartheta)}\left[\frac{1}{r}[D^2] - [D^2]_{,r}\right]\psi \qquad (24)$$

$$\omega_{,rr} = \left[-\tfrac{2}{r^3\sin(\vartheta)}[D^2] + \tfrac{1}{r^2\sin(\vartheta)}[D^2]_{,r} + \tfrac{1}{r^2\sin(\vartheta)}[D^2]_{,r} - \tfrac{1}{r\sin(\vartheta)}[D^2]_{,rr}\right]\psi =$$
$$= \tfrac{1}{r\sin(\vartheta)}\left[-\tfrac{2}{r^2}[D^2] + \tfrac{2}{r}[D^2]_{,r} - [D^2]_{,rr}\right]\psi \qquad (25)$$

$$\omega_{,\vartheta} = \left[\frac{\cos(\vartheta)}{r\sin^2(\vartheta)}[D^2] - \frac{1}{r\sin(\vartheta)}[D^2]_{,\vartheta}\right]\psi = \frac{1}{r\sin(\vartheta)}\left[\frac{\cos(\vartheta)}{\sin(\vartheta)}[D^2] - [D^2]_{,\vartheta}\right]\psi \qquad (26)$$

$$\omega_{,\vartheta\vartheta} = \left[\frac{-\sin^3(\vartheta) - \cos^2(\vartheta)2\sin(\vartheta)}{r\sin^4(\vartheta)}[D^2] + \frac{\cos(\vartheta)}{r\sin^2(\vartheta)}[D^2]_{,\vartheta} + \frac{\cos(\vartheta)}{r\sin^2(\vartheta)}[D^2]_{,\vartheta}\right.$$
$$\left. - \frac{1}{r\sin(\vartheta)}[D^2]_{,\vartheta\vartheta}\right]\psi =$$
$$= \frac{1}{r\sin(\vartheta)}\left[-\frac{1+\cos^2(\vartheta)}{\sin^2(\vartheta)}[D^2] + 2\frac{\cos(\vartheta)}{\sin(\vartheta)}[D^2]_{,\vartheta} - [D^2]_{,\vartheta\vartheta}\right]\psi \qquad (27)$$

Stokes solution for sphere

Finally eq (18) reads

$$\left(\nabla^2 - \frac{1}{r^2\sin^2(\vartheta)}\right)\omega = \omega_{,rr} + \frac{2}{r}\omega_{,r} + \frac{1}{r^2}\omega_{,\vartheta\vartheta} + \frac{\cos(\vartheta)}{r^2\sin(\vartheta)}\omega_{,\vartheta} - \frac{\omega}{r^2\sin^2(\vartheta)} = 0 \qquad (28)$$

After some manipulation we obtain

$$\left[[D^2]_{,rr} + \frac{1}{r^2}\left([D^2]_{,\vartheta\vartheta} - \frac{\cos(\vartheta)}{\sin(\vartheta)}[D^2]_{,\vartheta}\right)\right]\psi = 0 \qquad (29)$$

Equation for stream function ψ

$$\Longrightarrow \qquad [D^2][D^2]\psi = 0 \qquad (30)$$

Stokes solution for sphere

Boundary Conditions for stream function ψ

- Far from the body: $\mathbf{U} \to 1$ if $r \to \infty$

$$\begin{cases} U_r = \cos(\vartheta) = \frac{1}{r^2 \sin(\vartheta)} \psi_{,\vartheta} \\ U_\vartheta = -\sin(\vartheta) = -\frac{1}{r \sin(\vartheta)} \psi_{,r} \end{cases} \tag{31}$$

$$\begin{cases} \psi_{,\vartheta} = r^2 \sin(\vartheta) \cos(\vartheta) = \frac{r^2}{2} \sin(2\vartheta) \\ \psi_{,r} = r \sin^2(\vartheta) \end{cases} \tag{32}$$

$$\begin{cases} \psi = -r^4 \cos(2\vartheta) + F(r) = \frac{r^2}{2} \sin^2(\vartheta) + F^*(r) \\ \psi = \frac{r^2}{2} \sin^2(\vartheta) + G(\vartheta) \end{cases} \tag{33}$$

$$F^*(r) = G(\vartheta) = \mathrm{cost} = 0 \quad \Rightarrow \quad \psi = r^2 \sin^2(\vartheta)/2 \qquad r \to \infty$$

Stokes solution for sphere

Boundary Conditions for stream function ψ

- On the surface of the body: $\mathbf{U} \rightarrow 0$ if $r \rightarrow 1$

$$\Rightarrow \qquad \psi_{,r} = \psi_{,\vartheta} = 0 \qquad r = 1$$

- Separation of variables

$$\psi = \sin^2(\vartheta) f(r) \tag{34}$$

Finally

$$[D^2]\psi = \sin^2(\vartheta) f_{,rr} + \frac{1}{r^2}\left[\left(\sin^2(\vartheta)\right)_{,\vartheta\vartheta} - \frac{\cos(\vartheta)}{\sin(\vartheta)}\left(\sin^2(\vartheta)\right)_{,\vartheta}\right] f \tag{35}$$

Stokes solution for sphere

Knowing that

$$\left(\sin^2(\vartheta)\right)_{,\vartheta} = 2\sin(\vartheta)\cos(\vartheta) = \sin(2\vartheta) \quad ; \qquad \left(\sin^2(\vartheta)\right)_{,\vartheta\vartheta} = 2\cos(2\vartheta)$$

Equation (35) can be written as

$$\begin{aligned}
[D^2]\psi &= \sin^2(\vartheta) f_{,rr} + \frac{1}{r^2}\left[2\cos(2\vartheta) - 2\cos^2(\vartheta)\right] f = \\
&= \sin^2(\vartheta) f_{,rr} + \frac{1}{r^2}\left[2 - 4\sin^2(\vartheta) - 2\cos^2(\vartheta)\right] f = \\
&= \sin^2(\vartheta)\left[\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right] f = \sin^2(\vartheta)[R^2] f
\end{aligned} \qquad (36)$$

Stokes solution for sphere

Assuming

$$g(r) = [R^2]f \qquad A = [D^2]\psi \qquad A = \sin^2(\vartheta)g(r) \tag{37}$$

we obtain

$$[D^2][D^2]\psi = [D^2]A = \sin^2(\vartheta)[R^2]g = \sin^2(\vartheta)[R^2][R^2]f = 0 \tag{38}$$

$$[R^2][R^2]f = [R^2]g = 0 \tag{39}$$

- Boundary conditions for f[1]

$$f \to \frac{r^2}{2} \qquad r \to \infty \tag{40}$$

$$f = f_{,r} = 0 \qquad r = 1 \tag{41}$$

[1] $\psi = r^2 \sin^2(\vartheta)/2$

Stokes solution for sphere

- Solution procedure for $[R^2][R^2]f = [R^2]g = 0$

$$[R^2]g = \left[\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right] g = g_{,rr} - \frac{2}{r^2} g = 0 \qquad (42)$$

General solution for g can be written as[1] $\quad g = c_1 r^2 + c_2 r^{-1}$

$$\Rightarrow [R^2]f = g = c_1 r^2 + c_2 r^{-1} \qquad (43)$$

Finally the general solution for f is

$$f = c_1 r^4 + c_2 r^2 + c_3 r + c_4 r^{-1} \qquad (44)$$

[1] Verify it!

Stokes solution for sphere

- Boundary conditions define

$$c_1 = 0 \qquad c_2 = \frac{1}{2} \qquad \text{(stream function at } \infty) \tag{45}$$

$$\begin{cases} \frac{1}{2} + c_3 + c_4 = 0 & c_3 = -\frac{3}{4} \quad f = 0 \quad \text{for} \quad r \to 1 \\ 1 + c_3 - c_4 = 0 & c_4 = \frac{1}{4} \quad f_{,r} = 0 \quad \text{for} \quad r \to 1 \end{cases} \tag{46}$$

Finally

$$f = \frac{r^2}{2} - \frac{3}{4}r + \frac{1}{4r} \quad \Rightarrow \quad \psi = r^2 \sin^2(\vartheta) \left[\frac{1}{2} - \frac{3}{4}r^{-1} + \frac{1}{4}r^{-3}\right]$$

ψ is symmetric with respect to an orthogonal plane of U

Stokes solution for sphere

Velocity Field

$$U_r = \cos(\vartheta)\left[1 - \frac{3}{2}r^{-1} + \frac{1}{2}r^{-3}\right] \quad (47)$$

$$U_\vartheta = -\sin(\vartheta)\left[1 - \frac{3}{4}r^{-1} - \frac{1}{4}r^{-3}\right] \quad (48)$$

The perturbation of the flow field, given by the vector $\mathbf{U} - U$, is dumped as r^{-1}, i.e. very slowly: the flow field is modified even far away from the body. Basically the flow field is perturbed by viscous diffusion of the vorticity which is generated at the wall of the rigid body

Stokes solution for sphere

Vorticity

$$\omega = -\frac{1}{r\sin(\vartheta)}[D^2]\psi = -\frac{1}{r}\sin(\vartheta)[R^2]f =$$
$$= -\frac{\sin(\vartheta)}{r}\left[f_{,rr} - \frac{2}{r^2}f\right] =$$
$$-\frac{\sin(\vartheta)}{r}\left[1 + \frac{1}{2r^3} - \frac{2}{r^2}\left(\frac{r^2}{2} - \frac{3}{4}r + \frac{1}{4r}\right)\right] = -\frac{3}{2}\frac{\sin(\vartheta)}{r^2} \qquad (49)$$

Vorticity is dumped as r^{-2}. The only term of ψ contributing to vorticity is the term proportional to r, known as the *Stokeslet*. The other two terms represent irrotational motion (uniform stream and dipole)

Stokes solution for sphere

Pressure

$$\nabla P = \frac{1}{Re}\nabla^2 \mathbf{U} = -\frac{1}{Re}\nabla \times \omega \tag{50}$$

$$\frac{\partial P}{\partial r} = -\frac{1}{Re}\frac{1}{\sin(\vartheta)}\frac{\partial\left(\omega \sin(\vartheta)\right)}{\partial \vartheta} = -\frac{1}{Re}\frac{1}{\sin(\vartheta)}\left[\omega_{,\vartheta}\sin(\vartheta) + \omega\cos(\vartheta)\right] =$$
$$= -\frac{1}{Re}\frac{1}{r}\left[-\frac{3}{2r^2}\cos(\vartheta) + \frac{\cos(\vartheta)}{\sin(\vartheta)}\left(-\frac{3}{2}\frac{\sin(\vartheta)}{r^2}\right)\right] = \frac{1}{Re}\frac{1}{r^3}3\cos(\vartheta) \tag{51}$$

$$\Rightarrow \quad P = -\frac{1}{Re}\frac{3}{2}\frac{\cos(\vartheta)}{r^2} + F(\vartheta) \tag{52}$$

Stokes solution for sphere

$$\frac{1}{r}\frac{\partial P}{\partial \vartheta} = \frac{1}{Re}\frac{1}{r}\frac{\partial(\omega r)}{\partial r} = \frac{1}{Re}\frac{1}{r}\left[\omega + r\omega_{,r}\right] =$$

$$= \frac{1}{Re}\frac{1}{r}\left(-\frac{3}{2}\sin(\vartheta)\right)\left[\frac{1}{r^2} - \frac{2}{r^2}\right] = \frac{1}{Re}\frac{3}{2r^3}\sin(\vartheta) \quad (53)$$

$$\Rightarrow \quad P = -\frac{1}{Rer^2}\frac{3}{2}\cos(\vartheta) + G(r) \quad (54)$$

Cuopling eqs (52) and (54) we obtain

$$P = c_P - \frac{3}{2}\frac{1}{Re}\frac{\cos(\vartheta)}{r^2} \quad \text{from B.C.} \Longrightarrow P = P_0 - \frac{3}{2}\frac{1}{Re}\frac{\cos(\vartheta)}{r^2} \quad (55)$$

As for vorticity, just the rotational term in the solution gives a contribution for the pressure. Pressure is not symmetric: there is a net contribution along the direction of motion

Stokes solution for sphere

Evaluation of drag of the sphere
We must evaluate T_{rr} and $T_{r\vartheta}$ acting on a plane of normal $\hat{r}$ (the sphere surface)

$$D^* = \rho {U^*}^2 {a^*}^2 2\pi \int_0^{2pi} \{[T_{rr}]_{r=a} \cos(\vartheta) - [T_{r\vartheta}]_{r=a} \sin(\vartheta)\} \sin(\vartheta) d\vartheta \quad (56)$$

where

$$T_{rr} = -P + \frac{2}{Re} D_{rr} \qquad T_{r\vartheta} = \frac{2}{Re} D_{r\vartheta} \quad (57)$$

$$T_{rr}|_{r=1} = -P_0 + \frac{3}{2}\frac{1}{Re}\cos(\vartheta) \qquad T_{r\vartheta}|_{r=1} = -\frac{1}{Re}\frac{3}{2}\sin(\vartheta) \quad (58)$$

Stokes solution for sphere

$$D^* = \rho {U^*}^2 {a^*}^2 2\pi \int_0^{2\pi} \left\{ \left(-P_0 + \frac{3}{2}\frac{1}{Re}\cos(\vartheta) \right) \cos(\vartheta) + \frac{1}{Re}\frac{3}{2}\sin^2(\vartheta) \right\} \sin(\vartheta) d\vartheta =$$
$$= \rho {U^*}^2 {a^*}^2 2\pi \left[-P_0 \frac{\sin^2(\vartheta)}{2} + \frac{3}{2}\frac{1}{Re}\left(-\frac{\cos^3(\vartheta)}{3} \right) + \frac{1}{Re}\frac{3}{2}\left(-\cos(\vartheta) + \frac{\cos^3(\vartheta)}{3} \right) \right]_0^{\pi} =$$
$$= \rho {U^*}^2 {a^*}^2 2\pi \left[\frac{1}{Re} + \frac{2}{Re} \right] = 6\pi\mu U^* a^* \tag{59}$$

Drag Coefficient

$$C_D = \frac{D^*}{\frac{1}{2}\rho {U^*}^2 {a^*}^2 \pi} = \frac{12\nu}{U^* a^*} = \frac{24}{Re} \qquad \text{if} \qquad Re = \frac{U^* 2a^*}{\nu} \tag{60}$$

Stokes solution for sphere

- A simple application: small particle fall velocity in a fluid

Applying force balance

$$\text{Mass force} \qquad \frac{4}{3}\pi {a^*}^3 * (\rho_s - \rho) * g^* = \frac{1}{2}\rho {U^*}^2 \pi {a^*}^2 C_D \qquad \text{Drag force} \tag{61}$$

$$\Rightarrow \qquad U^* = \frac{2}{9}\frac{\rho_s - \rho}{\rho} g^* \frac{{a^*}^2}{\nu} \tag{62}$$

The corresponding value of the Reynolds number for a sphere falling with its terminal velocity is

$$Re = \frac{U^* 2a^*}{\nu} = \frac{4}{9}\frac{{a^*}^3 g^*}{\nu}\frac{\rho_s - \rho}{\rho} \tag{63}$$

Stokes solution for cylinder

The problem is posed by

$$\nabla^2 \omega = 0 \qquad (64)$$

We solve the problem in dimensionless form using the following dimensionless quantities

$$Re = \frac{a^* U^*}{\nu} \qquad r = \frac{r^*}{a^*} \qquad \mathbf{U} = \frac{\mathbf{U}^*}{U^*}$$

- Axial symmetry $\Rightarrow \frac{\partial}{\partial z} = 0 \;\; ; \qquad U_z = 0 \quad \Rightarrow \quad \omega = (0, 0, \omega_z)$
- Vorticity ω_z

$$\omega_z = \frac{1}{r}\left[\left(rU_\vartheta\right)_{,r} - U_{r,\vartheta}\right] = \frac{1}{r}\left[-\psi_{,r} - r\psi_{,rr} - \frac{1}{r}\psi_{,\vartheta\vartheta}\right] =$$

$$= -\left[\psi_{,rr} + \frac{1}{r}\psi_{,r} + \frac{1}{r^2}\psi_{,\vartheta\vartheta}\right] = -\nabla^2\psi \qquad (65)$$

Stokes solution for cylinder

Equation for stream function ψ

$$\nabla^2\nabla^2\psi = 0 \tag{66}$$

Boundary conditions for ψ

$$\psi \to r\sin(\vartheta) \quad \text{if} \quad r \to \infty \quad ; \qquad \psi_{,r} = \psi_{,\vartheta} = 0 \quad \text{if} \quad r = 1 \tag{67}$$

Separation of variables $\Rightarrow$ $\psi = \sin(\vartheta)f(r)$

$$\nabla^2\psi = \sin(\vartheta)f_{,rr} + \frac{1}{r}f_{,r} - \frac{1}{r^2}\sin(\vartheta)f = \sin(\vartheta)[R^2]f \tag{68}$$

where $[R^2] = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}$

Finally

$$\nabla^4\psi = \sin(\vartheta)[R^2][R^2]f = 0 \quad \Rightarrow \quad [R^2][R^2]f = 0 \tag{69}$$

Stokes solution for cylinder

- Boundary conditions for f

$$f \to r \quad \text{if} \quad r \to \infty \quad ; \qquad f = f_{,r} = 0 \quad \text{if} \quad r = 1 \tag{70}$$

The solution for the cylinder for f can be written as

$$f = c_1 r^3 + c_2 r \log(r) + c_3 r + c_4 r^{-1} \tag{71}$$

Boundary condition at infinity imposes $c_1 = 0$ and $c_3 = 1$. Let assign just c_1 and let proceed with b.c. on the surface of the cylinder

$$\begin{cases} c_3 + c_4 = 0 & c_3 = -\frac{c_2}{2} & f = 0 & \text{for} & r \to 1 \\ c_2 + c_3 - c_4 = 0 & c_4 = \frac{c_2}{2} & f_{,r} = 0 & \text{for} & r \to 1 \end{cases} \tag{72}$$

Finally

$$\psi = c_2 \sin(\vartheta) \left[r \log(r) - \frac{1}{2} r + \frac{1}{2} r^{-1} \right]$$

Stokes solution for cylinder

$$\psi = c_2 \sin(\vartheta) \left[r \log(r) - \frac{1}{2} r + \frac{1}{2} r^{-1} \right] \tag{73}$$

Second and third terms are irrotational (uniform stream and dipole). First term represents rotational contribution, known as *Stokeslet*

Stokes' Paradox (1851)

The boundary condition at infinity can not be complied for any value of c_2. The velocity is not limited at infinity. No steady slow flow exists past a cylinder

In contrast with the solution for the sphere, the stokeslest is now more singular at infinity than uniform stream and predicts velocities that are unbounded far from the body

Whitehead Paradox

Analogous difficulties arise with three-dimensional bodies, though they are deferred to the second approximation for finite shapes: usually flow disturbances are weaker in three dimensions than two

Whitehead's Paradox (1889)

The solution for the second order approximation to Stokes' solution for unbounded uniform flow past three-dimensional body does not exist

The full Navier-Stokes equations give

$$[D^4]\psi = \frac{Re}{r^2 \sin(\vartheta)} \left(\psi_{,\vartheta} \frac{\partial}{\partial r} - \psi_{,r} \frac{\partial}{\partial \vartheta} + 2 \frac{\cos(\vartheta)}{\sin(\vartheta)} \psi_{,r} - \frac{2}{r} \psi_{,\vartheta} \right) [D^2]\psi \quad (74)$$

where on the right handside convective terms are present

Whitehead Paradox

Introducing a direct perturbation $\psi = \psi_0 Re + \psi_1 Re + \mathcal{O}(Re^2)$ of Re and plugging it in (74) we obtain the solutions at different orders

$$\mathcal{O}\left(Re^0\right) \quad [D^4]\psi_0 = 0 \quad \psi_0 = \sin^2(\vartheta)\left(\tfrac{1}{2}r^2 - \tfrac{3}{4}r + \tfrac{1}{4}r^{-1}\right) \tag{75}$$

$$\mathcal{O}\left(Re^1\right) \quad [D^4]\psi_1 = -\tfrac{9}{4}\left(2r^{-2} - 3r^{3} + r^{-5}\right)\sin^2(\vartheta)\cos(\vartheta) \tag{76}$$

A particular integral of (76) satisfying surface condition is found to be

$$-\frac{3}{32}\left(2r^2 - 3r + 1 - r^{-1} + r^{-2}\right)\sin^2(\vartheta)\cos(\vartheta) \tag{77}$$

However velocity does not behave properly at infinity and no complementary function can be added to correct it. Whitehead postulated the rise of discontinuities associated with the formation of a dead-water wake. This explanation is known to be incorrect

Oseen Approximation

Oseen (1910) showed how Stokes and Whitehead paradoxes arise from the singular nature of the flow at low Reynolds number. He showed that convective effects can not be neglected everywhere in the domain

Convective terms $\propto U_r U_{r,r}$ Viscous Terms $\propto \frac{1}{Re} U_{r,rr}$

- Cylinder

$$U_r = \frac{1}{r}\psi_{,\vartheta} = \frac{1}{r} c\cos(\vartheta)\left(\frac{1}{2}r^{-1} - \frac{1}{2}r + r\log(r)\right) =$$

$$= c\cos(\vartheta)\left(\frac{1}{2}r^{-2} - \frac{1}{2} + \log(r)\right) \quad (78)$$

Oseen Approximation

$$\lim_{r\to\infty} U_r U_{r,r} = \lim_{r\to\infty} \left(\frac{1}{2}r^{-2} - \frac{1}{2} + \log(r)\right)\left(-r^{-3} + r^{-1}\right) = = \lim_{r\to\infty}\left(\frac{\log(r)}{r}\right) \quad (79)$$

$$\lim_{r\to\infty} U_{r,rr} = \lim_{r\to\infty}\left(3r^{-4} - r^{-2}\right) = \lim_{r\to\infty}\left(-r^{-2}\right) \quad (80)$$

Ratio of neglected terms to those retained

$$\lim_{r\to\infty}\left|\frac{\text{convective}}{\text{viscous}}\right| = \lim_{r\to\infty} Re\ r\log(r) \quad (81)$$

Oseen Approximation

- Sphere

$$U_r = \cos(\vartheta)\left[1 - \frac{3}{2}r^{-1} + \frac{1}{2}r^{-3}\right]$$

$$\lim_{r\to\infty} U_r U_{r,r} = \lim_{r\to\infty}\left(1 - \frac{3}{2}r^{-1} + \frac{1}{2}r^{-3}\right)\left(\frac{3}{2}r^{-2} - \frac{3}{2}r^{-4}\right) = \\ = \lim_{r\to\infty}\left(\frac{3}{2}r^{-2}\right) \quad (82)$$

$$\lim_{r\to\infty} U_{r,rr} = \lim_{r\to\infty}\left(-3r^{-3} + 6r^{-5}\right) = \lim_{r\to\infty}\left(-3r^{-3}\right) \quad (83)$$

Ratio of neglected terms to those retained

$$\lim_{r\to\infty}\left|\frac{\text{convective}}{\text{viscous}}\right| = \lim_{r\to\infty}\frac{1}{2}r\ Re \quad (84)$$

Oseen Approximation

Stokes approximation becomes invalid once $Re\ r$ is of the order of unity. This occurs at distances of the order of ν/U^*, meaning that the viscous length is then the significant reference dimension

$$Re\ r \sim \mathcal{O}(1) \quad \Rightarrow \quad \frac{r^*}{a^*} \sim \frac{\nu}{U^* a^*} \tag{85}$$

In three dimensional flow the difficulty tends to be concealed because the first approximation is sufficiently well behaved

Oseen Solution

Oseen approximated the convective terms by their linearized forms valid far from the body, where the difficulties arises. This constitutes an ad hoc uniformization of a non uniform direct perturbation

$$u u_{,x} + v u_{,y} + w u_{,z} = \begin{cases} 0 & \text{Stokes} \\ U u_{,x} & \text{Oseen} \end{cases} \tag{86}$$

The Oseen's equations provide a unformly valid first approximation for either plane or three-dimensinal flow at low Reynolds number
Assuming $\mathbf{U} = \mathbf{U_0} + \mathbf{u}$

$$\left(\mathbf{U} \cdot \nabla\right) \mathbf{U} = \left(\mathbf{U_0} \cdot \nabla\right) \left(\mathbf{U_0} + \mathbf{u}\right) + \left(\mathbf{u} \cdot \nabla\right) \left(\mathbf{U_0} + \mathbf{u}\right) \tag{87}$$

$$\nabla^2 \mathbf{U} = \nabla^2 \mathbf{U_0} + \nabla^2 \mathbf{u} \tag{88}$$

Oseen Solution

Being $\mathbf{U_0}$ an uniform flow we obtain

$$\nabla^2 \mathbf{U_0} = 0 \quad ; \qquad (\mathbf{U_0} \cdot \nabla)\, \mathbf{U_0} = (\mathbf{u} \cdot \nabla)\, \mathbf{U_0} = 0 \tag{89}$$

The problem is as the previous one, with the following extra terms

$$(\mathbf{U_0} \cdot \nabla)\, \mathbf{u} \qquad (\mathbf{u} \cdot \nabla)\, \mathbf{u} = 0 \tag{90}$$

The second term is still negligible with respect to viscous terms

$$U_r = \cos(\vartheta) \left[1 - \frac{3}{2} r^{-1} + \frac{1}{2} r^{-3} \right] = U_{0r} + u_r \tag{91}$$

Oseen Solution

$$\lim_{r\to\infty}\left|\frac{u_r u_{r,r}}{\frac{1}{Re}u_{r,rr}}\right| = \lim_{r\to\infty}\left|\frac{-\frac{9}{4}r^{-3}}{\frac{1}{Re}\left(-3r^{-3}\right)}\right| = \frac{3}{4}Re \ll 1 \qquad (92)$$

$$\lim_{r\to\infty}\left|\frac{U_{0r} u_{r,r}}{\frac{1}{Re}u_{r,rr}}\right| = \lim_{r\to\infty}\left|\frac{\frac{3}{2}r^{-2}-\frac{3}{2}r^{-4}}{\frac{1}{Re}\left(-3r^{-3}+6r^{-5}\right)}\right| = \frac{1}{2}Re\ r \sim \mathcal{O}(1) \qquad (93)$$

When $r \longrightarrow 1$, i.e. close to the body surface, both terms vanish and convective terms are negigible. In this case Oseen's solution and Stokes' solution are coincident

Oseen Solution

Although Oseen's equations are linear, their solution is sufficiently complex that no second approximations are known. At the first order of approximation for the sphere the stream function reads

$$\psi_0 = \left(\frac{1}{2}r^2 + \frac{1}{4}r^{-1}\right)\sin^2(\vartheta) - \frac{3}{Re}\left(1+\cos(\vartheta)\right) \cdot \left[1 - \exp\left(-\frac{Re\ r}{4}\left(1-\cos(\vartheta)\right)\right)\right] \tag{94}$$

$$\text{if} \quad r \to \infty \qquad \psi_0 \to \frac{1}{2}r^2\sin^2(\vartheta) \tag{95}$$

Oseen Solution

$$\text{if} \quad r \to 1 \qquad \exp\left(-\frac{Re\ r}{4}(1-\cos(\vartheta))\right) \to 1 - \frac{Re\ r}{4}(1-\cos(\vartheta)) \quad (96)$$

$$\psi_0 \to \left(\frac{1}{2}r^2 + \frac{1}{4}r^{-1}\right)\sin^2(\vartheta) - \frac{3}{4}r(1-\cos(\vartheta))(1+\cos(\vartheta)) \quad (97)$$

$$\psi_0 \to \left(\frac{1}{2}r^2 + \frac{1}{4}r^{-1} - \frac{3}{4}r\right)\sin^2(\vartheta) \qquad \text{Stokes Solution} \quad (98)$$

As done by Oseen, Lamb (1911) obtained the solution for the cylinder

Oseen Solution

Finally, once derived the solution for the sphere and the cylinder it is possible to evaluate drag coefficients

- Sphere

$$C_D = \frac{24}{Re}\left[1 + \frac{3}{16}Re\right] \tag{99}$$

- Cylinder

$$C_D = \frac{8\pi}{Re \log\left(7.4/Re\right)} \tag{100}$$

Oseen Solution

Second approximation far from sphere

Improvement of Stokes' solution applying the method of Matched Asymptotic Expansion (Kaplun and Lagerstrom, 1957; Proudman and Pearson, 1957)

More details in *Perturbation Methods in Fluid Mechanics*, Milton Van Dyke, 1975